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Properties of the symplecton calculus

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Abstract. The representation of the group SU(2) afforded by the symplecton calculus of Biedenharn and Louck is mathematically related to the older treatments of the representations of this group. The method used is similar to the phase space description of quantum mechanics, and considerably simplifies important calculations.

Keywords. Symplecton calculus; SU(2).

Introduction

In the construction of unitary representations of continuous groups that arise in physical problems, physicists frequently make use of the algebra of harmonic oscillator operators, or as they are also called, boson operators. Thus one starts with a certain number of independent boson creation and annihilation operators and tries to form functions of these which will obey the commutation rules of the Lie algebra of the continuous group one is interested in. The motivations for this procedure are clearly the ease with which one can compute commutators among functions of boson operators and the ease with which one can deal with the states of these oscillators, their scalar products, etc.

A continuous group whose representation theory finds very wide applications in physical problems is SU(2), the unitary unimodular group in two dimensions.* This happens because SU(2) is the universal covering group of the group of real orthogonal rotations in three-dimensional space. And it is in connection with SU(2) that one has the best-known example of the use of the boson operator technique for construction of group representations, namely, the Jordan–Schwinger construction (Jordan 1935; Schwinger 1952). This expresses the three hermitian generators of SU(2) as certain functions of the annihilation and creation operators associated with two independent harmonic oscillators, and one is thereby led to a specific unitary representation of SU(2). Actually, considered purely as a representation of SU(2), what one obtains *via* the Jordan–Schwinger construction is identical to a representation of SU(2) obtained by more elementary means not involving oscillator operators at all but merely certain polynomials in classical, commuting, complex variables. This latter is described, for example, in Weyl's

* For the representation theory of this group see, for example, Edmonds A R (1957).

book (Weyl 1931). Thus one may look upon the Jordan–Schwinger calculus as being a mathematically equivalent reformulation of Weyl's treatment, but which makes certain calculations simpler.

One might have thought that the Jordan–Schwinger construction was minimal in the sense that one could not do with less than two independent oscillators in building up the SU(2) structure. However, it has been shown by Biedenharn and Louck that this is not so, and that a truly minimal oscillator operator construction of the representations of SU(2) exists which works with one boson alone. (Biedenharn and Louck 1971). They have named this construction the "Symplecton calculus." (However, this reduction from the use of two bosons to just one is accompanied by some basic changes in the interpretation of the formalism; this will be briefly discussed later). When one identifies the representation of SU(2) provided by the symplecton calculus, one finds it to be the same as the one provided by the Jordan–Schwinger scheme or by Weyl's treatment. One would suspect therefore that there is a well-defined mathematical transformation that relates the symplecton calculus to the older ways of treating the group SU(2), which transformation brings out clearly the points of equivalence in the various treatments. The purpose of this paper is to develop this equivalence transformation in detail, and also to show how some calculations in the symplecton calculus can be simplified with its use.

The material of this paper is arranged as follows. Section 1 contains a brief resumé of the Weyl and Jordan–Schwinger treatments of SU(2), followed by an account of the symplecton technique. In section 2 we establish the equivalence between the symplecton calculus and the Weyl treatment of SU(2) which uses polynomials in classical commuting variables. With the help of this equivalence, we are able to give a closed expression for an important set of operator polynomials whose defining properties were set down by Biedenharn and Louck. This is an illustration of the usefulness of the equivalence mentioned above. Another illustration is contained in section 3 where we give a simple proof of an important multiplication law for the above-mentioned polynomials; the proof given by Biedenharn and Louck was based on induction while ours is somewhat more direct. In the concluding section, we make a few remarks comparing the different treatments of SU(2).

1. Various treatments of SU(2)

The defining representation of the group SU(2) consists of all complex unitary unimodular matrices in two dimensions. Thus a general element g can be written in the form

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{1.1}$$

in terms of two complex parameters α, β [The bar denotes complex conjugation]. Group multiplication corresponds to matrix multiplication in this defining representation. One usually identifies the three (hermitian) infinitesimal generators in this case with the Pauli matrices; writing J_k, $k = 1, 2, 3$ for them in general, one has here:

$$J_k \to \tfrac{1}{2}\sigma_k \tag{1.2}$$

In an arbitrary (unitary) representation of SU(2), the (hermitian) operators J_k obey the commutation relations

$$[J_k, J_l] = i\epsilon_{klm} J_m \tag{1.3}$$

As is well-known, there is one unitary irreducible representation (UIR) of SU(2) (up to equivalence) in each complex dimension $(2j+1)$, $j = 0, \tfrac{1}{2}, 1, \ldots$ And within any given UIR one can introduce a basis consisting of the eigenvectors of J_3, say, with well-known expressions for the matrix elements of the other two generators J_1, J_2.

The Weyl, the Jordan–Schwinger, and the symplecton calculi are three superficially different ways of realising the UIR's of SU(2). Each of them leads to a direct sum of all the UIR's of SU(2), once each. We describe them briefly in the above-named sequence.

