Pramāṇa, Vol. 3, No. 5, 1974, pp 306-310. © Printed in India.

Dual pole eikonal and s-channel singularities

S BISWAS and P DASGUPTA
Department of Physics, University of Kalyani, Kalyani 741235

MS received 26 July 1974

Abstract. Eikonalization of dual pole amplitudes, such as the Veneziano amplitude, is shown to lead to singularities in the impact parameter plane, which, in the eikonal approximation, may be interpreted as branch points in the direct channel angular momentum plane. This result is discussed in the light of dual absorption models.

Keywords. Eikonal approximation; Regge poles.

During the last few years there have been several attempts to construct models which incorporate both duality and absorptive corrections. Such a synthesis seems to go well with phenomenological evidences (Harari 1971 *a*, Cohen-Tannoudji *et al* 1972). The absorptive correction implies the existence of cuts in the complex angular momentum plane. As a consequence the duality principle in a typical dual absorption model should be stated as ' resonances = Regge poles + cuts '. In the analysis that follows we shall show that such a generalized pole-cut duality is strongly indicated within the framework of eikonal approximation.

It is well known that unitarization of Regge pole models usually leads to branch point singularities in the complex angular momentum plane in the crossed channel (Harari 1971 *a*). Thus Regge cuts in the crossed channel arise in the theories which attempt to sum multi-Regge exchange graphs or multiple scattering terms. As an example, the substitution of a Regge pole amplitude for the single scattering term in the Glauber representation in the s-channel leads to Regge cuts in the t-channel (Mandelstam 1963, Arnold 1965). Here we want to point out that if one assumes a dual amplitude such as the Veneziano beta function, for the single scattering term, then the eikonal approximation seems to imply the existence of cuts in the complex angular momentum plane for the direct channel also.

We proceed with the assumption that in the Glauber representation in the s-channel

$$A(s, \boldsymbol{q}) = \frac{i}{2\pi} \int d^2 b \exp(i\boldsymbol{q}\cdot\boldsymbol{b}) [\exp\{i\chi(s,b)\} - 1] \tag{1}$$

$\boldsymbol{q}$ being the momentum transfer ($t = -q^2$), the eikonal $\chi(s, b)$ is given by the Fourier transform of the Veneziano amplitude

$$\chi(s, b) = -\frac{\beta}{2\alpha'} \int_0^1 dx\,(1-x)^{-\alpha(s)} \frac{x^{-\alpha_0}}{\ln x} \exp(b^2/4\,\alpha' \ln x) \tag{2}$$

where α_0 and α' are respectively the intercept and slope of the linearly rising Regge trajectory, β is the multiplicative constant of the Veneziano amplitude. In eq. (2), we have used a standard integral representation of the beta function. That the amplitude (1) has moving Regge cut in the t-channel can be easily shown by expanding the exponential in the manner of Frautschi and Margolis (1968). In the eikonal approximation, the Glauber representation for small angles is supposed to represent the high energy limit of the partial wave expansion. Thus in this approximation $\frac{1}{2}\chi(s, b)$ is the phase shift at high energy corresponding to the partial wave with $l = kb - \frac{1}{2}$, k denoting the c.m. momentum. This enables one to derive the analytic properties of the partial wave amplitude at high energy by studying the analytic properties of $\chi(s, b)$ as a function of b for fixed s.

It may be observed in eq. (2) that the possible occurrence of singularities in b will depend on the behaviour of the integrand near $x = 1$. Thus putting $\nu = -1/\ln x$ and dividing up the integral into (0 to N) and (N to ∞), N being sufficiently large, the leading term of the latter part of the integral is

$$g(c, \lambda) = \int_N^\infty \mathrm{d}\nu\, \nu^\lambda \exp(-c\nu) \tag{3}$$

with $\lambda = -1 + \alpha(s)$ and $c = b^2/4\alpha'$. The integral in eq. (3) may be related to the incomplete gamma function (Luke 1962)

$$\Gamma(a, z) = \int_z^\infty \mathrm{d}t\, \mathrm{e}^{-t}\, t^{a-1}. \tag{4}$$

Using the formula,

$$\Gamma(a, z) = \Gamma(a) - a^{-1} z^a\, {}_1F_1(a, a+1, z) \tag{5}$$

we get from eqs (3) and (4)

$$g(c, \lambda) = \frac{1}{c^{\lambda+1}}\left[\Gamma(\lambda+1) - \frac{(cN)^{\lambda+1}}{\lambda+1}\, {}_1F_1(\lambda+1, \lambda+2, -cN)\right] \tag{6}$$

This shows that the eikonal has a fixed branch point at $c = 0$. Consequently the phase and, hence, the partial wave amplitude must have a fixed branch point in the complex l plane at $l = -\frac{1}{2}$. For negative integer λ, eq. (6) is not suitable. In that case one may directly proceed with eq. (3) and, after working out the integral, a logarithmic branch point at $c = 0$ may easily be noticed.

