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The second moment of the Fourier coefficients of triple product L-functions

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Abstract. Suppose that $\mathbb{H}^*$ is the set of all primitive cusp forms f of even integral weight $k \geq 2$ for the full modular group $SL_2(\mathbb{Z})$. In this paper, we establish asymptotic formulas for the second moment of Fourier coefficients of the triple product L-function $L(s, f \otimes f \otimes f)$ and the related L-function $L(s, \mathrm{sym}^2 f \otimes f)$ attached to f on average, which improves previous results.

Keywords. Fourier coefficients; triple product L-function; variance.

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1. Introduction

Suppose that $\mathbb{H}^*$ is the set of all primitive cusp forms of even weight $k \geq 2$ for the full modular group $SL_2(\mathbb{Z})$, where k is an integer. Then $\mathbb{H}^*$ consists of the common eigenfunctions f of all Hecke operators. For each $f \in \mathbb{H}^*$, at the cusp ∞, it has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z},$$

where $\mathfrak{Im}(z) > 0$ and $\lambda_f(n)$ denotes the n-th normalized Fourier coefficient. The Fourier coefficient $\lambda_f(n)$ is real-valued and satisfies

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

for any integers $m, n \geq 1$. It was shown by Deligne [4] that for any prime p, there are two complex numbers α_p and β_p such that

$$\alpha_p \beta_p = |\alpha_p| = |\beta_p| = 1 \quad \text{and} \quad \lambda_f(p) = \alpha_p + \beta_p. \tag{1}$$

For all integer $n \geq 1$, this follows the Deligne inequality

$$|\lambda_f(n)| \leq d(n) \ll n^{\varepsilon} \tag{2}$$

with the divisor function $d(n)$.

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For each $f \in H_k^*$, let $L(s, f)$ be the Hecke L-function defined by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1} \quad \Re(s) > 1.$$

In number theory, the average behavior of various sums of Fourier coefficient $\lambda_f(n)$ received many scholars' attention. In 1927, Hecke [8] showed that

$$\sum_{n \le x} \lambda_f(n) \ll_f x^{1/2}.$$

Later, the first improvement was given by Wilton [37] in which only the case of Ramanujan's τ-function was given and generalized by Walfisz [35] to other forms. Walfisz [35] showed that

$$\sum_{n \le x} \lambda_f(n) \ll_f x^{(1+\theta)/3}, \tag{3}$$

where θ is a constant such that

$$|\lambda_f(n)| \le n^{\theta}. \tag{4}$$

The works on the exponent θ in (4) of Kloosterman [15], Davenport [3], Salié [33], Weil [36] and Deligne [4] implied better corresponding results in (3). Halfner and Ivić [7] removed x^{θ} of Deligne's result and got

$$\sum_{n \le x} \lambda_f(n) \ll_f x^{1/3}.$$

Rankin [32] improved the above result further to

$$\sum_{n \le x} \lambda_f(n) \ll_f x^{1/3} (\log x)^{-0.0652}.$$

The best result until now is due to Wu [38], who showed that

$$\sum_{n \le x} \lambda_f(n) \ll_f x^{1/3} (\log x)^{-0.1185}.$$

For the second moment, Rankin [31] and Selberg [34] independently proved that

$$\sum_{n \le x} \lambda_f(n)^2 = c_f x + O_f(x^{3/5}),$$

where c_f is a suitable constant depending on f. Recently, the exponent $3/5$ was improved by Huang [9]. For higher moments, we refer to [5,16,17,20–22,26] for details.

Let $L(s, f \otimes f \otimes f)$ be the triple product L-function, which is defined in (9) and satisfies similar analytic properties to those of the Hecke L-functions. In 2017, Lü and Sankaranarayanan [24] investigated the average behavior of the coefficients $\lambda_{f\otimes f\otimes f}(n)$ of the triple product L-function $L(s, f \otimes f \otimes f)$. Precisely, they proved that, for any $\varepsilon > 0$,

$$\sum_{n \le x} \lambda_{f\otimes f\otimes f}(n) \ll_{f,\varepsilon} x^{7/10+\varepsilon}, \tag{5}$$

$$\sum_{n \le x} \lambda_{f\otimes f\otimes f}(n)^2 = x P(\log x) + O_{f,\varepsilon}(x^{175/181+\varepsilon}), \tag{6}$$