Let us start with Weyl's treatment.* We consider polynomials $f(\xi, \eta)$ in two complex variables ξ, η. For conciseness, we may think of ξ and η as constituting the two entries in a row-vector Ψ, and then write $f(\Psi)$ for the above polynomial. The set of all these polynomials obviously forms a complex linear vector space $\mathscr{P}$ of infinite dimension. In this space we define a set of linear operators T_g, $g \in$ SU(2), that give us a linear representation of SU(2):

$$(T_g f)(\Psi) = f(\Psi g) \tag{1.4}$$

Here, the element g should be thought of as the 2×2 matrix appearing in eq. 1.1. It is trivial to verify that we do have a representation of SU(2) here. The evaluation of the infinitesimal generators of this representation is straightforward. Writing $J_k^{(W)}$ for them (the superscript W is for Weyl), we identify their forms by setting

$$\begin{aligned} T_g &\simeq 1 + i\theta_k J_k^{(W)} + 0(\theta^2) \\ g &\simeq 1 + i\theta_k \sigma_k/2 + 0(\theta^2) \end{aligned} \tag{1.5}$$

We then find:

$$J_1^{(W)} = \frac{1}{2}\left(\xi\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\xi}\right), \; J_2^{(W)} = -\frac{i}{2}\left(\xi\frac{\partial}{\partial\eta} - \eta\frac{\partial}{\partial\xi}\right),$$

$$J_3^{(W)} = \frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) \tag{1.6}$$

The familiar raising and lowering combinations are given by:

$$\begin{aligned} J_+^{(W)} &= J_1^{(W)} + iJ_2^{(W)} = \xi\frac{\partial}{\partial\eta} \\ J_-^{(W)} &= J_1^{(W)} - iJ_2^{(W)} = \eta\frac{\partial}{\partial\xi} \end{aligned} \tag{1.7}$$

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* The details that follow may also be found in Bargmann V 1962.

This infinite dimensional representation of SU(2) is well-known to be a direct sum of the finite dimensional UIR's of SU(2), each appearing once. The reduction is achieved by considering polynomials $f(\xi, \eta)$ which are homogeneous of a given degree; this degree is clearly preserved by the transformations T_g in eq. (1.4). For each value of j, $j = 0, \frac{1}{2}, 1, \ldots$, a basis for homogeneous polynomials of degree $2j$ is provided by the monomials

$$f_{jm}(\xi, \eta) = \xi^{j+m}\, \eta^{j-m}/[(j+m)!\,(j-m)!]^{\frac{1}{2}},$$
$$m = j, j-1, \ldots, -j \tag{1.8}$$

which are $(2j+1)$ in number. The index m on f_{jm} is the eigenvalue of the generator $J_3^{(W)}$, and the $(2j+1)$ monomials for given j transform under the action of T_g *via* the spin j UIR of SU(2); the numerical factors ensure that in this basis each T_g is represented by a unitary matrix. Since these monomials taken together for all j and m obviously span the space of all polynomials $f(\xi, \eta)$, this construction has led to a direct sum of all the UIR's of SU(2), once each.

The Jordan–Schwinger method introduces two independent boson creation operators $a_\alpha{}^\dagger$ and their conjugates a_α obeying the commutation relations

$$[a_\alpha, a_\beta{}^\dagger] = \delta_{\alpha\beta}, \;\; [a_\alpha, a_\beta] = [a_\alpha{}^\dagger, a_\beta{}^\dagger] = 0; \;\; \alpha, \beta = 1, 2 \tag{1.9}$$

These operators act on a Hilbert space $\mathscr{H}$ with a positive-definite metric and the dagger denotes hermitian conjugation relative to this metric.* It follows that there is a unique vector $|0\rangle$ in the Hilbert space (unique up to a phase factor) satisfying

$$a_\alpha |0\rangle = 0, \;\; \alpha = 1, 2; \;\; \langle 0|0\rangle = 1 \tag{1.10}$$

All the vectors of the Hilbert space are generated by acting on the vector $|0\rangle$ with polynomials in the two (commuting) creation operators $a_\alpha{}^\dagger$. One now defines a unitary representation of SU(2) on $\mathscr{H}$ by choosing the generators to be:

$$J_+^{(J\text{-}S)} \equiv J_1^{(J\text{-}S)} + iJ_2^{(J\text{-}S)} = a_1{}^\dagger a_2$$
$$J_-^{(J\text{-}S)} \equiv J_1^{(J\text{-}S)} - iJ_2^{(J\text{-}S)} = a_2{}^\dagger a_1$$
$$J_3^{(J\text{-}S)} = \tfrac{1}{2}(a_1{}^\dagger a_1 - a_2{}^\dagger a_2) \tag{1.11}$$

(The superscript J-S is for Jordon-Schwinger). The hermiticity of the generators is obvious. To reduce this representation one only needs to notice that all the generators commute with the total number operator

$$N = a_1{}^\dagger a_1 + a_2{}^\dagger a_2 = N_1 + N_2 \tag{1.12}$$

Consequently all those states with a fixed eigenvalue $2j$ for N transform among themselves, and in fact irreducibly, under SU(2). A basis for the Hilbert space can be defined by

$$|j, m\rangle = (a_1{}^\dagger)^{j+m}(a_2{}^\dagger)^{j-m}|0\rangle/[(j+m)!\,(j-m)!]^{\frac{1}{2}},$$
$$m = j, j-1, \ldots, -j; \;\; j = 0, \tfrac{1}{2}, 1, \ldots \tag{1.13}$$

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* We appeal here to the standard operator treatment of the simple harmonic oscillator in quantum mechanics given, for example, in Messiah A (1970).