The discontinuity of the analytically continued partial wave amplitude $f(l, s)$ across the cut resulting from this branch point singularity may then be calculated in the eikonal approximation:

$$\begin{aligned} f(l, s) - f(l\mathrm{e}^{2\pi i}, s) = \frac{1}{2ik}\{&\exp[\mathrm{i}\rho\Gamma(-\alpha)(4\alpha')^{-\alpha}/b^{-2\alpha}] \\ &- \exp[\mathrm{i}\rho\Gamma(-\alpha)(4\alpha')^{-\alpha}\,\mathrm{e}^{4\pi i\alpha}/b^{-2\alpha}]\} \times \\ &\times \exp\left\{\mathrm{i}\rho\left[\int_{1/N}^\infty \mathrm{d}\mu\,\frac{\mathrm{e}^{-b^2/4\alpha'\mu}}{\mu}\,\mathrm{e}^{(\alpha_0-1)\mu}(1-\mathrm{e}^{-\mu})^{-\alpha}\right.\right. \\ &\left.\left. + \frac{N^{-\alpha}}{\alpha}\,{}_1F_1(-\alpha, -\alpha+1, b^2/4\alpha')\right]\right\} \end{aligned} \tag{7}$$

where

$$\rho = \beta/2\alpha'.$$

The argument of Kaus and Zachariasen (1971) may then be repeated to show that the result of this cut is again to produce complex conjugate poles in pairs on the angular momentum plane.

As another example, we may take the dual and crossing symmetric amplitude of Cohen-Tannoudji *et al* (1971) in a simplified form to show, in the eikonal approximation, the existence of the fixed s-channel cut in the complex angular momentum plane. We define the eikonal

$$\chi(s, b) = \frac{\beta}{2\pi} \int \mathrm{d}^2 q \exp(\mathrm{i}\, \boldsymbol{q} \cdot \boldsymbol{b})\, M(s, \boldsymbol{q}) \tag{8}$$

where

$$M(s, \boldsymbol{q}) = \int_0^1 (1-x)^{-\alpha(sx)-1} x^{-\alpha(tx')-1} f(sx) f(tx')\, \mathrm{d}x$$

and

$$x' = 1 - x, \quad f(y) = \exp(py).$$

$M(s, \boldsymbol{q})$ gives a dual pole model which has all the features of the Veneziano model and, in addition, can be made to possess the Mandelstam analyticity by suitably choosing α and f. We take here a simplified version of the model (and thus give up some attractive features of the general model) by choosing α to be real and linear with $f(y) = e^{py}$. Equation (8) then becomes

$$\chi(s, b) = -\beta \int_0^1 \mathrm{d}x\, (1-x)^{-\alpha(sx)-1} f(sx)\, x^{-(1+\alpha_0)} \exp\{b^2/4\, (p + \alpha' \ln x)\, x'\} \times \frac{1}{(p + \alpha' \ln x)\, x'} \tag{9}$$

Again the singularities in b will depend on the behaviour of the integrand near $x = 1$. As before, the analysis may be carried out to relate $\chi(s, b)$ to the incomplete gamma function. Again, one gets a branch point singularity at $b = 0$ which implies the existence of a fixed s-channel cut at $l = -\frac{1}{2}$.

It may be recalled that, as Blankenbecler and Goldberger (1962) have shown, the Fourier-Bessel transform of a scattering amplitude containing s-channel Regge pole has similar branch point singularity at $b = 0$. As a matter of fact, for large momentum transfer, the small b behaviour of the kernel of the Blankenbecler-Goldberger representation dominates and leads to the usual Regge-pole term $t^{\alpha(s)} [\sin \pi\alpha(s)]^{-1}$. This leads us to believe that any dual pole model (containing both the t- and s-channel Regge poles) treated under the eikonal approximation would lead to branch points in angular momentum in the s-channel, in addition to the branch points in the t-channel.

