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Partial diagonality of stress trace

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Abstract. The trace of the stress energy tensor is shown to be diagonal between two states whose total four-momenta are equal. Additional comments are made on certain matrix-elements of the time-derivative of the dilation charge.

Keywords. Stress energy tensor; dilation.

The matrix elements of the stress energy tensor $\theta_{\mu\nu}$ are of interest [Pagels (1966), Gross and Wess (1970)] in a variety of physical situations. Apart from their measurability in gravitational couplings, they also contribute to lepton-hadron reactions [Mack (1971), Chanowitz and Ellis (1972)] through the occurrence of $\theta_{\mu\nu}$ in short-distance operator-product expansions of two hadronic currents. In this note we state and prove a result on certain matrix elements of the trace of the stress energy tensor θ^{μ}_{μ} which may turn out to be a useful constraint in the context of the above applications. The result is that θ^{μ}_{μ} is diagonal between two states of equal four-momenta. It derives crucially from the fact that the volume integral of θ^{μ}_{μ} is the time-derivative of the dilation charge $D(t)$. We also show that $\mathrm{d}D(t)/\mathrm{d}t$ is diagonal between two states whose relative four-momenta are spacelike or lightlike.

The result that the trace of the stress energy tensor is diagonal between two states of equal four-momenta will find application as a low energy theorem in situations where θ^{μ}_{μ} plays a role. These include gravitational theories where there is a scalar component coupling to θ^{μ}_{μ} as well as theories with dilation fields proportional to θ^{μ}_{μ}. In these cases zero momentum-transfer transitions of a state to itself plus any number of soft photons are forbidden.

Let us first state our main result formally.

THEOREM. If the states $|m\rangle$ and $|n\rangle$ have four momenta p_m and p_n respectively, then $\lim_{p_n \to p_m} \langle n | \theta^{\mu}_{\mu} | m\rangle$, defined appropriately (*i.e.*, with the ratio $(p^0_n - p^0_m)\,(\vec{p}^{\,2}_n - \vec{p}^{\,2}_m)^{-1}$ maintained finite), is proportional to δ_{nm}.

Proof. It is sufficient to show that $\langle n | \theta^{\mu}_{\mu} | m \rangle$ vanishes for $|n\rangle \neq |m\rangle$ under the given conditions. Use the results [Gell-Mann (1969), Carruthers (1971)] that

$$\int d^3x\theta^{\mu}_{\mu}(x) = \frac{dD(t)}{dt}$$

$$[D(t), P_0] = i\,\frac{dD(t)}{dt} - iP_0$$

and

$$[D(t), \vec{P}^2] = -2i\vec{P}^2$$

to obtain

$$(2\pi)^3\,\delta^{(3)}\,(\vec{p}_m - \vec{p}_n)\langle n \mid \theta^{\mu}_{\mu} \mid m\rangle = p^0_m \langle n \mid m\rangle + i\,(p^0_n - p^0_m)\langle n \mid D(t) \mid m\rangle$$

or,

$$(2\pi)^3\,\delta^{(3)}(\vec{0})\langle n \mid \theta^{\mu}_{\mu} \mid m\rangle = \left[p^0_m - 2\lim_{p_n \to p_m} \frac{p^0_n - p^0_m}{\vec{p}^2_n - \vec{p}^2_m}\,\vec{p}^2_m\right]\langle n \mid m\rangle.$$

If $(p^0_n - p^0_m)(\vec{p}^2_n - \vec{p}^2_m)^{-1}$ is finite, the right hand side is proportional to δ_{nm} and since $\delta^{(3)}(\vec{0})$ is nonvanishing, the matrix-element $\langle n \mid \theta^{\mu}_{\mu} \mid m\rangle$ must be zero for $\mid n\rangle \neq \mid m\rangle$ QED.