These vectors in fact form an orthonormal basis for the space, and again m is the eigenvalue of $J_3^{(J-S)}$. And for fixed j, the $(2j+1)$ basis vectors $|j, m\rangle$ transform *via* the spin j UIR of SU(2). The SU(2)-representation contents are clearly identical in the Weyl and the Jordan-Schwinger schemes.

In fact, one can see that the Jordan–Schwinger calculus is essentially identical to the Weyl treatment: the creation operators $a_\alpha{}^\dagger$ in the former are analogous to the commuting complex variables ξ, η in the latter, and one can think of a_1, a_2 as being analogous to $\partial/\partial\xi$, $\partial/\partial\eta$ consistent with the commutation rules eq. (1.9). The orthonormal basis vectors $|j, m\rangle$ correspond to the basic monomials $f_{jm}(\xi, \eta)$. The chief advantage in the operator treatment is that the notion of scalar products among the vectors is built into the Hilbert space structure so that computation of matrix elements, etc., is made automatic. One could impose a Hermitian scalar product into the space $\mathscr{P}$ of classical polynomials $f(\xi, \eta)$ too if one wished, by *defining* the basic monomials to satisfy

$$(f_{j'm'}(\xi, \eta), f_{jm}(\xi, \eta)) = \delta_{j'j}\delta_{m'm} \tag{1.14}$$

Then the parallel is complete. In particular, no more than the definition eq. (1.14) is needed to specify the scalar product, though if one wished to picture the scalar product of two polynomials $f(\xi, \eta)$ and $f'(\xi, \eta)$ as being given by some kind of integration over the complex ξ and η planes, that too could be arranged consistent with eq. (1.14) (Bargmann 1962):

$$(f'(\xi, \eta), f(\xi, \eta)) = \int d\xi d\bar{\xi} d\eta d\bar{\eta} \exp(-\xi\bar{\xi} - \eta\bar{\eta}) \overline{f'(\xi, \eta)} f(\xi, \eta) \tag{1.15}$$

Equipped with this scalar product, the complete mathematical identity of the Weyl and the Jordan–Schwinger techniques is easily seen.

The symplecton calculus generates a representation of SU(2) (in fact the same one encountered twice above) using just one boson operator system. One considers an operator a and its "conjugate" $\bar{a}$ subject to the rule

$$[\bar{a}, a] = \bar{a}a - a\bar{a} = 1 \tag{1.16}$$

To a limited extent, $\bar{a}$ and a are like boson destruction and creation operators, namely, to the extent that the commutation rules coincide. However, the underlying linear space on which $\bar{a}$ and a operate is *not* a Hilbert space with a positive-definite metric (in fact it is an indefinite metric space), and there is no vector annihilated by $\bar{a}$. In any case, in building up the SU(2) representation, attention is focussed not on the underlying vector space on which $\bar{a}$ and a operate, but on the linear space of all *operators* constructed as arbitrary polynomials in the pair of operators $\bar{a}$, a subject to eq. (1.16). Let this linear space be called S. A particularly simple way to generate a linear operator acting *on* S is to consider the operation of taking the commutator of a fixed element in S with a general one: this defines an operator on S associated with the chosen fixed element (which is of course some polynomial in $\bar{a}$ and a). The three generators of SU(2) are built up in just this way. One defines three operators $X_\pm$, X_3 by:

$$X_+ = -\tfrac{1}{2}a^2,\ X_- = \tfrac{1}{2}\bar{a}^2,\ X_3 = \tfrac{1}{4}(a\bar{a} + \bar{a}a) \tag{1.17}$$

These X's are elements of S. For any element $f(a, \bar{a})$ in S, we define the associated linear operator $\Omega(f)$ acting on S by

$$\Omega(f)f'(a, \bar{a}) = [f(a, \bar{a}), f'(a, \bar{a})] \tag{1.18}$$

f' being a variable element in S. The SU(2) generators $J^{(S)}$ are then chosen to be the Ω's corresponding to the X's:

$$\begin{aligned} J_\pm^{(S)} &\equiv J_1^{(S)} \pm iJ_2^{(S)} = \Omega(X_\pm) \\ J_3^{(S)} &= \Omega(X_3) \end{aligned} \tag{1.19}$$

The X's can be easily checked to obey the commutation relations of SU(2); via the Jacobi identity this property is then ensured for the $J^{(S)}$'s. (The superscript S denotes symplecton).

