On the first simultaneous sign change and non-vanishing of Fourier coefficients of cusp forms

GUODONG HUA\textsuperscript{1,2}

\textsuperscript{1}School of Mathematics and Statistics, Weinan Normal University, Shaanxi 714099, Weinan, China
\textsuperscript{2}School of Mathematics, Shandong University, Shandong, Jinan 250100, China
E-mail: gdhuanumb@yeah.net

MS received 29 December 2021; revised 2 May 2022; accepted 3 December 2022

Abstract. Let $f, g$ be two distinct normalized primitive holomorphic cusp forms of weights $k_1, k_2$ and levels $N_1, N_2$, respectively. Denote by $\lambda_f(n)$ and $\lambda_g(n)$ the $n$-th normalized Fourier coefficients of $f$ and $g$, respectively. In this paper, we investigate the first change of sequences $\{\lambda_f(n_i)\}_{n \in \mathbb{N}}$ and $\{\lambda_f(n_j)\lambda_g(n_l)\}_{n \in \mathbb{N}}$ with positive integers $i \geq 2$ and $j, l \geq 1$. And we also derive the lower bounds for the sequences $\{\lambda_f(n_i)\}_{n \in \mathbb{N}}, \{\lambda_f(n_j)\lambda_f(n_l)\}_{n \in \mathbb{N}}$ of the same signs in the interval $[1, x]$, which remains the best possible results in terms of order of magnitude, where $i \geq 2, j, l \geq 1$ are positive integers.

Keywords. Cusp forms; Fourier coefficients; simultaneous sign changes.

2010 Mathematics Subject Classification. 11F11, 11F30, 11F66.

1. Introduction

The Fourier coefficients are important and interesting objects in number theory. In recent times, the sign change problems of Fourier coefficients has been the focus of many investigations in modular forms. Let $f$ be a normalized primitive holomorphic cusp form of even integral weight $k$ for the Hecke congruence group $\Gamma_0(N)$. Denote by $H_k^\ast(N)$ the set of all normalized primitive holomorphic cusp forms (newforms) of even integral weight $k$ for the Hecke congruence group $\Gamma_0(N)$ with trivial nebentypus, which are eigenfunctions of all Hecke operators $T_n$. For $N = 1$, we abbreviate by $H_k^\ast$. It is well-known that for $f \in H_k^\ast$, we have the Fourier expansion at the cusp $\infty$:

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{k-1}e(nz), \quad \Im(z) > 0, \quad (1.1)$$

where $e(z) = e^{2\pi iz}$ and $\lambda_f(1) = 1$. It is known from the theory of Hecke operator $\lambda_f(n)$ is real and satisfies the multiplicative property

© Indian Academy of Sciences
Published online: 05 May 2023
where \(m \geq 1\) and \(n \geq 1\) are positive integers. In 1974, Deligne [7] proved the celebrated Ramanujan–Petersson conjecture
\[
|\lambda_f(n)| \leq d(n),
\] where \(d(n)\) is the classical divisor function.

The sign change problem of Fourier coefficients attached to modular forms has a long history which goes back to Siegel [43]. We denote by \(Q_f = k^2N\). Let \(f \in H^*_k(N)\) be a primitive holomorphic cusp form of integral weight for \(\Gamma_0(N)\). By using a classical theorem of Landau and some analytic properties of the associated \(L\)-function \(L(s, f)\), one can prove that \(\{\lambda_f(n)\}_{n \in \mathbb{N}}\) has infinitely many sign changes (see [23]). Let \(f \in H^*_k(N)\) and take \(n_f\) to be the smallest integer \(n\) such that \(\lambda_f(n) < 0\) and \((n, N) = 1\). In 2007, Iwaniec et al. [15] using the convexity bound of \(L(s, f)\) and the Hecke relation proved that
\[
n_f \ll Q_f^{1/2+\varepsilon},
\]
where \(\varepsilon > 0\) is an arbitrarily small number. Later many authors sharpened (1.3) and the best records due to Matomäki [36] showed that
\[
n_f \ll Q_f^{1/2-1/8},
\]
which remains the best records up-to-date.

The sign changes of Fourier coefficients associated with modular form also have been studied in various aspects. The sign changes of the subsequence of Fourier coefficients at prime number was first studied by Ram Murty [33]. Later, Meher et al. [37] gave a quantitative version of sign changes of the sequences \(\{\lambda_f(n^j)\}_{n \in \mathbb{N}}\) in short intervals. Very recently, Lao and Luo [32] considered the number of sign changes of the sequences \(\{\lambda_f(n^j)\}_{n \in \mathbb{N}}\) for \(n \leq x\), where \(j \geq 3\) is any fixed positive integer.

In recent times, the study of simultaneous sign changes of two cusp forms have also been considered by a number of authors. Let \(f\) and \(g\) be two distinct holomorphic cusp forms, and denote its \(n\)-th Fourier coefficients by \(\lambda_f(n)\) and \(\lambda_g(n)\) respectively. Kohnen and Sengupta [26], Gun et al. [10] and, Kumari and Murty [27] considered the sign change problems of the sequence \(\{\lambda_f(n)\lambda_g(n)\}_{n \in \mathbb{N}}\). In particular, Kumari and Murty [27] also gave the number of sign changes of the sequence \(\{\lambda_f(n)\lambda_g(n)\}_{n \in \mathbb{N}}\) in short intervals. In 2019, Gun et al. [11] considered the first simultaneous sign change of the same sequence in a variant form. And in the same paper, they also considered the number of sign changes of the sequence \(\{\lambda_f(n)\lambda_g(n^2)\}_{n \in \mathbb{N}}\) in short intervals. In [32], Lao and Luo considered the number of sign changes of the sequences \(\{\lambda_f(n^i)\lambda_g(n^j)\}_{n \in \mathbb{N}}\) for \(n \leq x\), where \(i \geq 1, j \geq 2\) are positive integers.

