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Off-diagonal long-range order and currents in weakly coupled superconductors

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Abstract. Within the framework of quasi equilibrium approximation, Josephson's expression for current in a weakly coupled superconducting system is derived without the use of any specific microscopic model. It is based only on the existence of the off-diagonal long-range order in the two-particle reduced density matrix. Allowing for deviations from the quasi equilibrium approximation, generalised Josephson equations are obtained which include ohmic terms. The effect of relaxation and thermal fluctuations is examined in detail to emphasise the physical origin of various terms in the expression for current.

Keywords. ODLRO; superconductor; Josephson equations.

1. Introduction

With the possibility of high precision measurement of fundamental constants using the ac Josephson effect in weakly coupled superconducting systems (Finnegan *et al* 1971), it has become essential to understand the origin of this phenomenon with as much generality as possible. The original derivation of Josephson (1962, 1965) was based on the Ginzburg-Landau theory in which the superconducting state is described by a complex order parameter. He showed that when a voltage V is applied across a barrier separating two superconducting regions, a current I flows through it satisfying the equations

$$I = I_1 \sin \theta \tag{1}$$

$$d\theta/dt = 2eV/\hbar \tag{2}$$

where θ is the difference in the phase of the order parameter on either side of the barrier. We shall be referring to (1) and (2) as Josephson equations.

The above relations have been confirmed experimentally, not only for the so-called Josephson junctions, but also for various other systems such as Dayem bridges and point contacts. Josephson equations have thus come to be recognized as characteristic of any weakly coupled superconducting system. These equations were later derived on the basis of the BCS model as well.

Recently, Bloch (1970) has deduced these equations as a general consequence of fundamental principles, and not specifically due to any assumed microscopic model. However, his introduction of the pairing of the electrons due to the off-diagonal long-range order (ODLRO) present in the system is somewhat incomplete. Also, his derivation is restricted to the ring geometry in which an emf is induced when the magnetic flux through the ring is made time varying. Our derivation is for an open ended geometry where the voltage can be applied in the usual manner. The effect of the ODLRO is treated in a more direct manner, and Yang's proof (Yang 1962) for flux quantization is suitably extended to derive the Josephson equations. The derivation is made without any assumption regarding the details of the Hamiltonian of the system, except that it leads to ODLRO in the two-particle reduced density matrix in Yang's sense. In addition, to start with, a quasi stationary approximation for sufficiently slow variation of the external potential is used. Later, emphasis is laid on understanding the physical origin of various other terms in the expression for current that one uses in phenomenological models of such systems (Biswas and Jha 1970). To this end, we go beyond the quasi stationary approximation allowing for deviations from it leading to dissipative currents. No attempt is made to obtain explicit expressions for the constants describing the system. These can, in principle, be evaluated once an explicit form of the Hamiltonian is assumed.

In Section 2, we derive the Josephson equations considering a weakly coupled superconducting system subject to a scalar electric potential. We allow for an arbitrary but slow time dependence for the potential. The problem is reduced to a form in which the information regarding the ODLRO can be conveniently utilised, without any need to know the details of the Hamiltonian. The superconducting and normal systems are treated on the same footing and the reason why the supercurrent originates in the former is brought out.

Deviations from quasi equilibrium conditions are considered in Section 3. The density matrix describing these deviations is examined on the basis of the general theory of relaxation developed by Bloch (1957), and it is evaluated in the simple 'relaxation time' approximation. In this way, we not only obtain the usual ohmic current, but in addition an out of phase supercurrent when there is ODLRO. From the expression for the energy dissipated per unit time, the autocorrelation function of the fluctuating current is then introduced, using the fluctuation dissipation theorem. Thus one has a generalised Josephson equation which includes all these currents. We discuss our results in Section 4.

2. ODLRO and Josephson equations

The system is described by a Hamiltonian $\mathcal{H}_0(\mathbf{p}_k, \mathbf{X}_k)$ in the absence of external fields, where $\mathbf{X}_k$ and $\mathbf{p}_k$ refer to the coordinate and momentum of the kth particle and k runs from 1 to N, labelling the N particles of the system. We do not assume any particular form for the dependence on $\mathbf{p}_k$ and $\mathbf{X}_k$, thus retaining complete generality.

