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Fluctuation power spectrum of a Josephson junction

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Abstract. The fluctuation power spectrum of the Josephson junction has been evaluated in the limit of large energy barriers $[(U_n^{\max} - U_n^{\min})\gamma] \gg 1$ and small currents $[x \ll x_c]$. The result is valid for finite capacitance of the junction. The effect of the fluctuating Josephson current on the voltage $V(t)$ across the junction has also been taken into account.

Keywords. Josephson junction; Josephson current; fluctuation power spectrum.

1. Introduction

We consider here the case of capacitative loading in which the Josephson junction is kept in series with a large effective resistance R and a battery, so that it is driven by a constant current source I. It is well-known (Ambegaonkar and Halperin 1964, Ivanchenko and Zilberman 1969, Biswas and Jha 1970) that the equations satisfied by θ, the phase difference of the order parameter across the junction and V, the potential difference are equivalent to those of Brownian motion of a particle of coordinate θ, momentum $\hbar cV/2e$ in a potential

$$U(\theta) = -(\hbar/2\theta)(I\theta + I_1 \cos\theta) \tag{1}$$

where I_1 is the maximum value of the Josephson current. Let us introduce the dimensionless quantities

$$x = I/I_1, \quad v = V/I_1 R, \quad \tau = t/RC$$
$$\gamma = \hbar I_1/ek_B T, \quad \beta_0 = 2eI_1 R^2 C^2/\hbar C \tag{2}$$

2. Fokker-Planck equation

The corresponding Fokker-Planck equation for $P(v, \theta, \tau)$ giving the distribution of v, θ, τ is given by (Biswas and Jha 1970)

$$\frac{\partial P}{\partial \tau} = -\beta_0 v \frac{\partial P}{\partial \theta} + \frac{2}{\gamma k_B T}\frac{\partial U}{\partial \theta}\frac{\partial P}{\partial v} + \frac{\partial}{\partial v}(vP) + \frac{2}{\gamma\beta_0}\frac{\partial^2 P}{\partial v^2} \tag{3}$$

The potential $U(\theta)$ has well-defined maxima and minima at

$$\theta = \theta_n^{\max} = \pi - \sin^{-1} x + 2n\pi$$

and

$$\theta = \theta_n^{\min} = \sin^{-1} x + 2n\pi$$

respectively.

Near the n-th maximum and the n-th minimum the potential $U(\theta)$ can be approximated respectively as

$$U(\theta) \approx U_n^{\max} + \tfrac{1}{2}(\hbar/2e)(\theta - \theta_n^{\max})^2 \cos\theta_n^{\max}$$
$$= U_n^{\max} - \tfrac{1}{2}(\hbar/2e)(\omega_0^2/\beta_c) X^2 \tag{4}$$

with

$$U_n^{\max} = -(\hbar/2e)\left[x(\pi - \sin^{-1}x + 2n\pi) - (1 - x^2)^{\frac{1}{2}}\right]$$
$$X = \theta - \theta_n^{\max}$$
$$\omega_0 = \beta_c^{\frac{1}{2}}(1 - x^2)^{\frac{1}{4}} \tag{5}$$

and

$$U(\theta) \approx U_n^{\min} + \frac{\hbar}{2e}\frac{1}{2}\left(\frac{\omega_0^2}{\beta_c}\right)Y^2$$

with

$$U_n^{\min} = -(\hbar/2e)\left[x(\sin^{-1}x + 2n\pi) + (1 - x^2)^{\frac{1}{2}}\right]$$
$$Y = \theta - \theta_n^{\min}.$$

3. Solution of the F. P. equation and the power spectrum

One can now solve the F.P. equation (3) in the neighbourhood of $\theta = \theta_n^{\min}$ and $\theta = \theta_n^{\max}$ following the method of Chandrasekhar (1943). The power spectrum $P(\omega)$ can then be calculated from the formula

$$P(\omega) = \tfrac{1}{4} I^2 \int_{-\infty}^{\infty} dt \exp[i(\delta\omega)t] \exp[-\psi(t)]$$

where

$$\delta\omega = \omega - (2eV_0/\hbar) - (2e\bar{V}/\hbar)$$

$$\psi(t) = (2e^2/\hbar^2)\int_0^t dt_1 \int_0^t dt_2 \langle \tilde{V}(t_0 + t_1)\tilde{V}(t_0 + t_2)\rangle$$