This construction leads therefore to a representation of SU(2) acting on the space S of operators built up polynomially from $\bar{a}$ and a. Now the degree of homogeneity of a polynomial $f(a, \bar{a})$ is not definable in the normal way in view of eq. (1.16). But it again turns out that the representation of SU(2) occurring in S can be broken up and expressed as a direct sum of the UIR's of SU(2), each one appearing once. And corresponding to this one can set up a set of basic polynomials $\mathcal{P}_{jm}(a, \bar{a})$ which are linearly independent, span S, and effect the reduction of the representation. The defining equations for determining the $\mathcal{P}_{jm}(a, \bar{a})$ have been given by Biedenharn and Louck.

One sees that though all three methods lead to the same representation of SU(2), the Weyl and Jordan–Schwinger methods work with polynomials in *commuting* variables [in (ξ, η) and in $(a_1^\dagger, a_2^\dagger)$ respectively] and define the representation on these polynomials; whereas the symplecton calculus defines the representation on polynomials in *noncommuting* variables a, $\bar{a}$. The similarity lies in the use of polynomials in *two quantities*. We will develop a detailed one-to-one correspondence between the Weyl and the symplecton methods, which will then clarify the structure of the latter. The technique we use is essentially what is used to express quantum mechanics in classical phase space language (Wigner 1932, Moyal 1949).

2. Relation between Weyl and symplecton calculi

A general polynomial $f(a, \bar{a})$ in S does not have a unique form because of eq. (1.16). However it does acquire a unique form if with the help of eq. (1.16) it is rewritten in such a way that in each term the power of a stands to the left of the power of $\bar{a}$, when it will be said to be in ordered form. Thus by considering expressions of the form

$$\sum_{r,s} f_{r,s}\, a^r \bar{a}^s \tag{2.1}$$

the $f_{r,s}$ being complex numbers, one botains all elements of S, there being a one-to-one correspondence between elements of S and sets of coefficients $\{f_{r,s}\}$. Given this, one can now set up a one-to-one correspondence between elements of S and those of $\mathcal{P}$ by associating with the operator (2.1) the classical (*i.e.* non-operator) polynomial

$$\sum_{r,s} f_{r,s}\, \xi^r \eta^s \tag{2.2}$$

Conversely, given any polynomial $f(\xi, \eta)$ in $\mathcal{P}$, we shall denote by

$$(f\,(\xi, \eta))_+ \tag{2.3}$$

the *operator* belonging to S and obtained from $f(\xi, \eta)$ by writing powers of ξ to the left of powers of η in each term, then substituting a for ξ and $\bar{a}$ for η without disturbing the positions. Thus we have

$$f(\xi, \eta) = \sum_{r,s} f_{r,s}\, \xi^r \eta^s \Rightarrow (f(\xi,\eta))_+ = \sum_{r,s} f_{r,s}\, a^r \bar{a}^s \tag{2.4}$$

An important property of the $(.)_+$ symbol is expressed by the following:

$$(f(\xi, \eta))_+ = 0 \Rightarrow f(\xi, \eta) = 0 \tag{2.5}$$

With the help of this correspondence every linear operation on S can be transferred to act as a linear operation on $\mathcal{P}$ and conversely. We see this now for the SU(2) generators defined in the symplecton calculus. It suffices to work with monomials and then extend the result to polynomials by linearity. For the case of $J_+^{(s)}$, we have:

$$J_+^{(s)}\, a^r\, \bar{a}^s = -\tfrac{1}{2}\,[a^2, a^r\, \bar{a}^s] = -\tfrac{1}{2}\, a^{r+2}\, \bar{a}^s + \tfrac{1}{2}\, a^r\, \bar{a}^s\, a^2 \tag{2.6}$$

Using eq. (1.16) to order the term $\bar{a}^s\, a^2$ we find:

$$\bar{a}^s\, a^2 = a^2\, \bar{a}^s + 2sa\bar{a}^{s-1} + s\,(s-1)\, \bar{a}^{s-2} \tag{2.7}$$

so that we have:

$$\begin{aligned} J_+^{(s)}\, a^r\, \bar{a}^s &= s\, a^{r+1}\, \bar{a}^{s-1} + \frac{s\,(s-1)}{2}\, a^r\, \bar{a}^{s-2} \\ &= \left(s\xi^{r+1}\, \eta^{s-1} + \frac{s\,(s-1)}{2}\, \xi^r\, \eta^{s-2}\right)_+ \\ &= \left(\left(\xi\, \frac{\partial}{\partial\eta} + \frac{1}{2}\, \frac{\partial^2}{\partial\eta^2}\right) \xi^r\, \eta^s\right)_+ \\ &= \left(e^Q\, \xi\, \frac{\partial}{\partial\eta}\, e^{-Q}\, \xi^r\, \eta^s\right)_+, \\ Q &\equiv \frac{1}{2}\, \frac{\partial^2}{\partial\xi\partial\eta} \end{aligned}$$