Let \(k_1, k_2 \geq 2\) be even integers and \(N_1, N_2\) be square-free integers. We write \(N = \text{lcm}[N_1, N_2]\). In this paper, we let \(f \in H^*_{k_1}(N_1)\), \(g \in H^*_{k_2}(N_2)\) be two distinct Hecke eigenforms that are not equivalently twist, and denote by \(\lambda_f(n)\) and \(\lambda_g(n)\) the \(n\)-th normalized Fourier coefficients of \(f\) and \(g\), respectively. Denote by \(n_{f,i}\) the least positive integer such that \(\lambda_f(n^i) < 0\) and \((n, N_1) = 1\), where \(i \geq 2\) are positive integers. The first purpose in this paper is to consider the first sign change of the sequence \(\{\lambda_f(n^i)\}_{n \in \mathbb{N}}\) in terms of the parameters \(k_1, N_1, i\) by proving the following theorem.
Theorem 1.1. Let $N_1 \geq 1$ be square-free and $i \geq 2$ be a positive integer. Let $f \in H^\infty_{k_1}(N_1)$ be a Hecke eigenform with the $n$-th normalized Fourier coefficient denoted by $\lambda_f(n)$. Then

$$n_{f,i} \ll \max \left\{ \exp \left( c_1(i + 1)^{16} \log^2 \left( \sqrt{q(f) \otimes q(g)} \right) \right), \left( \left[ \frac{i + 1}{4} \right] + 1 \right)! \right\}^{2+\varepsilon} q(f)\left( \frac{1}{2} + \varepsilon \right),$$

unconditionally for $i = 2$ and under the assumption that there exists no Landau–Siegel zero for Rankin–Selberg $L$-function $L(f \otimes g, s)$ for $i \geq 3$. Here $c_1 > 0$ is an absolute constant and $q(f)\left( \frac{1}{2} + \varepsilon \right)$ denotes the analytic conductor of the $i$-th Rankin–Selberg $L$-function $L(f \otimes g, s)$ associated with $f$.

In the analogue manner as with Theorem 1.1, we also consider the first simultaneous sign change of the sequences $\{\lambda_f(n)\}_{n \in \mathbb{N}}$, where $j, l \geq 1$ are positive integers. Let $n_{f,g,j,l}$ denote the least positive integer $n$ such that $\lambda_f(n^j)\lambda_g(n^l) < 0$ and $(n, N) = 1$. For $j = l = 1$, we abbreviate by $n_{f,g,1,1} := n_{f,g}$. In this paper, we also consider the size of $n_{f,g,j,l}$ in terms of parameters $k_1, k_2, N_1, N_2, j, l$ of $f$ and $g$. We shall be able to establish the following theorem.

Theorem 1.2. Let $N_1, N_2 \geq 1$ be square-free integers. Let $f \in H^\infty_{k_1}(N_1), g \in H^\infty_{k_2}(N_2)$ be two distinct Hecke eigenforms and not equivalently twist. Denote the $n$-th normalized Fourier coefficients of $f, g$ by $\lambda_f(n), \lambda_g(n)$, respectively. Then we have

$$n_{f,g} \ll \max \left\{ \exp \left( c_2 \log^2 \left( \max\{q(f), q(g)\} \right) \right), q(f \otimes g)\left( \frac{1}{2} - \eta' \right) \right\}.$$

Here $c_2 > 0$ is an absolute constant and $\eta' > 0$ is some certain unspecified constant, and $q(f)\left( \frac{1}{2} + \varepsilon \right)$ and $q(g)\left( \frac{1}{2} + \varepsilon \right)$ denote the analytic conductors of the symmetric square $L$-functions $L(f \otimes g, s)$ and $L(g \otimes f, s)$ associated with $f$ and $g$, respectively. And $q(f \otimes g)$ denotes the analytic conductor of Rankin–Selberg $L$-function $L(f \otimes g, s)$.

Theorem 1.3. Let $N_1, N_2 \geq 1$ be square-free integers, and let $\mathcal{S} := \{(j, l) : j, l \geq 1\} \setminus \{(1, 1)\}$. Let $f \in H^\infty_{k_1}(N_1), g \in H^\infty_{k_2}(N_2)$ be two distinct Hecke eigenforms and not equivalently twist. Denote the $n$-th normalized Fourier coefficients of $f, g$ by $\lambda_f(n), \lambda_g(n)$, respectively. Then for the pair $(j, l) \in \mathcal{S}$, we have

$$n_{f,g,j,l} \ll \max \left\{ \exp \left( c_3((2j + 1)(2l + 1))^{8} \log^2 \left( \sqrt{q(f) \otimes q(g)} \right) \right), \left( \left[ \frac{(i + 1)(j + 1)}{4} \right] + 1 \right)! \right\}^{2+\varepsilon} q(f \otimes g)\left( \frac{1}{2} + \varepsilon \right),$$

which holds under the assumption that there exists no Landau–Siegel zeros for automorphic $L$-functions $L(f \otimes g, s)$, $L(f \otimes g, s)$, $L(f \otimes g, s)$ for $1 \leq j_1, j_2 \leq 2j, l \leq 1$. 
Let $\lambda_{\text{sym}^j f}(n)$ be the $n$-th normalized coefficient of the Dirichlet expansion of the $j$-th symmetric power $L$-function attached to $f$. Define

$$\mathcal{N}_{\text{sym}^j f}^\pm(x) := \sum_{n \leq x, (n,N_1) = 1, \lambda_{\text{sym}^j f}(n) \geq 0} 1,$$

the counting functions for the number of positive and negative coefficients $\lambda_{\text{sym}^j f}(n)$ of the same sign, respectively. where $j \geq 1$ is any fixed positive integer. For $j = 1$, we write $\mathcal{N}_{\text{sym}^1 f}^\pm(x) := \mathcal{N}_{f}^\pm(x)$. In 2008, Kohnen et al. [25] considered the case $j = 1$ and proved

$$\mathcal{N}_{f}^\pm(x) \gg f x \log x$$

for $x \geq x_0(f)$, where $x_0(f) > 0$ is some suitable positive constant that depends on $f$. Later, a number of authors has improved the result (see [29,50]). The best records in this direction is due to Lau and Wu [29] and they proved that

$$\mathcal{N}_{f}^\pm(x) \gg f x$$

for $x \geq x_0(f)$, where $x_0(f) > 0$ is some suitable constant depending on $f$.

Next we use the $B$-free method to prove some non-vanishing results concerning the cusp form coefficients associated to one form or two forms supported at certain sparse sequences. These results gives the best possible lower bounds in terms of the order of magnitude.