In order to maintain the finiteness of $\lim_{p_n \to p_m} (p^0_n - p^0_m)(\vec{p}^2_n - \vec{p}^2_m)^{-1}$, one has to follow some kind of a prescription in taking the limit in question. One prescription, for instance, is to put the masses equal ($p^2_n = p^2_m = \mu^2$) first and then take $p_n \to p_m$. This leads to the result

$$(2\pi)^3\,\delta^{(3)}(\vec{0})\langle n \mid \theta^{\mu}_{\mu} \mid m\rangle = \frac{\mu^2}{p^0_m}\langle n \mid m\rangle \tag{1}$$

which vanishes for $\mid n\rangle \neq \mid m\rangle$. It should be pointed out that the equality $p_n = p_m$ of the four momenta is essential to the derivation of the result, otherwise the ratio $(p^0_n - p^0_m)(\vec{p}^2_n - \vec{p}^2_m)^{-1}$ cannot in general be maintained to be finite. For instance, let $\mid m\rangle$ be the vacuum and $\mid n\rangle$ a particle-antiparticle pair state so that in the CM frame $\vec{p}_n = \vec{p}_m = 0$ but $p^0_n \neq p^0_m = 0$. In this frame the ratio in question becomes infinite and the proof fails. This can be verified explicitly. For particles of spin $\frac{1}{2}$ and mass μ, we have from Pagels (1966) that

$$\langle p, \bar{p} \mid \theta^{\mu}_{\mu} \mid 0\rangle = \left(\frac{1}{2\pi}\right)^3 \frac{\bar{u}(p, s)}{\sqrt{4p_0\bar{p}_0}}[2\mu G_1(Q^2)\,P\cdot\gamma + G_2(Q^2)$$

$$\times P^2 + 3\mu\,G_3(Q^2)\,Q\cdot\gamma]\,v(\bar{p}, \bar{s}), \tag{2}$$

where $Q^2 = (p + \bar{p})^2$ and $P^2 = (p - \bar{p})^2$ and $G_1(0) = 0$, $G_2(0) = 1$. The right-hand side of eq. (1) nonvanishing in the CM frame. On the other hand, if $\mid m\rangle$ and $\mid n\rangle$ are chosen to be single particle states of the same four momenta p, then

$$\langle p \mid \theta^{\mu}_{\mu} \mid p\rangle = \left(\frac{1}{2\pi}\right)^3 \frac{1}{4p_0}\,\bar{u}(p, s^1)\,4\mu^2 u(p, s) = \left(\frac{1}{2\pi}\right)^3 \frac{\mu^2}{p_0}\,\delta_{s^1 s}$$

in agreement with eq. (1).

The statement on the diagonality of the operator $\mathrm{d}D(t)/\mathrm{d}t$ (as opposed to θ^{μ}_{μ}) can be made more general by weakening the condition $p_n = p_m$ since one does not need to use the three-dimensional data function $\delta^{(3)}(\vec{0})$ here. Thus for $(p_n - p_m)^2 \leqslant 0$, the limit of the matrix element

$$\lim_{p_n \to p_m} \langle n | \frac{\mathrm{d}D(t)}{\mathrm{d}t} | m \rangle$$

defined in the above manner is proportional to δ_{nm}. This follows because

$$\left\langle n \left| \frac{\mathrm{d}D(t)}{\mathrm{d}t} \right| m \right\rangle = \left[p^0_m + \frac{2(p^0_n - p^0_m)}{\vec{p}^2_n - \vec{p}^2_m} \vec{p}^2_m \right] \langle n | m \rangle.$$

So long as $(p_n - p_m)^2 = 0^-$ as $p_n \to p_m$, the ratio $p^0_n - p^0_m / \vec{p}^2_n - \vec{p}^2_m$ is bounded from above by $\sqrt{(p_n - p_m)^2}/(\vec{p}_n - \vec{p}_m) \cdot (\vec{p}_n + \vec{p}_m)$; further this last ratio can be maintained to be finite provided $\vec{p}_n - \vec{p}_m$ is prescribed not to tend to zero in a direction orthogonal to $\vec{p}_n + \vec{p}_m$. For $(p_n - p_m)^2 > 0$, this argument does not hold and the diagonality of $\mathrm{d}D(t)/\mathrm{d}t$ gets destroyed. This may again be checked by taking the vacuum and the pair-state and using eq. (2).

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