where $P(t)$ is a polynomial of degree 4. There are also some related generalized problems (for example, see [19,23], etc.). The Rankin–Selberg L-function $L(s, \mathrm{sym}^2 f \otimes f)$ defined in (12) with $i = 2$, $j = 1$ is closely related to the triple product L-function $L(s, f \otimes f \otimes f)$. Let $\lambda_{\mathrm{sym}^2 f \otimes f}(n)$ be the n-th coefficient of $L(s, \mathrm{sym}^2 f \otimes f)$ in its Dirichlet series expansion in the region of absolute convergence. In the same paper [24], Lü and Sankaranarayanan also proved that

$$\sum_{n \le x} \lambda_{\mathrm{sym}^2 f \otimes f}(n) \ll_{f,\varepsilon} x^{2/3+\varepsilon}, \tag{7}$$

$$\sum_{n \le x} \lambda_{\mathrm{sym}^2 f \otimes f}(n)^2 = x Q(\log x) + O_{f,\varepsilon}(x^{17/18+\varepsilon}), \tag{8}$$

where $Q(t)$ is a polynomial of degree 1.

In this paper, we can refine the above results (6) and (8) by establishing the following results.

Theorem 1. *For any $\varepsilon > 0$, with the above notations, we have*

$$\sum_{n \le x} \lambda_{f \otimes f \otimes f}(n)^2 = x P(\log x) + O_{f,\varepsilon}(x^{3271/3391+\varepsilon}),$$

where $P(t)$ is a polynomial in t of degree 4.

Theorem 2. *For any $\varepsilon > 0$, with the above notations, we have*

$$\sum_{n \le x} \lambda_{\mathrm{sym}^2 f \otimes f}(n)^2 = x Q(\log x) + O_{f,\varepsilon}(x^{923/983+\varepsilon}),$$

where $Q(t)$ is a polynomial of degree 1.

Remark 3. Note that here the primitive cusp form f is of level 1. We can not take higher level for f in Theorems 1 and 2 due to a lack of automorphy of the symmetric power lift of the general f with level larger than 1. However, if we add the condition that the cuspidal automorphic representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to f is non-CM, Theorems 1 and 2 would also follow for the primitive cusp form f with higher level, due to the breakthrough of Newton and Thorne [28, Theorem A].

For comparison, we have

$$175/181 = 0.966\ldots > 3271/3391 = 0.964\ldots,$$
$$17/18 = 0.94\ldots > 923/983 = 0.93\ldots.$$

The improvement benefits from the following aspects. On one hand, the new decompositions (see the following Lemmas 5 and 6) of the corresponding L-functions play important roles in the improvement. On the other hand, the improvement also derives from the recent new results on subconvexity bounds of the Riemann zeta function and the symmetric square L-function. For Theorem 2, a different shift of the line of the corresponding integration to [24] is also crucial to the improvement.

Let X be a random variable defined on a countable sample space Ω. We denote by $\sigma^2(X)$ and $\mathrm{E}(X)$ the variance and the expectation of X, respectively. Then according to Theorems

1 and 2, (5), (6) and the variance formula [2, Theorem 3],

$$\sigma^2(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2,$$

we can get the following corollary easily.

COROLLARY 4

Under previous notations, we have

$$\begin{aligned}\sigma^2(\lambda_{f\otimes f\otimes f}(n))_{1\le n\le x} &= \mathrm{E}(\lambda_{f\otimes f\otimes f}(n)^2) - \mathrm{E}(\lambda_{f\otimes f\otimes f}(n))^2\\ &= P(\log x) + O_{f,\varepsilon}(x^{-120/3391+\varepsilon}),\\ \sigma^2(\lambda_{\mathrm{sym}^2 f\otimes f}(n))_{1\le n\le x} &= \mathrm{E}(\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2) - \mathrm{E}(\lambda_{\mathrm{sym}^2 f\otimes f}(n))^2\\ &= Q(\log x) + O_{f,\varepsilon}(x^{-60/983+\varepsilon}).\end{aligned}$$

In Section 2, we give some preliminary lemmas. In Sections 3 and 4, we complete the proofs of Theorems 1 and 2, respectively. And throughout the paper, we denote by ε a sufficiently small positive constant, whose value may not be necessarily the same in all occurrences.

2. Preliminary lemmas

This section is devoted to establish and recall some preliminary results for the proofs of Theorems 1 and 2. We first introduce some specific L-functions.