This result is immediately generalised to arbitrary polynomials belonging to S and reads:

$$J_+^{(S)}\, (f(\xi, \eta))_+ = \left(e^Q\, \xi\, \frac{\partial}{\partial\eta}\, e^{-Q} f(\xi, \eta)\right)_+ \tag{2.8}$$

In a similar fashion, one easily establishes the equations

$$J_-^{(S)}\, (f(\xi, \eta))_+ = \left(e^Q\, \eta\, \frac{\partial}{\partial\xi}\, e^{-Q} f(\xi, \eta)\right)_+,$$

$$J_3^{(S)}\left(f(\xi,\eta)\right)_+ = \left(e^Q \cdot \frac{1}{2}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right)e^{-Q}f(\xi,\eta)\right)_+, \tag{2.9}$$

in fact the SU(2) generators in the Weyl scheme and the symplecton scheme are related by:

$$J_k^{(S)}\left(f(\xi,\eta)\right)_+ = \left(e^Q J_k^{(W)} e^{-Q} f(\xi,\eta)\right)_+ \tag{2.10}$$

Now the operator e^Q is a linear, invertible one acting on the space $\mathcal{P}$ of polynomials in ξ, η. Equation (2.10) therefore shows that there is a complete equivalence of the SU(2) representation constructed in the symplecton calculus and the more elementary one constructed in the Weyl treatment. Since eqs (2.4, 5) have already established an equivalence between S and $\mathcal{P}$ as linear spaces, we can conclude that as far as linear and group representation aspects are concerned, the two structures are completely equivalent.

This equivalence gives a simple proof of the statement that the symplecton representation of SU(2) is also a direct sum of all the UIR's, once each. Otherwise, demonstration of this fact would entail construction of the basis polynomials $\mathcal{P}_{jm}(a,\bar{a})$ and verification of their properties. We can turn the situation to our advantage and use the equivalence to give a closed expression for these polynomials. Combining eqs (1.8, 2.4 and 2.10), we set:*

$$\mathcal{P}_{jm}(a,\bar{a}) = 2^j\left(e^Q f_{jm}(\xi,\eta)\right)_+ \tag{2.11}$$

The numerical factor included here is to obtain agreement with Biedenharn and Louck. By considering simple cases like $j=\frac{1}{2}$ and 1, one can check that one has agreement with the explicit expressions given by them. Since e^{-Q} exists, since $f_{jm}(\xi,\eta)$ form a basis for $\mathcal{P}$, and because of eq. (2.5), we have a simple proof that $\mathcal{P}_{jm}(a,\bar{a})$ do form a basis for S. And relations of the form

$$J_3^{(S)}\mathcal{P}_{jm}(a,\bar{a}) = m\mathcal{P}_{jm}(a,\bar{a}),$$
$$J_\pm^{(S)}\mathcal{P}_{jm}(a,\bar{a}) = [(j\mp m)(j\pm m+1)]^{\frac{1}{2}}\mathcal{P}_{j,m\pm1}(a,\bar{a}) \tag{2.12}$$

are automatic consequences of corresponding relations for the Weyl representation, and eq. (2.10).

3. The multiplication law

The equivalence between S and $\mathcal{P}$ so far established concerns the linear and group representation properties only and does not extend to multiplication. Clearly multiplication of two polynomials $f(\xi,\eta)$ and $f'(\xi,\eta)$ in $\mathcal{P}$ is commutative, while the operators $(f(\xi,\eta))_+$ and $(f'(\xi,\eta))_+$ do not commute. We need to express the multiplication law in S in terms of polynomials in $\mathcal{P}$; stated differently, given two polynomials $f(\xi,\eta)$ and $f'(\xi,\eta)$ in $\mathcal{P}$, we seek an expression for the polynomial $h(\xi,\eta)$ defined by

$$\left(f'(\xi,\eta)\right)_+\left(f(\xi,\eta)\right)_+ = \left(h(\xi,\eta)\right)_+ \tag{3.1}$$

Such an expression can be obtained by working first with monomials.

Consider then the product

$$(\xi^p\eta^q)_+(\xi^r\eta^s)_+ = a^p\bar{a}^q a^r\bar{a}^s \tag{3.2}$$

* It is clear that the polynomials $\mathcal{P}_{jm}(a,\bar{a})$ are uniquely determined up to j-dependent multiplicative constants by their SU(2) properties.

To move the factor $\bar{a}^q$ to the right with the help of eq. (1.16), we may think of $\bar{a}$ as $\partial/\partial a$ purely as a mnemonic; we then get:

$$(\xi^p \eta^q)_+ \; (\xi^r \eta^s)_+ = a^p \left(\sum_{k=0}^{\infty} \frac{r!}{(r-k)!} \frac{q!}{k!\,(q-k)!} a^{r-k} \bar{a}^{q-k} \right) \bar{a}^s$$

$$= a^p \left(\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial \xi^k} \frac{\partial^k}{\partial \eta^k} \xi^r \eta^q \right)_+ \bar{a}^s = (\xi^p \{e^{2Q} \xi^r \eta^q\} \eta^s)_+ \tag{3.3}$$