Let $i \geq 2$ be any fixed positive integer. Define

$$\mathcal{N}_{f,i}^\pm(x) := \sum_{n \leq x, (n,N_1) = 1, \lambda_f(n^i) \geq 0} 1,$$

which denotes the counting function which counts the number of sequence $\{\lambda_f(n^i)\}_{n \in \mathbb{N}}$ of the same sign for $n \leq x$. We prove the following results.

**Theorem 1.4.** Let $\mathcal{N}_{f,i}^\pm(x)$ be defined by (1.5). Then

$$\mathcal{N}_{f,i}^\pm(x) \gg f_i x$$

for $x \geq x_0(f)$, and $x_0(f) > 0$ is some suitable constant depending on $f$.

Let $j \geq 1, l \geq 1$ be any fixed positive integers, and set

$$\mathcal{N}_{f,g,j,l}^\pm(x) := \sum_{n \leq x, (n,N_1) = 1, \lambda_f(n^j)\lambda_g(n^l) \geq 0} 1,$$

the counting functions for the number of positive and negative signs for the sequences $\{\lambda_f(n^j)\lambda_g(n^l)\}_{n \in \mathbb{N}}$ in the interval $[0, x]$, respectively.
Theorem 1.5. Let $N_{f,g}^\pm(x)$ be defined by (1.6) and $f$, $g$ are not twist equivalent. Then

$$N_{f,g,j,l}^\pm(x) \gg_{f,g,j,l} x$$

for $x \geq x_0(f,g)$, and $x_0(f,g,j,k) > 0$ is some suitable constant depending on $f$ and $g$.

Throughout the paper, we always assume $f \in H_{k_1}^*(N_1)$, $g \in H_{k_2}^*(N_2)$ be two distinct Hecke eigenforms, and denote by $\lambda_f(n)$ and $\lambda_g(n)$ the $n$-th normalized Fourier coefficients of $f$ and $g$, respectively. And let $\varepsilon > 0$ denote an arbitrarily small number which may vary in different occurrence. And $p$ always denotes a prime number. The set of all primes is denoted by $\mathcal{P}$.

2. Preliminaries

In this section, we collect relevant analytic properties of automorphic $L$-functions which plays an important role in the proof of the main results in this paper.

We firstly review some standard facts about general $L$-functions arising from cuspidal automorphic representations and their Rankin–Selberg convolutions. Let $m \geq 1$ be an integer, and let $\mathcal{A}(m)$ be the set of all irreducible cuspidal automorphic representations of $GL_m$ over $\mathbb{Q}$. We consider each $\pi = \bigotimes_p \pi_p \in \mathcal{A}(m)$ to be normalized so that $\pi$ has an unitary central character which is trivial on the positive reals. For $\pi \in \mathcal{A}(m)$, the standard $L$-function $L(s, \pi)$ associated with $\pi$ is of the form

$$L(s, \pi) = \prod_{p < \infty} L_p(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}.$$ 

The Euler product converges absolutely for $\Re(s) > 1$. For each finite prime $p$, the inverse of the local factor $L_p(s, \pi)$ is given by

$$L_p(s, \pi)^{-1} = \prod_{j=1}^{m} \left( 1 - \frac{\alpha_{j,\pi}(p)}{p^s} \right).$$

Here $\alpha_{j,\pi}(p) \in \mathbb{C}$, $j = 1, 2, \ldots, m$ are the Satake parameters. Let $N_\pi$ denote the conductor of $\pi$. We have $\alpha_{j,\pi}(p) \neq 0$ for all $j$ whenever $p \nmid N_\pi$, and it might be the case that $\alpha_{j,\pi}(p) = 0$ for some $j$ when $p | N_\pi$. At the archimedean place of $\mathbb{Q}$, there are $m$ complex Landlands parameters $\mu_\pi(p)$ from which we define

$$L_\infty(s, \pi) = \pi^{-\frac{m}{2}} \prod_{j=1}^{m} \Gamma \left( s + \frac{\mu_\pi(j)}{2} \right).$$

Let $\tilde{\pi}$ denote the contragredient of $\pi \in \mathcal{A}(m)$, which is also an irreducible cuspidal automorphic representation in $\mathcal{A}(m)$. For each $p \leq \infty$, we have

$$\{ \alpha_{j,\tilde{\pi}}(p) : 1 \leq j \leq m \} = \{ \overline{\alpha_{j,\pi}(p)} : 1 \leq j \leq m \}$$

and

$$\{ \mu_{\tilde{\pi}}(j) : 1 \leq j \leq m \} = \{ \overline{\mu_\pi(j)} : 1 \leq j \leq m \}.$$ 

Define the completed $L$-function

$$\Lambda(s, \pi) = N_\pi^s L(s, \pi)L(s, \pi_\infty).$$
Then $\Lambda(s, \pi)$ can be analytically continued to the whole complex plane as an entire function (except in the case of $\zeta(s)$, which has a simple pole at $s = 1$). Moreover, $\Lambda(s, \pi)$ is bounded in the vertical strips and satisfies a functional equation of the form

$$\Lambda(s, \pi) = W(\pi)\Lambda(1 - s, \tilde{\pi}),$$

where $W(\pi)$ is a complex number of modulus 1. We define the analytic conductor of $\pi$ by

$$C(\pi, t) = N_{\pi} \prod_{j=1}^{m} (1 + |it + \mu_{\pi}(j)|), \quad C(\pi) = C(\pi, 0). \quad (2.1)$$

For any $\varepsilon > 0$, we have the convexity bound (cf. [34, (1.22)])

$$L\left(\frac{1}{2} + it, \pi\right) \ll (C(\pi, t))^{\frac{1}{2} + \varepsilon}. \quad (2.2)$$

Let $\pi = \otimes_{p} \pi_{p} \in \mathcal{A}(m)$ and $\pi' = \otimes_{p} \pi'_{p} \in \mathcal{A}(m')$. The Rankin–Selberg $L$-function associated with $\pi$ and $\pi'$ is defined by

$$L(s, \pi \times \pi') = \prod_{p} L_{p}(s, \pi \times \pi') = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \times \pi'}(n)}{n^{s}}.$$