Let the system be subjected to a scalar potential given by $\phi(\mathbf{X}, t)$ so chosen that at $t=0$, $\phi(\mathbf{X})=0$ for all values of $\mathbf{X}$. It is to be noted that the possibility of application of this external field itself puts certain restrictions on the system. In particular this can be achieved for the Josephson junction in which there is an insulating barrier separating the two superconducting regions. If the system were to exhibit ODLRO, then the barrier must be thin enough to transmit this order. Our derivation of the Josephson equations will hence be valid for all such weakly linked superconducting systems.

2.1. *Quasi stationary approximation*

In the presence of $\phi(\mathbf{X}, t)$, the Hamiltonian becomes

$$\mathcal{H}=\mathcal{H}_0(\mathbf{p}_k, \mathbf{X}_k) + \sum_k e_k\phi(\mathbf{X}_k, t) \tag{3}$$

where e_k is the charge on the kth particle. We have to solve the time-dependent Schroedinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi'(\mathbf{X}_1,\ldots,\mathbf{X}_N, t)=\mathcal{H}\Psi'(\mathbf{X}_1,\ldots,\mathbf{X}_N, t) \tag{4}$$

Under the gauge transformation

$$\Psi(\mathbf{X}_1,\ldots,\mathbf{X}_N, t)=\exp\left(-\sum_k \frac{e_k}{i\hbar}\int_o^t \phi(\mathbf{X}_k, t_1)\mathrm{d}t_1\right)\Psi'(\mathbf{X}_1\ldots,\mathbf{X}_N,t) \tag{5}$$

by using the identity

$$\exp\left(-\sum_k \frac{e_k}{i\hbar}\int_o^t \phi(\mathbf{X}_k, t_1)\mathrm{d}t_1\right)\mathbf{p}_j \exp\left(\sum_k \frac{e_k}{i\hbar}\int_o^t \phi(\mathbf{X}_k, t_1)\mathrm{d}t_1\right)$$

$$=\mathbf{p}_j-e_j\int_o^t \nabla_j\phi(\mathbf{X}_j, t_1)\mathrm{d}t_1 \tag{6}$$

we obtain

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{X}_1,\ldots,\mathbf{X}_N, t)=\mathcal{H}_S(t)\Psi(\mathbf{X}_1,\ldots,\mathbf{X}_N, t) \tag{7}$$

where

$$\mathcal{H}_S(t)\equiv\mathcal{H}_0(\mathbf{p}_k-e_k\int_o^t\nabla_k\phi(\mathbf{X}_k, t_1)\mathrm{d}t_1, \mathbf{X}_k) \tag{8}$$

The right hand side of (7) has time dependence through the $\nabla\phi$ term in $\mathcal{H}_S$. When $\nabla\phi$ is small, the time dependent part is small and one is justified in looking for quasi stationary solutions of this equation. In this case, we regard the 't' appearing therein as a parameter (Greenwood 1968) and seek solutions of

$$E(t)\Psi_e(t)=\mathcal{H}_S(t)\Psi_e(t) \tag{9}$$

The rate of change per unit time of the quasi equilibrium free energy F is then given by

$$\mathrm{d}F/\mathrm{d}t\equiv\dot{F}=\sum_n \dot{E}_n\, exp(-\beta E_n)/\sum_n exp\,(-\beta E_n)=\mathrm{Tr}\,\rho_e\,\dot{\mathcal{H}}_s \tag{10}$$

where the equilibrium density matrix ρ_e is

$$\rho_e=exp(-\beta\mathcal{H}_S)/\mathrm{Tr}\,exp\,(-\beta\mathcal{H}_S) \tag{11}$$

with $$\beta=1/kT \tag{12}$$

2.2. *Geometry and modified boundary conditions*

At this stage, we restrict ourselves to a geometry of length L, with $\nabla\phi$ applied along its length, chosen to be the x direction. Since the transverse directions are unimportant in our problem, we take $\mathbf{X}_j$ to refer to X_j, the x coordinate of the jth particle, and the gradient operator will be taken to mean $\partial/\partial x$. We can then impose periodic boundary conditions (Peierls 1954) over the length L on the solutions of (9) viz.

$$\Psi_e(X_1,\ldots, X_{j-1}, X_j+L, X_{j+1},\ldots, X_N, t)=\Psi_e\,(X_1\ldots, X_{j-1}, X_j, X_{j+1}, X_N,t) \tag{13}$$

for every particle coordinate X_j. We emphasise again that the set of energy eigenvalues E, to be obtained as solutions of equations (9) and (13) carry t as a parameter.