$$\tilde{V}(t) = V - \bar{V}, \quad \bar{v} = \bar{V}/I_1 R$$

$$\bar{v} = 1/\beta_c \{[1 + 4\beta_c(1 - x^2)^{\frac{1}{2}}]^{\frac{1}{2}} - 1\} \exp\{-\gamma[(1 - x^2)^{\frac{1}{2}} + x\sin^{-1}x]\}$$
$$\times \sinh\tfrac{1}{2}\pi\gamma x \tag{6}$$

$$\langle \tilde{V}_1(\tau + \tau_1)\tilde{V}_1(\tau_1)\rangle$$
$$= \mathcal{L}t_{\tau_1 - \tau_0 \to \infty} \int d\tilde{V} d\tilde{V}_1 d\theta_1 d\theta \tilde{V}_1 \tilde{V} P(\tilde{V}_1, \theta_1, | \tilde{V}, \theta, \tau_1)$$
$$\times P(\tilde{V}, \theta, \tau_1 | \tilde{V}_0, \theta_0, \tau_0) \tag{7}$$

In (7) under the condition that $[(U_n^{\max} - U_n^{\min})\gamma] \gg 1$ and $x \ll x_c$, (*see* Chandrasekhar 1943, and Ivanchenko and Zilberman 1969) we replace the integral over θ by summation over points $\theta = \theta_n^{\max}$ and $\theta = \theta_n^{\min}$. The resulting power spectrum is given by

$$P(\omega) \propto \coth\frac{\hbar\omega_0}{2k_B TRC}\left[BR^2C^2(\delta\omega)^2 + B(1 - B)^2 + \omega_0^2 B + 1\right]$$
$$\times \left\{[(\delta\omega)^2 - (\omega_0^2 + B + B^2)/R^2C^2]^2 + (\delta\omega)^2\frac{(1 + 2B)^2}{R^2C^2}\right\}^{-1} \tag{8}$$

where

$$B = 4\lambda^2\left(\lambda + \frac{3}{2}\right)(\lambda + 1)^{-2}\sinh^2 \tfrac{1}{2}\pi\gamma x \exp\{-2\gamma\,[(1 - x^2)^{\frac{1}{2}} + x \sin^{-1}x]\}$$
$$\lambda = (\tfrac{1}{4} + \omega_0^2)^{\frac{1}{2}} - \tfrac{1}{2}. \tag{9}$$

4. Conclusion

Since we are considering a temperature much below the critical temperature the particle only rarely goes out of the potential well. The case resembles that of a damped oscillator (Ferrel 1968) with friction coefficient $(1 + 2B)(RC)^{-1}$ and natural resonant frequency $[\omega_0^2 + B + B^2]^{\frac{1}{2}}/RC$. The contribution through B is due to the effect of the fluctuating Josephson current giving rise to an effective average voltage at finite temperature. Since the Josephson junction is used for precision measurements at low temperature, the above calculation is very important to understand the limitation of such a measurement. For a typical Josephson junction $x = 0{\cdot}7$, $\beta_c = 5{\cdot}4 \times 10^2$ and $\omega_0^2 \simeq 3{\cdot}86 \times 10^2$, $1/RC \simeq 1{\cdot}3 \times 10^5$ Hz.

For the above values of the parameters the calculated values of B (*see* equation 9) for different values of T/T_c (T_c = critical temperature) have been given below:

T/T_c	B	T/T_c	B
0·3	5×10^{-5}	0·7	$7{\cdot}2 \times 10^{-2}$
0·4	$1{\cdot}4 \times 10^{-3}$	0·8	$1{\cdot}4 \times 10^{-1}$
0·5	$1{\cdot}2 \times 10^{-3}$	0·9	$2{\cdot}4 \times 10^{-1}$
0·6	$2{\cdot}9 \times 10^{-2}$		

The ratio of the frequency shift due to noise to the noiseless resonance frequency is given by $\simeq B/\omega_0^2$. The line-width from (8) is given by

$$\Gamma = \frac{(\omega_0^2 + B + B^2)}{RC\,(1 + 2B)} \simeq \frac{\omega_0^2}{RC}(1 - 2B) \tag{10}$$

so that the ratio of the noise contribution to the line-width to the noiseless one is $\simeq 2B$.

The value of $|B|$ increases from values of the order of 10^{-5} to the order of 10^{-1} as the temperature ratio T/T_c increases from 0·3 to 0·9. The thermal noise effect is therefore very significant and must be taken into account in a precision measurement with a Josephson junction.

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