The Euler product and Dirichlet series converge absolutely for $\Re(s) > 1$. For each finite prime $p$, the inverse of the local factor $L_{p}(s, \pi \times \pi')$ is given by

$$L_{p}(s, \pi \times \pi')^{-1} = \prod_{j=1}^{m} \prod_{j'=1}^{m'} \left(1 - \frac{\alpha_{j, j', \pi \times \pi'}(p)}{p^{s}}\right)$$

for suitable complex numbers $\alpha_{j, j', \pi \times \pi'}(p)$. For $p \nmid N_{\pi} N_{\pi'}$, we have the set of equalities

$$\{\alpha_{j, j', \pi \times \pi'}(p) : 1 \leq j \leq m, 1 \leq j' \leq m'\} = \{\alpha_{j', \pi}(p) \alpha_{j, \pi'}(p) : 1 \leq j \leq m, 1 \leq j' \leq m'\}.$$

At the archimedean place of $\mathbb{Q}$, there are $mm'$ complex Langlands parameters $\mu_{\pi \times \pi'}(j, j')$ from which we define

$$L_{\infty}(s, \pi \times \pi') = \pi^{-\frac{mm's}{2}} \prod_{j=1}^{m} \prod_{j'=1}^{m'} \Gamma\left(\frac{s + \mu_{\pi \times \pi'}(j, j')}{2}\right).$$

The parameters satisfy the equality

$$\{\mu_{\pi \times \pi'}(j, j')\} = \{\mu_{\pi \times \pi'}(j', j)\}$$

for $1 \leq j \leq m, 1 \leq j' \leq m'$. The completed $L$-function

$$L(s, \pi \times \pi') = N_{\pi \times \pi}^{\frac{s}{2}} L(s, \pi \times \pi') L_{\infty}(s, \pi \times \pi')$$

has a meromorphic continuation and is bounded (away from its poles) in vertical strips. And under the normalization on the central characters, $\Lambda(s, \pi \times \pi')$ is entire if and only if $\tilde{\pi} \neq \pi'$. Moreover, $\Lambda(s, \pi \times \pi')$ satisfies the functional equation

$$\Lambda(s, \pi \times \pi') = W(\pi \times \pi')\Lambda(1 - s, \tilde{\pi} \times \tilde{\pi'}),$$

where $W(\pi \times \pi')$ is a complex number of modulus 1.
We define the analytic conductor of $\pi \times \pi'$ to be

$$C(\pi \times \pi', t) = N_{\pi \times \pi'} \prod_{j=1}^{m} \prod_{j'=1}^{m'} (1 + |it + \mu_{\pi \times \pi'}(j, j')|),$$

$$C(\pi \times \pi') = C(\pi \times \pi', 0).$$

The combined work of Bushnell and Henniart [2, Theorem 1] and Brumley [13, Lemma A.2] proves that

$$C(\pi \times \pi', t) \leq C(\pi \times \pi')(1 + |t|)^{m'm}, C(\pi \times \pi') \leq e^{O(m'm')}C(\pi)^mC(\pi')^m. \quad (2.3)$$

For the convexity bound of $L(s, \pi \times \pi')$, very recently, Jiang et al. [19, Lemma 3.2] proved that

$$L\left(\frac{1}{2} + it, \pi \times \pi'\right) \ll (C(\pi \times \pi', t))^{\frac{1}{2} + \varepsilon} \quad (2.4)$$

Let $f \in H_{k_1}^1(N_1)$ be a Hecke eigenform. According to Deligne [7], for any prime $p$ there are two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\begin{cases} \alpha_f(p) = \varepsilon_f(p)p^{-\frac{1}{2}}, \beta_f(p) = 0, & \text{if } p \mid N_1, \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N_1, \end{cases}$$

where $\varepsilon_f(p) = \pm 1$. For each prime $p$, there exists an angle $\theta_f(p) \in [0, \pi]$ such that $\lambda_f(p) = 2 \cos \theta_f(p)$. The $j$-th symmetric power $L$-function attached to $f$ is defined as

$$L(\text{sym}^j f, s) = \prod_p \prod_{m=0}^{j} \left(1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m}{p^s}\right)^{-1} = \prod_p L_p(\text{sym}^j f, s), \quad \Re(s) > 1. \quad (2.5)$$

And it can be expanded into a Dirichlet series

$$L(\text{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\text{sym}^j f}(p^k)}{p^{ks}}\right),$$

where $\lambda_{\text{sym}^j f}(n)$ is a real multiplicative function. It is standard to find that

$$|\lambda_{\text{sym}^j f}(n)| \leq d_{j+1}(n), \quad (2.6)$$

where $d_{j+1}(n)$ is the $(j + 1)$-th fold divisor function associated with the Dirichlet series $\zeta(s)^{j+1}$.

It is predicted by the Langlands functoriality conjecture that $\pi_f$ gives rise to a symmetric power lift $\text{sym}^j \pi_f$ – an automorphic representation whose $L$-function is the symmetric power $L$-function attached to $f$,

$$L(\text{sym}^j \pi_f, s) = L(\text{sym}^j f, s).$$

It is conjectured that there exists an automorphic cuspidal self-dual representation $\text{sym}^j \pi_f$ of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose $L$-function is the same as $L(\text{sym}^j f, s)$.

For $1 \leq j \leq 8$, this special Langlands functoriality conjecture that $\text{sym}^j f$ is automorphic cuspidal which is shown by a series of important works by Gelbart and Jacquet [9], Kim [22], Kim and Shahidi [20,21], Shahidi [47], Clozel and Thorne [4–6]. Very
recently, Newton and Thorne [39,40] proved that \( \text{sym}^j f \) corresponds with a cuspidal automorphic representation of \( GL_{j+1}(\mathbb{A}_Q) \) for all \( j \geq 1 \) (with \( f \) being holomorphic cusp forms). Then we know that for all \( j \geq 5 \), there exists an automorphic cuspidal self-dual representation, denoted by \( \text{sym}^j \pi_f = \bigotimes \text{sym}^j \pi_{f,v} \) of \( GL_{j+1}(\mathbb{A}_Q) \) whose local \( L \)-factors \( L(\text{sym}^j \pi_{f,p}, s) \) agree with the local \( L \)-factors \( L_p(\text{sym}^j f, s) \) in (2.5). In particular, \( L(\text{sym}^j f, s) \) has an analytic continuation to the whole complex plane as an entire function and satisfies a certain Riemann-type functional equation for all \( j \geq 1 \).