Though equations (9) and (13) contain all the information regarding the external field within the quasi stationary approximation, these are not very convenient to discuss the effect of ODLRO in the system without asking for the details of the Hamiltonian. For this purpose it is better to deal with a completely equivalent set, where the effect of the external field enters only through modified boundary conditions. Using the identity (6), a reverse gauge transformation on Ψ_e leads to the equivalent problem

$$E\Psi_e' = \mathcal{H}_0 (\mathbf{p}_k, \mathbf{X}_k)\, \Psi_e' \tag{14}$$

with

$$\Psi_e' (X_1, \ldots, X_j+L, \ldots, X_N, t) = \exp\left(\frac{e_j}{i\hbar}\int_0^t V(t_1)\,\mathrm{d}t_1\right)\Psi_e'(X_1 \ldots, X_j, \ldots, X_N, t) \tag{15}$$

where we have defined

$$V(t) = \phi(X+L, t) - \phi(X, t) \equiv \phi(L, t) - \phi(0, t) \tag{16}$$

as the potential drop across the specimen.

Equation (14) is nothing but the Schroedinger equation in the absence of the external potential, but now it has to be solved under the boundary conditions (15) to take care of the external field in the quasi equilibrium approximation. For studying the effect of ODLRO on the rate of change of the quasi equilibrium free energy F it is desirable to use the new set of equations (14) and (15), whereas for discussing the deviations from quasi equilibrium, the equivalent set of equations (9) and (13) is found to be more convenient.

Writing $e_j = e\eta_j$, where e is the elementary charge and η_j takes positive or negative integer values depending on the magnitude and sign of the charge carried by the jth particle, and defining*

$$\theta(t) = \frac{2e}{\hbar}\int_0^t V(t_1)\mathrm{d}t_1 \tag{17}$$

the boundary conditions (15) may be rewritten as

$$\Psi_e (X_1, \ldots, X_j+L, \ldots, X_N, t) = \exp\left(-\frac{i}{2}\eta_j\, \theta(t)\right)\Psi_e'(X_1, \ldots, X_j, \ldots, X_N, t) \tag{18}$$

2.3. *Effect of* ODLRO

In order to investigate the effect of the modified boundary conditions (18) on a system characterised by the presence of ODLRO, following Yang (1962) we define

$$R = \exp(-\beta\mathcal{H}_0) \tag{19a}$$

in terms of which the partition function Q and the density matrix ρ_e are given by

$$Q = \mathrm{Tr}\, R \tag{19b}$$

$$\rho_e = R/Q \tag{19c}$$

The matrix elements in the coordinate representation of the second (two-particle) reduced density matrix are defined either by

$$\langle X_1, X_2 | \rho_2 | X_1' X_2' \rangle = \mathrm{Tr}\,(\psi(X_1)\,\psi(X_2)\rho_e\,\psi^\dagger(X_2')\,\psi^\dagger(X_1')) \tag{20a}$$

where $\psi(x)$ is the field operator, or by

$$\langle X_1 X_2 | \rho_2 | X_1' X_2' \rangle = \mathcal{N}(\mathcal{N}-1)\int \langle X_1 X_2 X_3 \ldots X_N | \rho_e | X_1' X_2' X_3 \ldots X_N \rangle \times \mathrm{d}X_3\, \mathrm{d}X_4 \ldots \mathrm{d}X_N. \tag{20b}$$

*The introduction of a factor 2 in equation (17) looks artificial at this stage, but its role becomes clear later on when the effect of ODLRO in ρ_2 is discussed.

Also, we define

$$\langle X_1 X_2 | R_2 | X_1', X_2' \rangle = Q \langle X_1 X_2 | \rho_2 | X_1' X_2' \rangle \quad (20c)$$

The boundary conditions (18) on the $\mathcal{N}$-particle wave function imply in turn,

$$\langle X_1 + mL, X_2 | R_2 | X_1', X_2' \rangle = \exp\left(-\frac{i}{2}\eta_1 m\theta\right) \langle X_1 X_2 | R_2 | X_1' X_2' \rangle \quad (21)$$

where m is an integer.