The operator Q here was defined in the equation preceding eq. (2.8). Now, to avoid splitting up the factors in each monomial we use a familiar device (Mehta, 1964) and write eq. (3.3) as:

$$(\xi^p \eta^q)_+ (\xi^r \eta^s)_+ = \left(\left\{ \exp\left(\frac{\partial^2}{\partial \xi_2 \partial \eta_1}\right) \cdot \xi_1^p \eta_1^q \xi_2^r \eta_2^s \right\}_{\xi_i = \xi,\ \eta_i = \eta} \right)_+ \tag{3.4}$$

One can now extend this result by linearity to any polynomial and one has

$$(f'(\xi, \eta))_+ (f(\xi, \eta))_+ = (h(\xi, \eta))_+$$

$$h(\xi, \eta) = \left\{ \exp\left(\frac{\partial^2}{\partial \xi_2 \partial \eta_1}\right) f'(\xi_1, \eta_1) f(\xi_2, \eta_2) \right\}_{\xi_i = \xi,\ \eta_i = \eta} \tag{3.5}$$

This is the desired expression for the non-commutative multiplication law for the operator polynomials in S.

We now make use of eq. (3.5) to derive an important multiplication law for the symplecton calculus polynomials $\mathcal{P}_{jm}(a, \bar{a})$. As noted by Biedenharn and Louck, this law is essential in the further development of their calculus. Consider then the product

$$\mathcal{P}_{jm}(a, \bar{a})\, \mathcal{P}_{j'm'}(a, \bar{a}) = 2^{j+j'} (e^Q f_{jm}(\xi, \eta))_+ (e^Q f_{j'm'}(\xi, \eta))_+ \tag{3.6}$$

Now the left hand side is again a polynomial in a and $\bar{a}$; after ordering it, it must necessarily be expressible as a linear combination of these same polynomials $\mathcal{P}_{j''m''}(a, \bar{a})$ since they span the space S. From general angular momentum theory we can say quite a lot about the form this linear combination must take. Keeping in mind eq. (2.12) as well as the fact that the SU(2) generators $J_k^{(S)}$ are the Ω-operators associated with the X_k [*see* eq. (1.19)], we can express the behaviour of $\mathcal{P}_{jm}(a, \bar{a})$ under finite elements of SU(2) thus:

$$\exp(i\theta_k J_k^{(S)})\, \mathcal{P}_{jm}(a, \bar{a}) \equiv \exp(i\theta_k X_k)\, \mathcal{P}_{jm}(a, \bar{a}) \exp(-i\theta_k X_k)$$

$$= \sum_{m'} D^j_{m'm}(\theta_k)\, \mathcal{P}_{jm'}(a, \bar{a}) \tag{3.7}$$

Here we have used axis-angle parameters for SU(2), and the matrices $D^j_{m'm}(\theta_k)$ are the usual D-functions of angular momentum theory. Let us now also recall the following facts: (i) the polynomials $\mathcal{P}_{jm}(a, \bar{a})$ are uniquely determined up to j-dependent numerical factors by their SU(2) transformation properties; (ii) the Clebsch-Gordan coefficients for SU(2) are unique, normalised, invariant three-index tensors for this group; (iii) these C–G coefficients also enjoy the usual orthonormality and completeness properties.* Combining all this with eq. (3.7), we immediately arrive at the following structure for the right hand side of eq. (3.6):

$$\mathcal{P}_{jm}(a,\bar{a})\,\mathcal{P}_{j'm'}(a,\bar{a}) = \sum_{j''=|j-j'|}^{j+j'} C(j'jj''\mid m', m, m'+m)\times$$
$$\times \langle j'' \mid j \mid j'\rangle\, \mathcal{P}_{j'', m+m'}(a,\bar{a}) \tag{3.8}$$

The unknowns here to be determined are the factors $\langle j'' \mid j \mid j'\rangle$ which have no dependence on the magnetic quantum numbers. To find them let us write eq. (3.8) in the equivalent form:

$$2^{j+j'}[(j+m)!\,(j-m)!\,(j'+m')!\,(j'-m')!]^{-\frac{1}{2}}\,(e^{Q}\xi^{j+m}\eta^{j-m})_{+}(e^{Q}\,\xi^{j'+m'}\eta^{j'-m'})_{+}$$
$$= \sum_{j''} C(j'jj''\mid m', m, m'+m)\,\langle j'' \mid j \mid j'\rangle\, 2^{j''}\times$$
$$\times\,[(j''+m+m')!\,(j''-m-m')!]^{-\frac{1}{2}}(e^{Q}\,\xi^{j''+m+m'}\,\eta^{j''-m-m'})_{+} \tag{3.9}$$