From the works of Rankin–Selberg theory associated with two automorphic cuspidal representations developed by Jacquet et al. [18], Jacquet and and Shalika [16,17], Shahidi [44–46,48], and the reformulation of Rudnick and Sarnak [41], we know the analytic properties for the Rankin–Selberg \( L \)-functions \( L(\text{sym}^j f \otimes \text{sym}^l g, s) \) with \( j, l \geq 1 \), where \( f \in H_{k_1}^*(N_1), g \in H_{k_2}^*(N_2) \) are two Hecke eigenforms.

Let \( f \in H_{k_1}^*(N_1) \) be a Hecke eigenform, according to Cogdell and Michel [3, Section 3.2.1]. The gamma factors of \( L(\text{sym}^j f, s) \) is given by

\[
L_\infty(\text{sym}^j f, s) = \begin{cases} \prod_{c=0}^n \Gamma_C(s + (v + \frac{1}{2})(k - 1)), & \text{if } j = 2n + 1, \\ \prod_{c=0}^n \Gamma_R(s + \delta_{2n}(s + v(k - 1))), & \text{if } j = 2n, \end{cases}
\]

where \( \Gamma_R = \pi^{-\frac{k}{2}} \Gamma(\frac{k}{2}), \Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s), \) here \( \Gamma_C(s) = \Gamma_R(s) \Gamma_R(s + 1), \) and

\[
\delta_{2n} = \begin{cases} 1, & \text{if } 2 \nmid n, \\ 0, & \text{otherwise}. \end{cases}
\]

From the automorphy of \( \text{sym}^j \pi_f \) for all \( j \geq 1 \) and the Rankin–Selberg method, we know that the function

\[
\Lambda(\text{sym}^j f, s) = N_{k_1}^{j/2} L_\infty(\text{sym}^j f, s) L(\text{sym}^j f, s)
\]

is entire on the whole complex plane \( \mathbb{C} \) and satisfies the functional equation

\[
\Lambda(\text{sym}^j f, s) = \epsilon_{\text{sym}^j f} \Lambda(\text{sym}^j f, 1 - s),
\]

where \( \epsilon_{\text{sym}^j f} = \pm 1 \). We can define the analytic conductor of \( L(\text{sym}^j f, s) \) in the sense of (2.1) such that

\[
q(\text{sym}^j f, t) = \begin{cases} N_{k_1}^{j/2} \prod_{r=0}^n (1 + |it + (v + \frac{1}{2})(k_1 - 1)|)^2, & \text{if } j = 2n + 1, \\ N_{k_1}(1 + |it + (v + \frac{1}{2})(k_1 - 1)|)^2 \prod_{r=1}^n (1 + |it + (v + \frac{1}{2})(k_1 - 1)|)^2, & \text{if } j = 2n. \end{cases}
\]

Here we denote \( q(\text{sym}^j f) := q(\text{sym}^j f, 0) \).

Similarly, if \( g \in H_{k_2}^*(N_2) \) be a Hecke eigenform, we can also define the symmetric power \( L \)-function \( L(\text{sym}^j g, s) \) associated with \( g \) for positive integers \( l \geq 1 \). It is well-known from the Rankin–Selberg theory that \( L(\text{sym}^j f \otimes \text{sym}^l g, s) \) can be extended to the whole complex plane as an entire function (except possibly for simple poles at \( s = 0, 1 \) when \( \text{sym}^j \pi_f \cong \text{sym}^j \pi_g \)) and satisfies the functional equation of Riemann-type, here \( j, l \geq 1 \) are positive integers.

It is standard to find that

\[
\lambda_f(p^j) = \lambda_{\text{sym}^j f}(p) = \frac{\alpha_f(p)j+1 - \beta_f(p)^j+1}{\alpha_f(p) - \beta_f(p)} = \sum_{m=0}^j \alpha_f(p)^{j-m} \beta_f(p)^m, \quad (p, N) = 1,
\]

...
which can be written as
\[ \lambda_f(p^j) = \lambda_{\text{sym}^j f}(p) = U_j(\lambda_f(p)/2), \quad (p, N) = 1, \] (2.7)
where \( U_j(x) \) is the \( j \)-th Chebyshev polynomial of the second kind. For any prime number \( p \), we also have
\[ \lambda_{\text{sym}^j f \otimes \text{sym}^l g}(p) = \lambda_{\text{sym}^j f}(p)\lambda_{\text{sym}^l g}(p) = \lambda_f(p^j)\lambda_g(p^l), \quad (p, N) = 1. \] (2.8)

We denote the analytic conductor of \( L(\text{sym}^j f \otimes \text{sym}^l g, s) \) by \( q(\text{sym}^j f \otimes \text{sym}^l g, t) \), and we write \( q(\text{sym}^j f \otimes \text{sym}^l g) := q(\text{sym}^j f \otimes \text{sym}^l g, 0) \). From (2.3), we know that
\[ q(\text{sym}^j f \otimes \text{sym}^l g) \leq e^{O((j+1)(l+1))} q(\text{sym}^j f)^{l+1} q(\text{sym}^l g)^{j+1}. \]

### 3. Auxiliary results

In this section, we introduce some lemmas which are useful in the proof of the main results in this paper.

The following result of Kowalski et al. [24] (see also [11, Lemma 6]) plays an important role in the proof of Theorem 1.4 and Theorem 1.5.

**Lemma 3.1.** Let \( f \in H_k^*(N) \) be a normalized non-CM Hecke eigenform. For \( \nu \geq 1 \), let
\[ P_{f,\nu} := \{ p \in \mathcal{P} \mid p \nmid N, \lambda_f(p^\nu) = 0 \}. \]
Then for any \( \nu \geq 1 \), we have
\[ \#(P_{f,\nu} \cap [1, x]) \ll_{f, \delta} x \left( \frac{\log x}{(\log x)^{1+\delta}} \right), \]
for any \( x \geq 2 \) and \( 0 < \delta < \frac{1}{2} \).

Next, we cite an important result regarding the sub-convexity bound of Rankin–Selberg \( L \)-function \( L(f \otimes g, s) \) due to Michel and Venkatesh [35].

**Lemma 3.2.** [35, Theorem 1.2] Let \( L(f \otimes g, s) \) be the Rankin–Selberg function associated to cusp forms \( f \) and \( g \) which are not equivalently twist. Then
\[ L \left( f \otimes g, \frac{1}{2} + it \right) \ll q(f \otimes g, t)^{\frac{1}{2} - \eta}, \]
where \( q(f \otimes g, t) \) is the analytic conductor of \( L(f \otimes g, s) \), and \( \eta > 0 \) is some positive absolute constant.