Suppose we confine our attention to the case in which X_1, X_2, X_1', X_2' all refer to the electronic coordinates, with $\eta_1 = \eta_2 = \eta_1' = \eta_2' = -1$. Eq. (17) then leads to

$$\begin{aligned} \langle X_1 + mL, X_2 | R_2 | X_1' X_2' \rangle &= \langle X_1 X_2 + mL | R_2 | X_1' X_2' \rangle \\ &= \langle X_1 X_2 | R_2 | X_1' - mL, X_2' \rangle \\ &= \langle X_1 X_2 | R_2 | X_1' X_2' - mL \rangle \\ &= \exp\left(\frac{i}{2} m\theta\right) \langle X_1 X_2 | R_2 | X_1' X_2' \rangle \end{aligned} \quad (22)$$

Since Tr $\rho_2 = \mathcal{N}(\mathcal{N}-1)$, from (20c) we find

$$Q = \text{Tr}\, R_2 / \mathcal{N}(\mathcal{N}-1) \quad (23)$$

If the system possesses ODLRO in the coordinate representation of ρ_2, it implies in particular that $\langle X_1 X_2 | R_2 | X_1' X_2' \rangle \neq 0$ in the microscopic neighbourhood of

$$X_1 = X_2; \; X_1' = X_2'; \; | X_1 - X_1' | \text{ arbitrary.} \quad (24)$$

It is convenient to express the regions of non-vanishing matrix elements of R_2 in the coordinate system

$$\xi = X_1 - X_2; \; \eta = X_1' - X_2'; \; \zeta = X_1 + X_2 - X_1' - X_2' \quad (25)$$

introduced by Yang (1962). The points for which the corresponding matrix elements of R_2 are related to each other by a multiplicative phase factor demanded by (22), lie on a fcc lattice of cubic side $2L$ as shown in figure 1. Equation (24) implies non-vanishing values for the matrix elements of R_2 along the lines parallel to ζ axis, with ξ and η taking values which are any integer multiples of L. Along any of these lines the points separated by a distance of $2L$ have their values related by the periodicity condition

$$R_2(\xi = n_1 L, \eta = n_2 L, \zeta + 2L) = \exp(i\theta) R_2(\xi = n_1 L, \eta = n_2 L, \zeta) \quad (26)$$

In order to satisfy the above condition, R_2 in the presence of the external potential

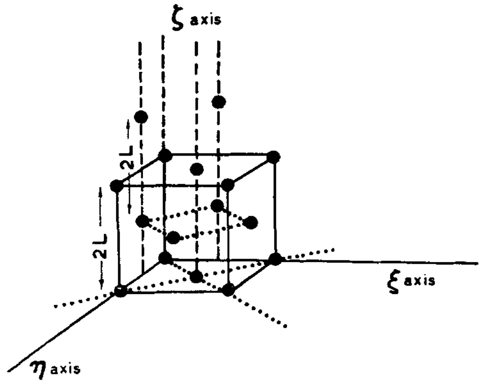

Figure 1. Matrix elements of R_2 related to one another by a multiplicative phase lie on a fcc lattice in (ξ, η, ζ) space. The phase depends only on their ζ separation.
● such equivalent points; .... regions where R_2 is non-vanishing even in the absence of any ODLRO; ———————— Regions of non-vanishing R_2 only when there is ODLRO in ρ_2.

must depend on θ. But since θ occurs only as exp $(i\theta)$, the dependence has to be periodic with a period 2π. Because of (23), this means that the partition function Q, and hence the quasi equilibrium free energy $F=-kT\ln Q$, are periodic functions of θ with a period 2π. Further, time reversal invariance requires F to be an even function of θ and so one may write

$$F=\sum_{n=0}^{\infty}F_n\cos n\theta \tag{27}$$

2.4. *Josephson equations*

The θ defined by (17) contains t as a parameter with

$$\mathrm{d}\theta/\mathrm{d}t=2eV/\hbar \tag{28}$$

which is identical to the Josephson equation (2). We interpret the free energy $F(\theta(t))$ as $F(t)$, representing a thermodynamically reversible process. $\mathrm{d}F/\mathrm{d}t$ then refers to the reversible work done on the system per unit time. Equating $\mathrm{d}F/\mathrm{d}t$ to the current $I(t)$ times the voltage $V(t)$ gives the expression for current. Thus, using (28), we obtain

$$I=\frac{2e}{\hbar}\frac{\mathrm{d}F}{\mathrm{d}\theta}=\sum_{n=1}^{\infty}I_n\sin n\theta \tag{29}$$

where $\quad I_n=-2enF_n/\hbar$

Just as in Bloch's treatment, we cannot evaluate the coefficients of expansion, viz F_n or I_n without going into the details of the Hamiltonian $\mathcal{H}_0$.