We can solve for the factors $\langle j'' \mid j \mid j'\rangle$ by using the law of multiplication eq. (3.5) to evaluate the left hand side. But let us first specialise eq. (3.9) to the case $m=j$, $m'=-j'$; this will be sufficient to ensure that none of the possible values of j'' are missed, and also simplifies the calculation. So we work with

$$2^{j+j'}[(2j)!\,(2j')!]^{-\frac{1}{2}}(\xi^{2j})_{+}(\eta^{2j'})_{+}$$
$$= \sum_{j''} 2^{j''}\,[(j''+j-j')!\,(j''-j+j')!]^{-\frac{1}{2}}\,C\,(j'jj''\mid -j', j, j-j')\times$$
$$\times\,\langle j'' \mid j \mid j'\rangle\,(e^{Q}\,\xi^{j''+j-j'}\,\eta^{j''-j+j'})_{+} \tag{3.10}$$

Application of eq. (3.5) to the left hand side is trivial; on the other hand, using eq. (2.5), we can convert eq. (3.10) into an equation involving ξ and η alone. Doing so and shifting the operator e^{Q} to the left, we get:

$$\sum_{j''} 2^{j''}\,[(j''+j-j')!\,(j''-j+j')!]^{-\frac{1}{2}}\,C(j'jj''\mid -j', j, j-j')\times$$
$$\times\,\langle j'' \mid j \mid j'\rangle\,\xi^{j''+j-j'}\,\eta^{j''-j+j'} =$$
$$2^{j+j'}[(2j)!\,(2j')!]^{-\frac{1}{2}}\times\exp\left(-\frac{1}{2}\frac{\partial^2}{\partial\xi\partial\eta}\right)\xi^{2j}\,\eta^{2j'} \tag{3.11}$$

The polynomial standing on the extreme right is easily computed:

$$\exp\left(-\frac{1}{2}\frac{\partial^2}{\partial\xi\partial\eta}\right)\xi^{2j}\,\eta^{2j'} = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,2^r}\,\frac{(2j)!\,(2j')!}{(2j-r)!\,(2j'-r)!}\,\xi^{2j-r}\,\eta^{2j'-r} \tag{3.12}$$

The actual upper limit to the index r is the lesser of $2j$ and $2j'$. If we change from the variable r to $j''=j+j'-r$ in eq. (3.12), then the range of j'' and the exponents of ξ and η agree exactly with the left hand side of eq. (3.11), so that we may directly compare terms and equate the coefficients. In this manner we get:

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* For the definitions and properties of the SU(2) Clebsch-Gordan coefficients *see*, for example, Edmonds A R (1957).

$$C(j'jj''\,|-j',j,j-j')\,\langle j''\,|\,j\,|\,j'\rangle = \frac{(-1)^{j+j'-j''}}{(j+j'-j'')!}\left[\frac{(2j)!\,(2j')!}{(j''+j-j')!\,(j''+j'-j)!}\right]^{\frac{1}{2}} \tag{3.13}$$

The particular C–G coefficient appearing here has the value (Edmonds 1957)

$$C(j'jj''\,|-j',j,j-j') = (-1)^{j+j'-j''}\left[\frac{(2j''+1)\,.\,(2j)!\;(2j')!}{(j+j'-j'')!\;(j+j'+j''+1)!}\right]^{\frac{1}{2}} \tag{3.14}$$

leading to the value

$$\langle j''\,|\,j\,|\,j'\rangle = \frac{(2j''+1)^{-\frac{1}{2}}\,[(j+j'-j'')!\;(j''+j-j')!\,(j'+j''-j)!}{(j+j'+j''+1)!]^{-\frac{1}{2}}} \tag{3.15}$$

for the unknowns in the product rule eq. (3.8). These results agree with those of Biedenharn and Louck. They have been derived here however by a slightly more direct and economical method, and suggest the value of utilising the relation between S and $\mathscr{P}$ that we have established.

If one examines the arguments that led to the general form eq. (3.8) for products of the basic polynomials $\mathscr{P}_{jm}(a, \bar{a})$, one realises that they are of a general group-theoretic nature and should therefore apply to the space $\mathscr{P}$ and the polynomials $f_{jm}(\xi, \eta)$ as well. That is, the relation

$$f_{jm}(\xi,\eta)\,f_{j'm'}(\xi,\eta) = \sum_{j''} C(j'jj''|m',m,m'+m)\,\langle j''\,|\,j\,|\,j'\rangle'\,f_{j'',m+m'}(\xi,\eta) \tag{3.16}$$

involving some new coefficients $\langle j''\,|\,j\,|\,j'\rangle'$ must necessarily hold. This is indeed the case, but one finds that only the term $j''=j+j'$ survives. In fact one has:

$$\langle j''\,|\,j\,|\,j'\rangle' = \delta_{j'',j+j'}\,[(2j)!\;(2j')!/(2j+2j')!]^{-\frac{1}{2}} \tag{3.17}$$

This characteristic difference between the product laws for the polynomials in S and in $\mathscr{P}$ clearly arises because the former are polynomials in non-commuting variables obeying eq. (1.16) while the latter are polynomials in the commuting variables ξ and η. This is an elucidation of the remark of Biedenharn and Louck that the product law eq. (3.8) with the coefficients in eq. (3.15) is a unique property of the symplecton calculus.