Define \( L \)-functions
\[ L_{f,i}(s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n^i)}{n^s}, \quad \Re(s) > 1 \] (3.1)
Lemma 3.4. Let $L_f, i(s)$ and $L_{f,g,j,l}(s)$ defined by (3.1) and (3.2), respectively. Let $i \geq 2$ and $j, l \geq 1$ be positive integers, then

$$L_f, i(s) = L(\text{sym}^i f, s)U_i(s)$$

and

$$L_{f,g,j,l}(s) = L(\text{sym}^i f \otimes \text{sym}^j g, s)U_{j,l}(s),$$

where $U_i(s)$, $U_{j,l}(s)$ are Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$ and $U_i(s)$, $U_{j,l}(s) \neq 0$ for $\Re(s) = 1$.

Proof. Since $\lambda_f(n^i)$, $\lambda_f(n^j)\lambda_g(n^l)$ are real multiplicative functions and satisfies the trivial bound $O(n^\varepsilon)$, we have

$$L_f, i(s) = \prod_{p \mid N} 1 + \sum_{k \geq 1} \frac{\lambda_f(p^k)}{p^{ks}} U_i(s)$$

and

$$L_{f,g,j,l}(s) = \prod_{p \mid N} 1 + \sum_{k \geq 1} \frac{\lambda_f(p^k)\lambda_g(p^l)}{p^{ks}} U_{j,l}(s),$$

where the functions $U_i(s)$, $U_{j,l}(s)$ admit the Dirichlet series which converges uniformly and absolutely in the half-plane $\Re(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

In the half-plane $\Re(s) > 1$, the corresponding coefficients of $p^{-s}$ determine analytic properties of $L_f, i(s)$ and $L_{f,g,j,l}(s)$. We also have the relations

$$\lambda_f(p^j) = \lambda_{\text{sym}^j f}(p),$$

$$\lambda_f(p^j)\lambda_g(p^l) = \lambda_{\text{sym}^j f \times \text{sym}^l g}(p) = \lambda_{\text{sym}^j f \times \text{sym}^l g}(p), \ (p, N) = 1.$$  

Then the required results follow from these identities. Since the symmetric power lift $\text{sym}^j f$ are automorphic for all $j \geq 1$ due to Newton and Thorne [39, 40], a more general result follows from [8, Proposition 2].

Lemma 3.4. Let $f \in H_{k_1}^\times(N_1)$ be a Hecke eigenform. Let $i \geq 2$ be a positive integer, and assume that $\lambda_f(n^i) \geq 0$ for all $n \leq x$. Then for $x \geq \exp(c_1 (i + 1)^{16} \log^2(\sqrt{q(\text{sym}^i f \otimes \text{sym}^i f)}))$, we have

$$\sum_{n \leq x, \ (n, N_1) = 1 \ n \text{ square-free}} \lambda_f(n^i) \gg \frac{x}{\log^2 x}$$
unconditionally for \( i = 2 \), and under the assumption that there exists no Landau–Siegel
zero for Rankin–Selberg \( L \)-function \( L(\text{sym}^i f \otimes \text{sym}^i f, s) \) for \( i \geq 3 \). Here \( c_1 \) is an absolute
constant and \( q(\text{sym}^i f \otimes \text{sym}^i f) \) is the analytic conductor of the Rankin–Selberg \( L \)-
function \( L(\text{sym}^i f \otimes \text{sym}^i f, s) \) associated with \( f \).

**Proof.** We prove the lemma by following the argument of Gun et al. [11, Lemma 12] with
slight modifications. Define

\[
\psi(\text{sym}^i f \otimes \text{sym}^i f, x) = \sum_{n \leq x} \Lambda_{\text{sym}^i f \otimes \text{sym}^i f}(n)
\]
as in the sense of [14, (5.46)]. Then by [14, (5.49)] and the prime number theorem for
automorphic \( L \)-functions [14, (5.52)], we can derive that

\[
\sum_{n \leq x} \lambda_{\text{sym}^i f \otimes \text{sym}^i f}(p) \log p = x + O\left(\sqrt{x}(i + 1)^4 \log^2(xq(\text{sym}^i f \otimes \text{sym}^i f))\right)
+ O\left(x \sqrt{q(\text{sym}^i f \otimes \text{sym}^i f)} \exp\left(-\frac{c}{2(i + 1)^8 \sqrt{\log x}}\right)\right),
\]

under the assumption that there exist no Landau–Siegel zeros for Rankin–Selberg \( L \)-
function \( L(\text{sym}^i f \otimes \text{sym}^i f, s) \) for all \( i \geq 2 \), where \( c > 0 \) is some absolute constant. For
the case \( i = 2 \), it has been proved by Ramakrishnan and Wang [42] the non-existence of
Landau–Siegel zero.

On noting that in (3.5) the first error term is dominated by the second error term and the
multiplicative relations (2.8), we have

\[
\sum_{n \leq x} \lambda^2_f(p^i) \log p = x + O\left(x \sqrt{q(\text{sym}^i f \otimes \text{sym}^i f)} \exp\left(-\frac{c}{2(i + 1)^8 \sqrt{\log x}}\right)\right),
\]

(3.6)

From the given hypothesis that \( \lambda_f(n^i) \geq 0 \) for all \( n \leq x \),

\[
\sum_{n \leq x} \lambda_f(n^i) \geq \frac{1}{2} \sum_{p, q \leq \sqrt{x}, p \neq q} \lambda_f(p^i q^i) = \frac{1}{2} \left( \sum_{p \leq \sqrt{x}} \lambda_f(p^i) \right)^2 - \frac{1}{2} \sum_{p \leq \sqrt{x}} \lambda^2_f(p^i).
\]

By the relation (2.6), (3.6) and partial summation, we then have

\[
\sum_{n \leq x} \lambda_f(n^i) \gg \left( \sum_{p \leq \sqrt{x}} \lambda^2_f(p^i) \right)^2 + O\left(\frac{\sqrt{x}}{\log x}\right)
= \frac{x}{\log^2 x},
\]