Bloch (1970) has shown that for a wide barrier, the lowest n term in the summation in (29) predominates and then we recover the Josephson equation (1). In particular, in the presence of a steady applied voltage V_0, we get an alternating current of frequency $2eV_0/h$. Allowing for terms in addition to $n=1$ in the summation in (29) amounts to a non-sinusoidal (but still periodic) wave form for the current and recently there has been experimental evidence in favour of this (Simonds and Parker 1970).

For normal systems which do not exhibit any ODLRO, the non-vanishing regions of the matrix elements of R_2, given by (24), are absent. Consequently the periodicity condition (26) does not lead to a dependence of the quasi equilibrium free energy F on θ. This leads to $\mathrm{d}F/\mathrm{d}t=0$ and within the quasi equilibrium approximation no current flows in such systems.

3. Corrections to the quasi stationary approximation: generalized Josephson equations

In the previous section, we have seen how the presence of ODLRO leads to the so called supercurrent in superconducting weak links. But one has, in addition, other types of current too in such systems. For derivation of these we go beyond the quasi stationary approximation made in Section 2. This approach is applicable equally well to normal systems which do not exhibit any ODLRO.

3.1. *Deviations from the quasi stationary approximation*

Considering the system in contact with a thermal reservoir which accounts for its approach to equilibrium and for maintaining steady state flows, we write the total Hamiltonian as

$$\mathcal{H}_T=\mathcal{H}_S+\mathcal{H}_R+\mathcal{H}_{S-R} \tag{30}$$

where $\mathcal{H}_S(t)$ given by equation (8) and $\mathcal{H}_R$ are Hamiltonians of the system and reservoir respectively and $\mathcal{H}_{S-R}$ is the interaction between the two. One assumes that the reservoir is in thermal equilibrium at all times. The equation of motion of the density matrix ρ of the system alone, obtained by tracing over the reservoir variables, is written down by a forward integration retaining terms upto the second order in $\mathcal{H}_{S-R}$. The resulting equation, derived in detail by Bloch and Wangsness (1956) and Bloch (1957) is capable of dealing with an irreversible process. Neglecting small correction terms to $\mathcal{H}_S$, it may be written as

$$\frac{d\rho}{dt} \approx \frac{1}{i\hbar}\left[\mathcal{H}_S, \rho\right] + \Gamma(\rho) \tag{31}$$

where the relaxation term $\Gamma(\rho)$, given by equation 3.18 of Bloch (1957) is linear in ρ. For a sufficiently small relaxation time, $\Gamma(\rho)$ is such that for the equilibrium density matrix ρ_e given by equation (11), it satisfies the relation

$$\Gamma(\rho_e) = 0 \tag{32}$$

Let

$$\chi = \rho - \rho_e, \ (\mathrm{Tr}\ \chi = 0) \tag{33}$$

represent the deviation from the quasi stationary approximation made in the last section. This satisfies the equation (Bloch 1957)

$$\frac{d\chi}{dt} = \frac{1}{i\hbar}\left[\mathcal{H}_S, \chi\right] + \Gamma(\chi) - \frac{d\rho_e}{dt} \tag{34}$$

so that the deviation from the quasi equilibrium distribution is proportional to the time rate of change of ρ_e, i.e. to the rate of change of $\mathcal{H}_S(t)$. In our approach we will neglect higher derivatives of $\mathcal{H}_S(t)$. The term $\Gamma(\chi)$ in (34) carries all the complicated information about the relaxation processes. We shall, however, make the simplest approximation for this term by writing it in the form $-\chi/\tau_c$, where τ_c is a positive constant with the dimension of time. Equation (34) is an operator equation for χ, and has to be solved subject to the condition Trace $\chi = 0$. In order to obtain χ, let us choose a complete set of states which are the quasi stationary eigenstates of the Hamiltonian $\mathcal{H}_S(t)$ in equation (9):

$$\mathcal{H}_S \mid n \rangle = E_n \mid n \rangle \tag{35}$$

In this representation

$$\rho_e \mid n \rangle = f_n \mid n \rangle \tag{36}$$

where

$$f_n = \exp(-\beta E_n) \Big/ \sum_n \exp(-\beta E_n) \tag{37}$$

Using the orthogonality relations, it is straight forward to show by differentiating both sides of (35) and (36) with respect to time that

