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Application of holographic addition and subtraction to quadratic motion analysis

P C GUPTA* and K SINGH**
Department of Physics, Indian Institute of Technology, New Delhi 110029

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Abstract. Hologram interferometry of objects moving with constant acceleration has been investigated with special reference to the techniques of holographic addition and subtraction, which results in an extension of the range of measurement of conventional time average holography. Distribution of intensity in the fringes has been plotted for some typical cases.

Keywords. Hologram interferometry; time average holography.

Introduction

The technique of time-average hologram interferometry, pioneered by Powel and Stetson (1965) is now well known for its ability to measure motion and vibrations of three-dimensional objects with interferometric precision. The technique has been applied to study different types of motions such as sinusoidal vibration, linear motion, sinusoidal oscillations with uniform slow drift, step motion and damped sinusoidal oscillations, etc. Detailed bibliographies on the subject and results of theoretical and experimental investigations are available (Fryer 1970; Aleksoff 1971; Singh 1972, 1973).

Under certain conditions, the motion of the object may have quadratic dependence on time. This motion has been analyzed holographically and the characteristic fringe function has been evaluated by Neumann (1967) and later on by Vikram and Sirohi (1971). It is now well known that in the reconstructed image the intensity at any point is the intensity of the image of the static object times the square of the absolute value of the characteristic fringe function. Moreover, the equivalence between the characteristic fringe function and the optical transfer function of image motion is also well known.

The transfer function for this type of motion has been derived by Rosenau (1961) and its effect on the diffraction images of one-dimensional periodic objects has been considered by Rattan and Singh (1971). Som (1971) has derived an expression for the transfer function that is valid for any linear smear irrespective of whether or not the smear is due to uniform relative motion and has treated quadratic motion as a special case.

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* Present address: Advanced Centre for Electronic Systems, Indian Institute of Technology Kanpur 208016.

** To whom all correspondence should be addressed.

However, it is to be noted that conventional time-average hologram interferometry is not suitable for the measurement of severe motions. In such cases, a complete dark reconstruction is obtained and from it no information about the motion can be extracted. This is true in the case of quadratic motion also (Vikram and Sirohi 1971).

Many methods for overcoming this limitation have been suggested in the literature. Two of these, namely, holographic addition (Wall 1969, 1970) and holographic subtraction (Hariharan 1973) have been analyzed here to study holographic images of objects moving with constant acceleration.

Theory

Let us consider an object the motion of which is governed by the equation

$$x(t) = at^2 + bt + c \tag{1}$$

Using the approach of Zambuto and Lurie (1970) the characteristic fringe function for this type of motion may be written in the form

$$C = \frac{1}{T} \int_0^T \exp\left\{ i \frac{2\pi}{\lambda} (\cos\alpha + \cos\beta)(at^2 + bt + c) \right\} dt \tag{2}$$

where α is the angle between the axis of observation and the displacement vector and β is the angle between the direction of propagation of the incident light upon the object and the displacement vector. It is assumed that the recording time T is very large compared to $2\pi/\omega$, ω and λ being respectively the angular frequency and wavelength of monochromatic radiation used.

Putting $(2\pi/\lambda)$ $(\cos\alpha + \cos\beta)\, a = a'$, $(2\pi/\lambda)$ $(\cos\alpha + \cos\beta)\, b = b'$ and $(2\pi/\lambda)$ $(\cos\alpha + \cos\beta)\, c = c'$ expression (2) becomes

$$C = \frac{1}{T} \int_0^T \exp\{ i\, [a't^2 + b't + c'] \}\, dt.$$

On solving this, C is given by

$$\begin{aligned} C &= \frac{\exp\left[-\, i\left(\frac{b'^2}{4a'} - c' \right) \right] (P + iQ)}{(x - y)} \\ &= \frac{\exp\left[-\, i\left(\frac{b'^2}{4a'} - c' \right) \right] \sqrt{P^2 + Q^2}\, \exp i\left[\tan^{-1}\left(\frac{Q}{P} \right) \right]}{(x - y)} \\ &= \frac{\sqrt{P^2 + Q^2}\, \exp\left[i\left(\tan^{-1}\frac{Q}{P} - \frac{\pi y^2}{2} + c' \right) \right]}{(x - y)} \end{aligned} \tag{3}$$

