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A study of the variable moment of inertia models

NAZAKAT ULLAH
Tata Institute of Fundamental Research, Bombay 400005

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Abstract. The variable moment of inertia (VMI) model proposed by Holmberg and Lipas has been shown to be a special case of the VMI model of Mariscotti *et al.* The solution of Mariscotti's model is expressed in terms of hypergeometric functions, which directly give the rotational energies or their expansions in terms of the quantity $\mathcal{J}(\mathcal{J}+1)$, where $\mathcal{J}$ is the total angular momentum. The present way of looking at the VMI model also tells us how to write the general dependence of the vibrational energy and the moment of inertia on the energy E_J.

Keywords. Variable moment of inertia; Holmberg-Lipas model; Mariscotti model; vibrational energy.

1. Introduction

For sometime now it has been realized (Holmberg and Lipas 1968; Mariscotti *et al* 1969) that the parameters E_0 and I in the usual expression for the rotational energies

$$E_J = E_0 + \frac{\hbar^2}{2I}\,\mathcal{J}(\mathcal{J}+1) \tag{1}$$

could themselves be energy dependent. This gives rise to the Variable Moment of Inertia (VMI) model, which has been theoretically justified by many workers (Klein *et al* 1970; Volkov 1971).

Holmberg and Lipas (1968) had used Taylor series expansion for the energy dependent parameters E_0 and I to arrive at the new expression for E_J. Mariscotti and coworkers (1969) arrived at their VMI model by relating the collective vibrational deformation parameter with the moment of inertia paramater. Thus the two VMI models look quite different and it is hard to see whether they are equivalent or one of them is better than the other for fitting the values of E_J.

The purpose of the present work is to recast Mariscotti's expressions which lead us to a new way of looking at the VMI model. This new way of looking at the VMI model is different from the use of infinite order expansions of Klein *et al* (1970). Using the present method we shall compare the Holmberg-Lipas model with Mariscotti's model. We shall also show that the energies E_J given by Mariscotti's model can be expressed by hypergeometric functions.

2. Mathematical formulation

The two expressions which determine the values of E_J in Mariscotti's model are given by

$$E_J = E_0 + \tfrac{1}{2}\,C(I_J - I_0)^2 + \frac{1}{2I_J}\,\mathcal{J}(\mathcal{J}+1) \tag{2a}$$

$$C(I_J - I_0) - \frac{1}{2I_J^2} \mathcal{J}(\mathcal{J}+1) = 0 \tag{2b}$$

where C and I_0 are constants which are determined by fitting the observed spectrum.

As had been remarked in the introduction, the Holmberg-Lipas model uses Taylor series expansion for I. The value of E_J in this model are given by

$$E_J = E_0 + \frac{1}{2[I_0 + I_1(E_J - E_0)]} \mathcal{J}(\mathcal{J}+1) \tag{3}$$

where I_0 and I_1 are the unknown parameters. In both equations (2) and (3) we have put $\hbar = 1$.

From equations (2b) and (3), it is obvious that in one case I_J is given as a function of $\mathcal{J}(\mathcal{J}+1)$, while in the other it is a function of $(E_J - E_0)$ and therefore the comparison of the models is difficult to carry out in this form, except using the infinite series expansion for $(E_J - E_0)$ in terms of $\mathcal{J}(\mathcal{J}+1)$.

An elegant way to handle the present problem is to rewrite equation (2a) and (2b) in the following form:

$$X_J = \tfrac{1}{2} C I_0 (I_0 - I_J) + \frac{3}{4I_J} \mathcal{J}(\mathcal{J}+1) \tag{4a}$$

$$I_J^2 - \tfrac{4}{3} I_0 I_J + \tfrac{1}{3} I_0^2 - \frac{2}{3C} X_J = 0 \tag{4b}$$

where $X_J = (E_J - E_0)$. We shall show later that the above form also tells us how to write in general the dependence of I_J, or the first term in expression (4a), on the energy X_J in a closed form.