(3.7)
provided \( x \geq \exp \left( c_1 (i + 1)^{16} \log^2 \left( \sqrt{q(\text{sym}^1 f \otimes \text{sym}^1 f)} \right) \right) \), where \( c_1 > 0 \) is an absolute constant.

\[ \Box \]

**Lemma 3.5.** Let \( f \in H_{k_1}^*(N_1) \), \( g \in H_{k_2}^*(N_2) \) be two distinct non-CM type Hecke eigenforms and not equivalently twist. Assume that \( \lambda_f(n)\lambda_g(n) \geq 0 \) for all \( n \leq x \). Then for \( x \geq \exp(c_2 \log^2(\sqrt{\max\{q(\text{sym}^2 f), q(\text{sym}^2 g)\}})) \), we have

\[
\sum_{\substack{n \leq x, \ (n,N)=1 \ \text{n square-free}}} \lambda_f(n)\lambda_g(n) \gg \frac{x}{\log^2 x},
\]

where \( c_2 > 0 \) is an absolute constant, and \( q(\text{sym}^2 f) \) and \( q(\text{sym}^2 g) \) are the analytic conductors associated with the Rankin–Selberg \( L \)-functions \( L(\text{sym}^2 f, s) \) and \( L(\text{sym}^2 g, s) \).

**Proof.** This follows essentially the argument of Kumari and Sengupta [28, Remark 3.2], together with the celebrated work of Banks [1] that the non-existence of Landau–Siegel zero for the symmetric square \( L \)-functions \( L(\text{sym}^2 f, s) \), \( L(\text{sym}^2 g, s) \).

\[ \Box \]

**Lemma 3.6.** Let \( \mathcal{D} := \{(j,1) : j,1 \geq 1\} \backslash \{(1,1)\} \). Let \( f \in H_{k_1}^*(N_1) \), \( g \in H_{k_2}^*(N_2) \) be two distinct non-CM type Hecke eigenforms and not equivalently twist with the \( n \)-th normalized Fourier coefficients denoted by \( \lambda_f(n) \), \( \lambda_g(n) \), respectively. Assume that \( \text{sym}^j \pi_f \not\cong \text{sym}^j \pi_g \) for all \( 1 \leq j' \leq \min\{2j, 2l\} \) and \( \lambda_f(n^j)\lambda_g(n^{j'}) \geq 0 \) for all \( n \leq x \). Then for \( x \geq \exp(c_3 ((2j+1)(2l+1))^8 \log^2(\sqrt{\max\{q(\text{sym}^2 f), q(\text{sym}^2 g)\}})) \), we have

\[
\sum_{\substack{n \leq x, \ (n,N)=1 \ \text{n square-free}}} \lambda_f(n^j)\lambda_g(n^{j'}) \gg \frac{x}{\log^2 x},
\]

holds under the assumption that there exists no Landau–Siegel zeros for automorphic \( L \)-functions \( L(\text{sym}^{l_1} f, s) \), \( L(\text{sym}^{l_1} g, s) \), \( L(\text{sym}^{l_2} f \otimes \text{sym}^{l_2} g, s) \) for \( 1 \leq j_1, j_2 \leq 2j, 1 \leq l_1, l_2 \leq 2l \) (few cases are known without Landau–Siegel zeros, cf. [34, pp. 200–211]). Here \( c_3 > 0 \) is an absolute constant and \( q(\text{sym}^j f \otimes \text{sym}^j g) \) denotes the analytic conductor of the Rankin–Selberg \( L \)-functions \( L(\text{sym}^j f \otimes \text{sym}^j g, s) \) associated with \( f \) and \( g \).

**Proof.** By the prime number theorem for \( \zeta(s) \), we have

\[
\sum_{n \leq x} \log p = x + O(xe^{-\tilde{c}\sqrt{\log x}}),
\]

where \( \tilde{c} \) is some absolute constant. Noting the multiplicative relations (2.8) and

\[
\lambda_f^2(p^j)\lambda_g^2(p^{j'}) = (1 + \lambda_f(p^2) + \cdots + \lambda_f(p^{2j}))(1 + \lambda_g(p^2) + \cdots + \lambda_g(p^{2j})),
\]

along with applying the prime number theorem for the automorphic \( L \)-functions \( L(\text{sym}^{l_1} f, s) \), \( L(\text{sym}^{l_1} g, s) \), \( L(\text{sym}^{l_2} f \otimes \text{sym}^{l_2} g, s) \) with \( j_1, l_1, j_2, l_2 \geq 1 \) together with the partial summation formula under the given hypothesis, the proof of the lemma follows essentially the argument as in Lemma 3.4.
4. Proofs of Theorems 1.1–1.3

To prove Theorem 1.1, we consider the sum
\[
S_i(x) := \sum_{\substack{n \leq x, \ (n, N_1) = 1 \\ n \ square-free}} \lambda_f(n^i) \log^{\left\lfloor \frac{i+1}{4} \right\rfloor+2} \left( \frac{x}{n} \right).
\] (4.1)

Then the desired result will follow from upper and lower bound estimates for \( S_i(x) \) under the assumption that
\[
\lambda_f(n^i) \geq 0 \quad \text{for all } n \leq x.
\]

**PROPOSITION 4.1**

Let \( N_1 \geq 1 \) be square-free and \( i \geq 2 \) be a positive integer. Let \( f \in H_{k_1}^s(N_1) \) be a Hecke eigenform and \( S_i(x) \) be defined by (4.1), then we have
\[
S_i(x) \ll_f \left( \left\lfloor \frac{i+1}{4} \right\rfloor + 1 \right)! q(\text{sym}^i f)^{\frac{1}{4}+\varepsilon} x^{\frac{1}{4}+\varepsilon}.
\]