where

$$P = \int_y^x \cos\left(\frac{\pi}{2} u'^2 \right) du'$$

$$Q = \int_y^x \sin\left(\frac{\pi}{2} u'^2 \right) du'$$

Here

$$x = \left(\frac{2a'}{\pi}\right)^{\frac{1}{2}} \left(T + \frac{b'}{2a'}\right)$$

and

$$y = \left(\frac{2a'}{\pi}\right)^{\frac{1}{2}} \left(\frac{b'}{2a'}\right) \tag{4}$$

Here the parameter x is a function of a', b' and T, whereas y is a function of a' and b'. Since a and b (and therefore a' and b') govern the shape of the displacement curve [$x(t)$ *vs* t curve], y is a measure of the shape of the curve, whereas x in addition to governing the shape also contains T (the recording time) and therefore decides the maximum displacement. The intensity in the reconstructed image is now given by

$$I_1 = I_0 \mid C \mid^2$$

where I_0 is the intensity distribution of the static object. Thus normalized intensity in the image becomes

$$\bar{I}_1 = I_1/I_0 = \mid C \mid^2$$
$$= \frac{P^2 + Q^2}{(X - Y)^2} \tag{5}$$

This expression is identical to that derived by Vikram and Sirohi (1971). It is clear from expression (3) that the characteristic fringe function contains a phase term which modulates the phase of the reconstructed fringes. It has been stated by Neumann (1967) that this phase modulation term merely superimposes a Fresnel lens on the hologram which displaces the position of the blurred image without further changing its nature. This factor plays quite an important role in the case of holographic addition and subtraction as shown in the following sections.

Holographic addition

In the case of holographic addition two holograms of the same object are recorded in succession on the same recording media. The first exposure is made while the object is under motion and the second exposure is made at $t = 0$ while the object is stationary. Due to coherent addition of amplitudes, the intensity in the reconstructed image is given by the expression

$$I_1 = I_0 \mid e^{ic'} + C \mid^2$$

or

$$\bar{I}_1 = I_1/I_0 = \mid e^{ic'} + C \mid^2 \tag{6}$$

On substituting the value of C from expression (3) in expression (6), we have

$$\bar{I}_1 = \left| e^{ic'} + \frac{\sqrt{P^2 + Q^2} \exp\left[i\left(\tan^{-1}\left(\frac{Q}{P}\right) - \frac{\pi y^2}{2} + C'\right)\right]}{(x - y)} \right|^2$$

$$= \frac{(x - y)^2 + (P^2 + Q^2) + \left[2P(x - y)\cos\left(\frac{\pi}{2}y^2\right) + 2Q(x - y)\sin\left(\frac{\pi}{2}y^2\right)\right.}{(x - y)^2} \tag{7}$$