4. Concluding remarks

Guided by the fact that the SU(2) representation content in the symplecton calculus is identical with what one has in the older Weyl and Jordan-Schwinger treatments, we have set up the mathematical transformation that establishes the equivalence of the linear and group representation aspects of the two approaches. One important motivation behind the symplecton calculus is its use of a single boson operator system, in contrast to the two pairs used in the Jordan–Schwinger treatment. But this reduction in the number of operators used is in some sense superficial, since there is a change in the interpretation too. In the Jordan–Schwinger case, one has a definite solution to the boson operator commutation relations in a positive—definite Hilbert space, that is, a linear space of vectors exists on which the oscillator operators act linearly and irreducibly. And the transformations corresponding to elements of SU(2) also act on this space of vectors. Thus, the oscillator operators and the group representation act on the same space, and the vectors of this space can obviously be identified with polynomials in the two commuting creation operators. On the other hand, in the symplecton calculus, the initial step is not to find a solution to the commutation relation eq. (1.1a) in some suitable vector space, but rather to directly consider the linear space of polynomials in a and $\bar{a}$. And the SU(2) representation consists of linear operators acting on these polynomials rather than on an underlying vector space on which a and $\bar{a}$ themselves act. That is why the SU(2) generators $J_k^{(S)}$ are chosen as the "Ω-operators" corresponding to suitable polynomials $X_k(a, \bar{a})$. Thus, as noted before, one essentially ends up with a representation defined on polynomials in two quantities.

In this connection the following remark should be made. If one sets up a solution to the SU(2) commutation relations eq. (1.3) by means of hermitian operators J_k in a well-defined Hilbert space, then one need have no doubts that on exponentiation one will obtain a genuine representation of SU(2). This is the situation in the Jordan–Schwinger treatment. Alternatively, if one has any matrix solution at all to the commutation relations eq. (1.3) in a *finite-dimensional* vector space, then too one is assured that on exponentiation a representation of SU(2) will result; and in fact in a suitable basis the generators *will* appear as hermitian matrices. This is the case with the Weyl treatment if one approached it from the expressions in eq. (1.6) for the generators: these leave invariant subsets of polynomials homogeneous of a given degree, and such subsets are finite dimensional. [Of course, in this case one has the finite transformations already given in eq. (1.4)]. However, if one exhibits a solution to eq. (1.3) by means of linear operators in an infinite-dimensional space, without securing the Hermitian property for them, then in general one will not be dealing with a representation of SU(2) at all. A simple instance of this situation is the following: take any non-trivial unitary representation of the non-compact group SU(1, 1), possessing hermitian generators in a positive-definite Hilbert space (Bargmann 1947). By multiplying two of the SU(1, 1) generators (the "non-compact" ones) by factors of i and leaving the third alone, one immediately produces a solution to the SU(2) commutation relations but no representation of SU(2) at all.* At first sight the symplecton calculus solution to the SU(2) commutation rules would appear to belong to this third category. The space S of polynomials is infinite-dimensional, is not *a priori* equipped with a positive-definite scalar product, and the operators $\Omega(X_j)$ therefore have no verifiable hermiticity properties. There is therefore no guarantee that the $\Omega(X_j)$ will exponentiate to give a representation of SU(2). The equivalence we have established between the $J_k^{(S)}$ and the $J_k^{(W)}$ helps solve this problem for the symplecton calculus. It proves the existence of, and leads to explicit expressions for, the basic polynomials $\mathscr{P}_{jm}(a, \bar{a})$. Purely within the symplecton calculus itself, this important question could be settled by the realisation that the action of any

* The finite dimensional representations of the two groups SU(2) and SU(1, 1) are closely related: from a representation of the one we can get a representation of the other by analytic continuation.

$\Omega(X_l)$ on a polynomial $f(a, \bar{a})$ does not increase the degree of the leading terms in f.

Finally, it is clear that just as the symplecton calculus has a natural extension to the groups $Sp(2n)$, so does the analysis presented in this paper. This would bring in the language of classical Hamiltonian dynamics and phase space, and the Moyal scheme for transcription of quantum mechanics to classical language. The present work may also suggest generalisations of the symplecton calculus itself.

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References

Bargmann V 1947 *Ann. Math.* **48** 568

Bargmann V 1962 *Rev. Mod. Phys.* **34** 829

Biedenharn L C and Louck J D 1971 *Ann. Phys.* (*N.Y.*) **63** 459

Edmonds A R 1957 *Angular momentum in quantum mechanics* (Princeton University Press, Princeton, N.J.)

Jordan P 1935 *Z. Phys.* **94** 531

Mehta C L 1964 *J. Math. Phys.* **5** 677

Messiah A 1970 *Quantum mcehanics* Vol. 1 432–441 (North-Holland Publishing Company, Amsterdam)

Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99

Schwinger J 1952 USAEC Report NYO-3071 (unpublished); reprinted in *The quantum theory of angular momentum* eds L C Biedenharn and H Van Dam (Academic Press, New York) 1965

Weyl H 1931 *The Theory of groups and quantum mechanics* (Dover Publications, New York) Chapter III

Wigner E P 1932 *Phys. Rev.* **40** 749