**Proof.** From (3.3) in Lemma 3.3 and Perron’s formula [49, p. 228, Exercise 169], together with shifting the line of integration to the parallel line with \( \frac{1}{2} + \varepsilon \), we then have
\[
\sum_{\substack{n \leq x, \ (n, N_1) = 1 \\ n \ square-free}} \lambda_f(n^i) \log^{\left\lfloor \frac{i+1}{4} \right\rfloor+1} \left( \frac{x}{n} \right)
\ll \left( \left\lfloor \frac{i+1}{4} \right\rfloor + 1 \right)! \int_{2-i\infty}^{2+i\infty} L(\text{sym}^i f, s) \frac{x^s}{s^{\left\lfloor \frac{i+1}{4} \right\rfloor+2}} ds
\ll \left( \left\lfloor \frac{i+1}{4} \right\rfloor + 1 \right)! x^{\frac{1}{2}+\varepsilon} q(\text{sym}^i f)^{\frac{1}{4}+\varepsilon} \int_{-\infty}^{\infty} \frac{(3 + |t|)^{\left\lfloor \frac{i+1}{4} \right\rfloor+2}}{(1 + |t|)^{\left\lfloor \frac{i+1}{4} \right\rfloor+2}} dt
\ll \left( \left\lfloor \frac{i+1}{4} \right\rfloor + 1 \right)! q(\text{sym}^i f)^{\frac{1}{4}+\varepsilon} x^{\frac{1}{4}+\varepsilon},
\]
where we have use the convexity bound given by (2.2) and the Phragmén–Lindelöf principle. \( \square \)

Now we give the proof of Theorem 1.1 in detail. First of all, we assume that \( x \geq \exp \left( c_1 (i + 1)^{16} \log^2 \left( \sqrt{q(\text{sym}^i f \otimes \text{sym}^i f)} \right) \right) \) be a real number such that \( \lambda_f(n^i) \geq 0 \) for all \( n \leq x \). Then from Lemma 3.4 and Proposition 4.2, we obtain
\[
\frac{x}{\log^2 x} \ll S_i(x) \ll \left( \left\lfloor \frac{i+1}{4} \right\rfloor + 1 \right)! q(\text{sym}^i f)^{\frac{1}{4}+\varepsilon} x^{\frac{1}{4}+\varepsilon},
\]
which is equivalent to the assertion that
\[
x \ll \left( \left\lfloor \frac{i+1}{4} \right\rfloor + 1 \right)! q(\text{sym}^i f)^{\frac{1}{4}+\varepsilon}.
\]
Then the required results can be obtained under the given hypothesis concerning the Landau–Siegel zeros.

Theorem 1.2 and Theorem 1.3 can be obtained by a similar argument by noting Lemmas 3.2–3.6 and the convexity bound (2.4).

5. Proofs of Theorems 1.4 and 1.5

In this section, we give the proofs of Theorem 1.4 and Theorem 1.5. And we give the proof of Theorem 1.5 by following the argument of [29] with some modifications, since Theorem 1.4 can be handled in a similar approach.

Let \( p' \) be the least prime that \( \lambda_f(p'^j)\lambda_g(p'^l) < 0 \). Let

\[
\mathcal{B} := \{ p : \lambda_f(p^j)\lambda_g(p^l) = 0 \}
\]

\[
\cup \{ p : p | N \} \cup \{ p' \} \cup \{ p'^2 : p' \nmid p'N, \lambda_f(p^j)\lambda_g(p^l) \neq 0 \}
\]

\[
:= \{ b_i \}_{i \geq 1}
\]

(5.1)

with increasing order.

By Lemma 3.1, for any \( x \geq 2 \) and \( 0 < \delta < \frac{1}{2} \), one has

\[
\#\{ p : \lambda_f(p^j)\lambda_g(p^l) = 0 \} \ll \frac{x}{(\log x)^{1+\delta}}.
\]

(5.2)

Note that \((b_i, b_j) = 1\) for all \( b_i, b_j \in \mathcal{B} \) with \( b_i \neq b_j \). Applying (5.2) and the partial summation formula, we have

\[
\sum_{b_i \leq x \atop b_i \in \mathcal{B}} \frac{1}{b_i} \ll \frac{1}{x} \sum_{p \leq x} \frac{1}{p} + \int_2^x \frac{1}{t^2} \left( \sum_{p \leq t} \frac{1}{p} \right) dt + O(1) \ll \mathcal{B} 1.
\]

Let \( \mathcal{A} := \{ a_i \}_{i \geq 1} \) (with increasing order) be the sequence of all \( \mathcal{B} \)-free numbers, i.e., the integers indivisible by any element in \( \mathcal{B} \). From the result of Erdös, the set \( \mathcal{A} \) is positive density

\[
\lim_{x \to \infty} \frac{\#(\mathcal{A} \cap [1, x])}{x} = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{b_i} \right) > 0.
\]

From the multiplicative property of \( \lambda_f(n^j)\lambda_g(n^l) \) and the definition of \( \mathcal{B} \), we have \( \lambda_f(a^j)\lambda_g(a^l) \neq 0 \) for all \( a \in \mathcal{A} \). Let

\[
\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-,
\]

where

\[
\mathcal{A}^\pm := \{ a_i \in \mathcal{A} : \lambda_f(a_i^j)\lambda_g(a_i^l) \geq 0 \}.
\]

Consider the set

\[
\mathcal{N}^\pm := \mathcal{A}^\pm \cup \{ a; p' a_i \in \mathcal{A}^\mp \}.
\]

It is not hard to find that \( \lambda_f(a^j)\lambda_g(a^l) \geq 0 \) and \((a, N) = 1\) for all \( a \in \mathcal{N}^\pm \) and

\[
\mathcal{N}^\pm_{f,g,j,l}(x) \geq \#(\mathcal{N}^\pm \cap [1, x]) \geq \{ \mathcal{A} \cap [1, x/p'] \}
\]

for all \( x \geq 2 \). This completes the proof of Theorem 1.5.
Let $p''$ be the least prime that $\lambda_f(p''i) < 0$. For the proof of Theorem 1.4, we can prove similarly by defining

$$\mathcal{B}^* := \{p : \lambda_f(p^i) = 0\} \cup \{p : p | N\} \cup \{p''\} \cup \{p^2 : \lambda_f(p^i) \neq 0\}$$

and then argue in a similar manner as above.

Acknowledgements

The author would like to express his gratitude to Professor Guangshi Lü and Professor Bin Chen for their constant encouragement and valuable suggestions. The author is extremely grateful to the anonymous referees for their meticulous checking, for thoroughly reporting countless typos and inaccuracies as well as for their valuable comments. These corrections and additions have made the manuscript clearer and more readable. This work was supported in part by The National Key Research and Development Program of China (Grant No. 2021YFA1000700) and Natural Science Basic Research Program of Shaanxi (Program Nos 2023-JC-QN-0024, 2023-JC-YB-077).

References


**COMMUNICATING EDITOR:** Sanoli Gun