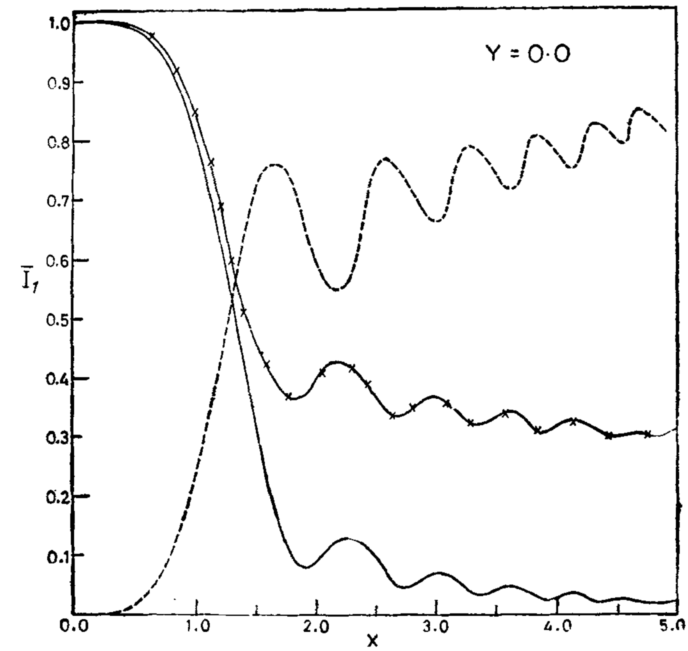

Figure 1. Variation of intensity as a function x, for $y = 0{\cdot}0$
—— for conventional time-average hologram interferometry.
× × × × for holographic addition.
- - - - for holographic subtraction.

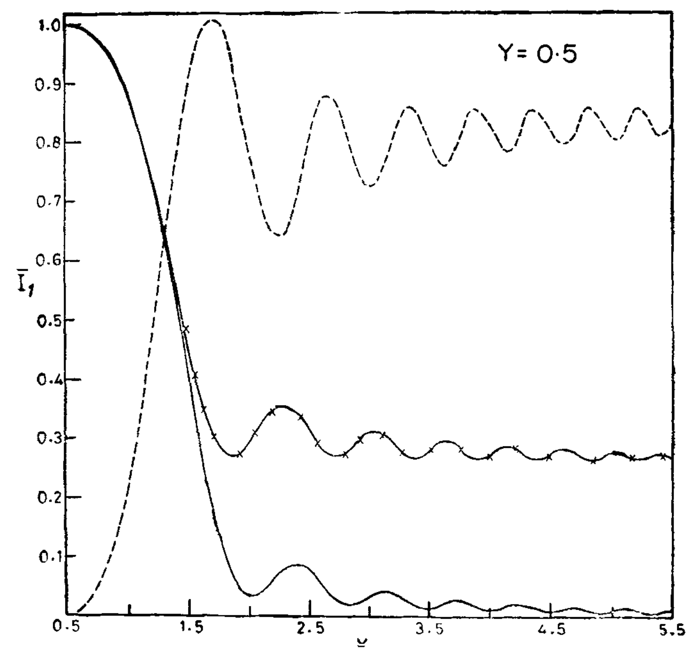

Figure 2. Same as in figure 1, for $y = 0{\cdot}5$.

Holographic subtraction

In the case of holographic subtraction also two holograms of the same object are recorded in succession on the same recording media. First exposure is made when the object is under motion and the second exposure is made when the object is stationary (*i.e.*, at $t = 0$) but in this exposure the phase of the reference beam is shifted by 180°. Due to coherent subtraction of amplitudes the intensity in the reconstructed image is given by the expression

$$I_1 = I_0 \mid e^{i\epsilon'} - C \mid^2$$

or

$$\bar{I}_1 = I_1/I_0 = \mid e^{i\epsilon''} - C \mid^2 \tag{8}$$

Substituting the value of C from expression (3) and on solving we have

$$\bar{I}_1 = \frac{(x-y)^2 + (P^2+Q^2) - \left[2P(x-y)\cos\left(\frac{\pi}{2}y^2\right) + 2Q(x-y)\sin\left(\frac{2}{\pi}y^2\right)\right]}{(x-y)^2} \tag{9}$$

Results and discussion

Expressions (5), (7) and (9) have been used to plot the results in graphical form for the intensity $\bar{I}_1$ in the reconstructed image *versus* X in figures 1 to 4 for various values of y. The solid line curves are for normal time-average holography and curves with crosses and dots are for holographic addition and subtraction respectively. Tabulated values of Fresnel integrals were taken from Abramowitz and Stegun. The fact that $x > y$ is used in plotting all the figures. The solid line curves in all the figures for various values of y agrees well with the results of Vikram and Sirohi (1971). In the case of holographic addition the image intensity has been properly normalized so that it is unity for $x = 0$ and in the case of holographic subtraction the curves have been normalized so that the intensity for first bright fringe is unity except when $y = 0$.