The conclusion is that a mechanism which generates Regge cuts in the crossed channel from a dual pole model (*i.e.*, satisfying the relations ' sum over the s-channel poles = sum over the t-channel poles ') also gives rise to a cut in the complex angular momentum plane for the direct channel. This suggests that an approximate duality may be supposed to hold between the branch cuts as well as poles of the s- and t-channels. This, of course, would be duality with a difference: whereas the t-channel cut is a moving cut, the s-channel cut turns out to be a fixed one.

That we are not getting moving branch points in both the channels, as it would be the case in a perfectly dual theory, may be due to the approximations involved in the analysis. In particular we have tried to spot singularities near the origin of the angular momentum plane by looking at a function which closely approximates the partial wave amplitude for large angular momentum. This might have altered the nature of the singularity considerably and does in no way disprove the possible existence of a strict duality among Regge cuts.

Further, if we suppose that branch points represent unitarity corrections to pole models, then the above examples show that unitarization in the s-channel automatically achieves at least partial unitarity corrections in the t-channel.

The structure of scattering amplitude on the impact parameter plane has been analyzed in detail by Kupsch and Stamatescu (1973). They show that the amplitude can be decomposed into two parts, one of which is singular at $b = 0$, the other being singular at $b \neq 0$. While the former determines the large t behaviour, the latter determines the large s behaviour. In fact, a simple pole at some non-zero complex b (with suitable choice for the s-dependence of the pole position) is able to reproduce the peripheral resonance dominance of Harari (1971 *b*) and the effective s-channel trajectory of Schrempp and Schrempp (1973, 1974) which have attained quite a lot of phenomenological success in the high energy region. One way of understanding the origin of the effective s-channel trajectory is to look at the effect of superposition of infinitely many resonances in one partial wave of the Veneziano amplitude. It turns out that the narrow resonances join with each other to produce a single broad resonance which seems to lie on an effective trajectory with the same energy dependence as postulated by Schrempp and Schrempp (1973, 1974). In the light of the analysis presented in this paper, going over to the impact parameter plane, this would mean that infinitely many terms (the Fourier-Bessel transforms of the Veneziano parents and daughters) which are singular at $b = 0$ may join together to generate a term having simple pole at $b = b_p \neq 0$. Therefore, it should be possible to obtain a clear connection between the terms singular at $b = 0$ and the terms singular at $b = b_p$ implying thereby a connection between the large s-behaviour and the large t-behaviour. At present, this suggests two lines of investigation. Firstly, it might be interesting to study a model in which one deals with a superposition of Veneziano terms with trajectories smeared out over a finite interval (*i.e.*, the intercepts and slopes of the trajectories are allowed to take values over a finite range) and to see how an effective peripheral trajectory originates in such a model. Secondly, one might also try to formulate a bootstrap theory in the impact-parameter language. The bootstrap mechanism connects the large s-behaviour with the large t-behaviour by demanding that the s-channel and the t-channel processes are governed by the same strong interaction. This connection as we have emphasized, should be reflected on the b-plane in a simple connection between the terms which are singular at $b = 0$ and the terms which are singular at $b = 0$. Therefore, it will not be surprising if the bootstrap theory takes a simple form on the impact parameter plane.

Acknowledgements

The authors would like to thank Dr S Mallik for valuable discussions. One of the authors (S B) acknowledges financial assistance from UGC.

References

Arnold R C 1965 *Phys. Rev.* **140 B** 1022
Blankenbecler R and Goldberger M L 1962 *Phys. Rev.* **126** 765
Cohen-Tannoudji G, Henyey F and others 1971 *Phys. Rev. Lett.* **26** 112
Cohen-Tannoudji G, Lacaze R and others 1972 *Nucl. Phys.* **B 45** 109
Frautschi S and Margolis B 1968 *Nuovo Cimento* **56 A** 115
Harari H 1971 *Phys. Rev. Lett.* **26** 1400
Harari H 1971 *b Proc. Int. Conf. Duality and Symmetry in Hadron Physics*; Tel-Aviv
Kupsch J and Stamatescu I O 1973 *Nuovo Cimento* **15** 663
Luke Y L 1962 *Integrals of Bessel Function* (McGraw-Hill, New York)
Mandelstam S 1963 *Nuovo Cimento* **30** 1148
Schrempp B and Schrempp F 1973 CERN Preprint TH 1736; 1974 CERN Preprint TH 1812
Zachariasen F 1971 CERN Preprint TH 1290