It is an easy matter now to compare the two models by simply looking at equations (3) and (4). From these equations we find that while the Holmberg-Lipas model uses a linear relation between I_J and X_J, Mariscotti's expression uses a quadratic to determine the dependence of I_J on X_J. As a matter of fact, on expanding I_J up to the first power in X_J in equation (4b), one gets

$$I_J = I_0 + \frac{1}{CI_0} X_J \tag{5}$$

Substitution of equation (5) in equation (4a) gives us exactly the Holmberg-Lipas expression for X_J, the relation between I_1 and C being $I_1 = (CI_0)^{-1}$. Thus the Holmberg-Lipas model can be derived as a special case of Mariscotti's model. It is interesting to note here that if one looks at the fitted spectra for the rare earth region (Holmberg and Lipas 1968, Mariscotti *et al* 1969), one finds that these numerical results also indicate that the Mariscotti model gives a better fit than the Holmberg-Lipas model. What we have shown in this section is that this is qutie a general result and one can construct a mathematical proof for it.

3. Solution in terms of hypergeometric functions

In this section we show that the energies E_J given by Mariscotti's relations (Eq. 2a and 2b) can be expressed in terms of hypergeometric functions. The advantage of this solution is twofold: (i) Instead of solving equation (2b) for I_J for each value of $\mathcal{J}$ and then substituting it in equation (2a) to calculate the values of E_J, one gets the energies E_J directly in terms of hypergeometric functions; (ii) The solution immediately enables us to get an expansion of E_J in terms of $\mathcal{J}(\mathcal{J}+1)$, if such an expansion needed, without recourse to the expansion of I_J in terms of $J(J+1)$.

Eliminating I_J (Burnside and Panton 1960) from equations (2a) and (2b) we get the following cubic equation which determines X_J.

$$X_J^3 - CI_0^2 X_J^2 + \tfrac{1}{4} CI_0 [CI_0^3 + 9J(J+1)] X_J - \tfrac{1}{32} CJ(J+1)[4CI_0^3 + 27J(J+1)] = 0 \qquad (6)$$

We now use the result (Hille 1959) that a general cubic

$$X^3 + a_1 X^2 + a_2 X + a_3 = 0 \qquad (7)$$

can be reduced to the form

$$Z^3 + 3Z - \omega = 0 \qquad (8)$$

using the transformation $3X = (3a_2 - a_1^2)^{\frac{1}{2}} Z - a_1$, $(3a_2 \neq a_1^2)$, the solutions of which can be expressed in terms of hypergeometric functions. Using this result the solution of equation (6) which is appropriate to the rotational energies X_J can be written as

$$X_J = \tfrac{1}{3} CI_0^2 + \frac{i}{6} CI_0^2 \left[1 - \frac{27}{CI_0^3} J(J+1)\right]^{\frac{1}{2}} \times \left\{ i\sqrt{3}\, F\left(-\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{2}, -\tfrac{1}{4}\omega^2\right) - \tfrac{1}{6}\omega\, F\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{2}, -\tfrac{1}{4}\omega^2\right) \right\} \qquad (9)$$

where

$$\omega = \left(\frac{2}{i}\right) \left[1 - \frac{27}{CI_0^3} J(J+1)\right]^{-3/2} \left\{1 + \frac{135}{2CI_0^3} J(J+1) - \frac{729}{8C^2 I_0^6} [J(J+1)]^2 \right\}$$

and $F(a, b, c, \xi)$ is a hypergeometric function. Because of the special values of the arguments a, b, c in equation (9), the hypergeometric functions can be expressed in terms of sine functions (Abramowitz and Stegun 1965).

4. Conclusions

We have shown that Mariscotti's relations (equations 2a and 2b) could be written in a different form, which can be used to show that Holmberg-Lipas model is a special case of Mariscotti's model.

In general one can write the following expression for the energies X_J,

$$X_J = V(X_J) + \frac{1}{2I(X_J)} J(J+1) \qquad (10)$$

where V is the vibrational energy and I the moment of inertia. For the Mariscotti model, we find that the dependence of $V(X_J)$ and $I(X_J)$ on X_J can be written as

$$V^2 - \tfrac{1}{3} CI_0^2 V - \tfrac{1}{6} CI_0^2 X_J = 0 \qquad (11a)$$

$$I^2 - \tfrac{8}{9} I_0 I + \left(\tfrac{4}{27} I_0^2 - \tfrac{8}{27} X_J\right) = 0 \qquad (11b)$$

The functional forms of V and I given by equations (11a and 11b) are interesting and hold the promise for a possible generalization of the VMI models.

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