It is clear from the figures that in the case of conventional time-average holography, the intensity in the higher order fringes decreases rapidly. Moreover with the application of techniques of holographic addition and subtraction the contrast of higher order fringes improves. This improvement in contrast is more in the case of holographic subtraction than in the case of holographic addition.

Since in the case of holographic subtraction there is only one single minimum which occurs at $x = 0$, deviation from this point can be detected easily. Thus, motions characterized by small values of x can be analyzed with ease. It is also obvious from the figures that with an increase in the value of y, the central *bright* fringe narrows down and the intensity of the higher order bright fringes decreases in the case of holographic addition. In the case of holographic subtraction, narrowing of the central *dark* fringe takes place. Since increase in y (small a, large b) means that the displacement curve approaches a straight line, intensity distribution for conventional time average case approaches the well-known sine-square function.

Thus in the two techniques the fringe brightness above the hologram noise is increased because the bright fringes are brighter than the background illumination.

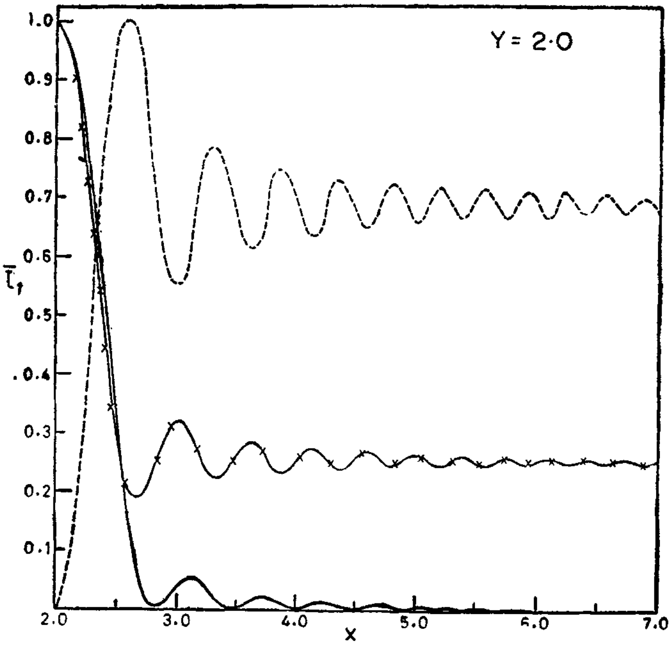

Figure 3. Same as in figure 1, for $y = 2 \cdot 0$.

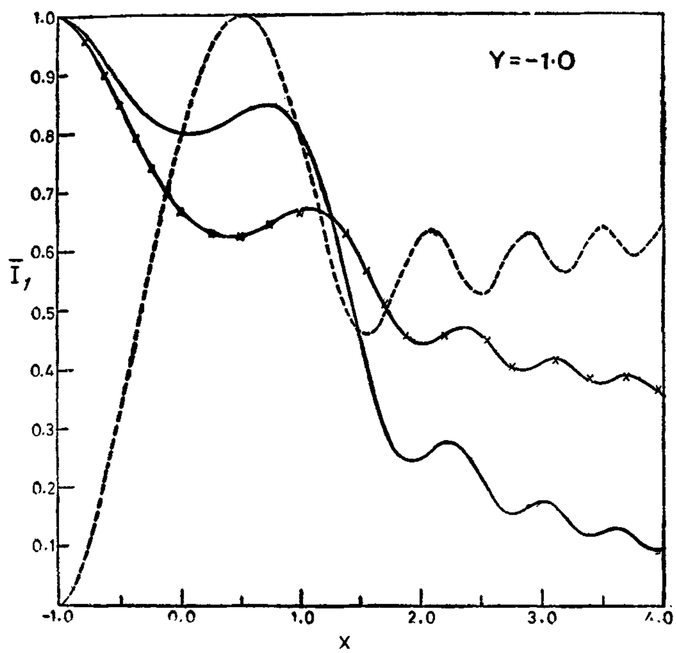

Figure 4. Same as in figure 1, for $y = -1 \cdot 0$.

Fringe counting is made easier if D.C. bias in position is resolved and thus the measurement range extended.

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