$$\dot{E}_n = \langle n \mid \dot{\mathcal{H}}_S \mid n \rangle \tag{38}$$

$$\langle m \mid \frac{d}{dt} \mid n \rangle = -\frac{\langle m \mid \dot{\mathcal{H}}_S \mid n \rangle}{E_m - E_n}(1 - \delta_{mn}) \tag{39}$$

$$\langle m \mid \dot{\rho}_e \mid n \rangle = \dot{f}_n \delta_{mn} + \frac{f_m - f_n}{E_m - E_n} \langle m \mid \dot{\mathcal{H}}_S \mid n \rangle \ (1 - \delta_{mn}) \tag{40}$$

To the lowest order in the external field, the steady state solution of Eq. (34) is given by

$$\langle m \mid \chi \mid n \rangle = \frac{i\hbar \langle \mathrm{m} \mid \dot{\rho}_e \mid \mathrm{n} \rangle}{E_m - E_n - i\hbar/\tau_c} \tag{41}$$

The rate of change of energy of the system may be calculated from the relation

$$\mathrm{d}W/\mathrm{d}t = \mathrm{Tr}\, \rho \dot{\mathcal{H}}_S = \mathrm{d}F/\mathrm{d}t + \mathrm{Tr}\, \chi \dot{\mathcal{H}}_S \tag{42}$$

where F is the quasi equilibrium free energy, the behaviour of which was discussed in the last section. Equations (38), (40) and (41) then lead to

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}t} - \tau_c \sum_n \dot{E}_n \dot{f}_n - {\sum_{n,m}}' | < m | \dot{\mathcal{H}}_S | n > |^2 \frac{f_m - f_n}{E_m - E_n} \left[\frac{\tau_c}{1 + \tau_c^2 (E_m - E_n)^2/\hbar^2} \right] \tag{43}$$

Since we are neglecting $\ddot{E}_n$ throughout, using equations (10) and (37) the above result may be rewritten in a more convenient form as

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}t} - \tau_c \frac{\mathrm{d}^2F}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}W}{\mathrm{d}t}\right)_R \tag{44}$$

where $$\left(\frac{\mathrm{d}W}{\mathrm{d}t}\right)_R = -{\sum_{n,m}}' | <m | \dot{\mathcal{H}}_S | n> |^2 \frac{f_m - f_n}{E_m - E_n} \left[\frac{\tau_c}{1 + \tau_c^2 (E_m - E_n)^2/\hbar^2} \right] \tag{45}$$

3.2. *Out of phase supercurrent*

As explained in Section 2, for normal systems $\mathrm{d}F/\mathrm{d}t = \mathrm{d}^2F/\mathrm{d}t^2 = 0$. However, when there is ODLRO in the system, the quasi equilibrium free energy F has the form

$$F = \sum_{n=o}^{\infty} F_n \cos n\theta$$

with non-vanishing coefficients. This implies that

$$\mathrm{d}F/\mathrm{d}t = -\sum_{n=1}^{\infty} 2eV\, F_n\, n \sin n\theta/\hbar \tag{46}$$

$$-\tau_c \frac{\mathrm{d}^2F}{\mathrm{d}t^2} = \tau_c \sum_{n=1}^{\infty} F_n\, n^2 \left(\frac{2eV}{\hbar}\right)^2 \cos n\theta \tag{47}$$

Thus, equating the first two terms in (44) to $I_S V$ gives

$$I_S = \sum_{n=1}^{\infty} I_n \sin n\theta + \sum_{n=1}^{\infty} (V/R_n) \cos n\theta \tag{48}$$

where I_n is the same as in Eq. (29), and where

$$\frac{1}{R_n} = \frac{4e^2n^2}{\hbar^2} \tau_c F_n = -\frac{2en}{\hbar} \tau_c I_n \tag{49}$$

The second term in (48) is a correction to the quasi equilibrium supercurrent (29). This additional current does not lead to any dissipation, but is proportional to the applied voltage. For a sufficiently small value of $eV\tau_c/\hbar$, it may always be neglected in comparison with the first term in (48). Such a correction term has already been predicted by Josephson (1965) and Bloch (1970).