For each $f \in H_k^*$, the triple product L-function $L(s, f\otimes f\otimes f)$ is defined by

$$\begin{aligned}L(s, f\otimes f\otimes f) &= \prod_p \left(1-\frac{\alpha_p^3}{p^s}\right)^{-1}\left(1-\frac{\alpha_p}{p^s}\right)^{-3}\left(1-\frac{\beta_p}{p^s}\right)^{-3}\left(1-\frac{\beta_p^3}{p^s}\right)^{-1}\\ &= \sum_{n=1}^{\infty}\frac{\lambda_{f\otimes f\otimes f}(n)}{n^s}\end{aligned} \tag{9}$$

for $\Re\mathrm{e}(s) > 1$. We can define the j-th symmetric power L-function $L(s, \mathrm{sym}^j f)$ attached to f as

$$L(s, \mathrm{sym}^j f) = \prod_p \prod_{u=0}^{j}\left(1-\frac{\alpha_p^{j-u}\beta_p^u}{p^s}\right)^{-1} \tag{10}$$

for $\Re\mathrm{e}(s) > 1$. It can also be represented as the following Dirichlet series, for $\Re\mathrm{e}(s) > 1$,

$$L(s, \mathrm{sym}^j f) = \sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p\left(1+\sum_{m\ge 1}\frac{\lambda_{\mathrm{sym}^j f}(p^m)}{p^{ms}}\right). \tag{11}$$

It is well-known that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function, and

$$L(s, \mathrm{sym}^0 f) = \zeta(s), \quad L(s, \mathrm{sym}^1 f) = L(s, f).$$

The Rankin–Selberg L-function $L(s, \mathrm{sym}^i f \otimes \mathrm{sym}^j f)$ attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ can be defined as, for $\Re(s) > 1$,

$$\begin{aligned} L(s, \mathrm{sym}^i f \otimes \mathrm{sym}^j f) &= \prod_p \prod_{u=0}^{i} \prod_{v=0}^{j} \left(1 - \frac{\alpha_p^{i-u} \beta_p^u \alpha_p^{j-v} \beta_p^v}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \otimes \mathrm{sym}^j f}(n)}{n^s}. \end{aligned} \tag{12}$$

In the recent result of Newton and Thorne [27], they showed that for all $j \in \mathbb{N}^+$, $\mathrm{sym}^j f$ is an automorphic cuspidal representation of $GL(j+1)$. Hence we can establish the following Lemmas 5 and 6, which gives new decompositions of the corresponding L-functions and they play important roles in the proofs of Theorems 1 and 2.

Lemma 5. *For* $\Re(s) > 1$, *let*

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f \otimes f \otimes f}(n)^2}{n^s}.$$

Then we have

$$L(s) = \zeta(s)^5 L(s, \mathrm{sym}^2 f)^9 L(s, \mathrm{sym}^4 f)^5 L(s, \mathrm{sym}^6 f) U(s),$$

where the function $U(s)$ *is a Dirichlet series absolutely convergent in* $\Re(s) > 1/2$ *and* $U(s) \neq 0$ *for* $\Re(s) = 1$.

Proof. Note that the coefficient $\lambda_{f \otimes f \otimes f}(n)^2$ is multiplicative and has the trivial upper bound $O(n^{\varepsilon})$. Thus for $\Re(s) > 1$, we have

$$L(s) = \prod_p \left(1 + \sum_{m \geq 1} \frac{\lambda_{f \otimes f \otimes f}(p^m)^2}{p^{ms}}\right).$$

The corresponding coefficients of the term p^{-s} determine the analytic properties of $L(s)$ in the half plane $\Re(s) > 1/2$. From [24, Lemma 2.1], we have

$$\lambda_{f \otimes f \otimes f}(p) = 2\lambda_f(p) + \lambda_{\mathrm{sym}^3 f}(p)$$

Thus from (10), (11) and (12) we have

$$\begin{aligned} \lambda_{f \otimes f \otimes f}(p)^2 &= 4\lambda_f(p)^2 + 4\lambda_f(p)\lambda_{\mathrm{sym}^3 f}(p) + \lambda_{\mathrm{sym}^3 f}(p)^2 \\ &= 4\lambda_{f \otimes f}(p) + 4\lambda_{\mathrm{sym}^3 f \otimes f}(p) + \lambda_{\mathrm{sym}^3 f \otimes \mathrm{sym}^3 f}(p). \end{aligned}$$