3.3. *The ohmic current*

The last term $\left(\frac{dW}{dt}\right)_R$ in equation (44) represents the ohmic loss. From the definition of $\mathcal{H}_S$ in Eq. (8),

$$\begin{aligned}\dot{\mathcal{H}}_S &= \tfrac{1}{2}\sum_k e_k[\mathbf{v}_k \cdot \mathbf{E}(\mathbf{X}_k, t) + \mathbf{E}(\mathbf{X}_k, t)\cdot \mathbf{v}_k] \\ &= \tfrac{1}{2}\int\sum_k e_k\,[\mathbf{v}_k\,\delta(\mathbf{X}-\mathbf{X}_k)+\delta(\mathbf{X}-\mathbf{X}_k)\mathbf{v}_k]\cdot\mathbf{E}(\mathbf{X}, t)\mathrm{d}^3X \\ &= \int \mathrm{d}^3X\,\mathbf{j}(\mathbf{X})\cdot\mathbf{E}(\mathbf{X}, t)\end{aligned} \tag{50}$$

where $\mathbf{j}(\mathbf{X})$ is the current density operator, and $\mathbf{E}(\mathbf{X}, t) = -\nabla\phi(\mathbf{X}, t)$ is just a space-time function. Thus

$$\langle m \,|\, \dot{\mathcal{H}}_S \,|\, n\rangle = \int \mathrm{d}^3X\,\mathbf{j}_{mn}(\mathbf{X})\cdot\mathbf{E}(\mathbf{X}, t) \tag{51}$$

Following Wannier (1966), for non-uniform current distribution one can introduce curvilinear coordinates q, q^*, q^{**} with q running along the field lines and q^*, q^{**} along an equipotential surface with an element of area d$\mathbf{A}$. We can integrate (51) over such an equipotential surface of fixed q, by breaking up the volume. integral into one over q and another over A, so that

$$\langle m \,|\, \dot{\mathcal{H}}_S \,|\, n\rangle = \int \mathrm{d}\mathbf{A}\,.\int \mathbf{j}_{mn}(\mathbf{X})\mathrm{d}q\,E(\mathbf{X}, t) \tag{52}$$

Ignoring the slow dependence of $\mathbf{j}_{mn}(\mathbf{X})$ on q, $E(\mathbf{X}, t)$ can be integrated over q from one equipotential surface to a neighbouring one, yielding their potential difference which is independent of the point chosen on the surface. Thereafter we can integrate $\mathbf{j}_{mn}$ over d$\mathbf{A}$, and call it I_{mn} for a purely resistive circuit. This is independent of $\mathbf{X}$. Proceeding this way from one equipotential surface to another, covering the whole region, we obtain

$$< m \,|\, \dot{\mathcal{H}}_S \,|\, n > = I_{mn} V \tag{53}$$

For a uniform $\mathbf{E} = \frac{V}{L}\hat{\mathbf{X}}$, the operator I is nothing but $\frac{1}{L}\sum_k e_k v_{kx}$

Comparing $\left(\frac{dW}{dt}\right)_R$ with $I_R V$, we find the ohmic part to be $I_R = \frac{V}{R}$ (54)

where

$$\frac{1}{R} = -\sum_{n,m}{}' \,|\, I_{mn}\,|^2\,\frac{f_m - f_n}{E_m - E_n}\,\frac{\tau_c}{[1+\tau_c^2(E_m-E_n)^2/\hbar^2]} \tag{55}$$

For a sufficiently small τ_c, this reduces to

$$\frac{1}{R} = -\sum_{n,m}{}' \,|\, I_{mn}\,|^2\,\frac{f_m - f_n}{E_m - E_n}\,\tau_c \tag{56}$$

In the limit $\beta \to 0$, equation (37) leads to

$$\begin{aligned}\frac{1}{R} &\to \beta\sum_{n,m} f_m\,|\,I_{mn}\,|^2\,\tau_c = \beta\,\langle\,|I|^2\,\rangle_e\,\tau_c \\ &= \beta\int_0^\infty \mathrm{d}\tau\,\langle\, I(0)I(\tau)\,\rangle\end{aligned} \tag{57}$$

where $\langle\, I(0)I(\tau)\,\rangle \equiv \langle\,|I|^2\,\rangle_e\,(\exp - |\tau|/\tau_c) = (kT/R\tau_c)\exp(-|\tau|/\tau_c)$ (58)

3.4. *Thermal fluctuations*

In addition to the currents I_S and I_R discussed above, there is a current due to thermal fluctuations. This is present even in the absence of the applied voltage. It is already clear from (54) and (58) that in this case although the mean value of such a current $\mathcal{I}(t)$ is zero, its autocorrelation function is non-vanishing. Indeed for a very short correlation time τ_c, (58) gives

$$<\mathcal{I}(0)\mathcal{I}(\tau)> \approx \frac{2kT}{R}\delta(\tau) \tag{59}$$

The above result may be derived more generally from the expression for the energy dissipation when the system is subject to an external field, via the fluctuation dissipation theorem. For a resistive circuit the spectral density of mean square fluctuations in current is given by

$$(\mathcal{I}^2)_\omega = \frac{\hbar\omega}{2\pi R}\coth\frac{\hbar\omega}{2kT} \tag{60}$$

For $\hbar\omega \ll kT$, this reduces to

$$(\mathcal{I}^2)_\omega = kT/\pi R \tag{61}$$

implying $$<\mathcal{I}(t)\ \mathcal{I}(t+\tau)> = 2kT\,\delta(\tau)/R \tag{62}$$

It is to be noted that (61) applies only for frequencies small compared with the reciprocal of the relaxation time τ_c.

Putting together the results of (3·2), (3·3) and (3·4), we, therefore, obtain the generalised Josephson equations for a weakly coupled superconductor subject to a voltage V. These together with equations (49), (55) and (59) may be written as

$$\mathcal{I} = \sum_{n=1}^{\infty} I_n \sin n\theta + \sum_{n=1}^{\infty}\frac{V}{R_n}\cos n\theta + \frac{V}{R} + \mathcal{I}(t) \tag{63}$$

and $$d\theta/dt = 2eV/\hbar \tag{64}$$

where the mean value $<\mathcal{I}(t)> = 0$.

4. Conclusion

In the preceding sections we have shown how the Josephson current follows quite naturally from the ODLRO present in weakly coupled superconducting systems. In our derivation, the only description of the superconducting state was introduced through the assumption of ODLRO in Yang's sense, and this seems to be the best way in which the Josephson equations can be obtained retaining complete generality. The factor 2 in equation (2) which one normally introduces by invoking the paired motion of the electrons, comes automatically in our treatment. This type of derivation also makes it clear why such currents flow only in a superconducting system, and not in a normal metal. The Josephson relation in the form of equation (29) is capable of explaining recent experimental results, which call for a generalisation of the original Josephson equation (eq 1).

In Section 3, we went beyond the quasi stationary approximation and described the deviations by a density matrix which took into account the relaxation. This led to an out of phase supercurrent and an ohmic current. For very short relaxation times, it was shown that the out of phase supercurrent can always be neglected in comparison with the usual Josephson term. Although we made the most simplifying assumption about the relaxation term, the method affords a very general way of tackling such problems. Formal expressions for the capacitance and the inductance of the weak link can also be introduced using this formalism.

The derivation of the ohmic current and the thermal noise introduced later, does not make use of the property of ODLRO and hence is equally well applicable to normal systems. They have been included for completeness as well as for clarifying the physical origin of the various terms in the expression for current.

Several questions may be asked regarding the nature and validity of the approximations made in Sections 2.1 and 2.2 and the extent to which these affect the generality in the derivation of the Josephson equations. Firstly, one may question why the Hamiltonian $\mathcal{H}_S$ of Eq. (8) was chosen for the quasi stationary approximation instead of the Hamiltonian $\mathcal{H}$ of Eq. (3). This is because our intention was to derive results valid for small electric fields, i.e. for small values of $\nabla\phi$. From this point of view, it is obviously more convenient to investigate the quasi stationary states of $\mathcal{H}_S$. Since in our problem, electrons must enter at one end of the sample and leave at the other end, even for dc fields the pure stationary states of the Hamiltonian $\mathcal{H}$ are not relevant. Once the quasi stationary approximation is made, one may inquire whether the specification of the boundary conditions (Eq. 13) is appropriate to the problem. The use of periodic boundary conditions on Ψ_e over a large length L is not inconsistent with the Hamiltonian $\mathcal{H}_S$, once $\nabla\phi$ has been chosen to have the same periodicity. We note that the potential drop occurs only in the thin non-superconducting region. As we require the system as a whole to exhibit ODLRO, without any loss of generality, the full length L may be taken as the periodicity interval.

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