Moreover, we can deduce that

$$\begin{aligned} \lambda_{f \otimes f \otimes f}(p)^2 &= 4(1 + \lambda_{\mathrm{sym}^2 f}(p)) \\ &\quad + 4(\lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p)) \\ &\quad + (1 + \lambda_{\mathrm{sym}^2 f}(p) + \lambda_{\mathrm{sym}^4 f}(p) + \lambda_{\mathrm{sym}^6 f}(p)) \\ &= 5 + 9\lambda_{\mathrm{sym}^2 f}(p) + 5\lambda_{\mathrm{sym}^4 f}(p) + \lambda_{\mathrm{sym}^6 f}(p). \end{aligned}$$

Therefore for $\Re\mathrm{e}(s) > 1$, we can write

$$\zeta^5(s)L^9(s, \mathrm{sym}^2 f)L^5(s, \mathrm{sym}^4 f)L(s, \mathrm{sym}^6 f)$$

as the Euler product

$$\prod_p \left(1 + \sum_{m=1}^{\infty} \frac{c(p^m)}{p^{ms}}\right)$$

with $c(p) = \lambda_{f\otimes f\otimes f}(p)^2$. Since $\lambda_{f\otimes f\otimes f}(n)^2$ is multiplicative, we can get

$$\begin{aligned} L(s) &= \zeta^5(s)L^9(s, \mathrm{sym}^2 f)L^5(s, \mathrm{sym}^4 f)L(s, \mathrm{sym}^6 f) \\ &\quad \times \prod_p \left(1 + \frac{\lambda^8_{\mathrm{sym}^2 f}(p) - c(p^2)}{p^{2s}} + \cdots\right) \\ &:= \zeta^5(s)L^9(s, \mathrm{sym}^2 f)L^5(s, \mathrm{sym}^4 f)L(s, \mathrm{sym}^6 f)U(s). \end{aligned}$$

The absolute convergence of $U(s)$ for the half plane $\Re\mathrm{e}(s) > 1/2$ can be deduced easily by the Deligne inequality (2). Thus we complete the proof of this lemma. □

Lemma 6. *For* $\Re\mathrm{e}(s) > 1$, *let*

$$D(s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2}{n^s}.$$

Then we have

$$D(s) = \zeta(s)^2 L(s, \mathrm{sym}^2 f)^4 L(s, \mathrm{sym}^4 f)^3 L(s, \mathrm{sym}^6 f)V(s),$$

where the function $V(s)$ *is a Dirichlet series absolutely convergent in* $\Re\mathrm{e}(s) > 1/2$ *and* $V(s) \neq 0$ *for* $\Re\mathrm{e}(s) = 1$.

Proof. The proof of this lemma is similar to that of Lemma 5. Thus we just sketch its proof. Since the coefficient $\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2$ is multiplicative and satisfies the trivial upper bound $O(n^{\varepsilon})$, we have that, for $\Re\mathrm{e}(s) > 1$,

$$D(s) = \prod_p \left(1 + \sum_{m\geq 1} \frac{\lambda_{\mathrm{sym}^2 f\otimes f}(p^m)^2}{p^{ms}}\right).$$

In the half plane $\Re\mathrm{e}(s) > 1/2$, the corresponding coefficients of the term p^{-s} determine the analytic properties of $D(s)$. By [24, (6.1)], we have

$$\lambda_{\mathrm{sym}^2 f\otimes f}(p) = \lambda_f(p) + \lambda_{\mathrm{sym}^3 f}(p)$$

Thus from (10), (11) and (12) we have

$$\begin{aligned} \lambda_{\mathrm{sym}^2 f\otimes f}(p)^2 &= \lambda_f(p)^2 + 2\lambda_f(p)\lambda_{\mathrm{sym}^3 f}(p) + \lambda_{\mathrm{sym}^3 f}(p)^2 \\ &= \lambda_{f\otimes f}(p) + 2\lambda_{\mathrm{sym}^3 f\otimes f}(p) + \lambda_{\mathrm{sym}^3 f\otimes \mathrm{sym}^3 f}(p). \end{aligned}$$

Further, it is easy to deduce that

$$\begin{aligned}\lambda_{f\otimes f\otimes f}(p)^2 &= (1+\lambda_{\mathrm{sym}^2 f}(p)) \\ &\quad +2(\lambda_{\mathrm{sym}^2 f}(p)+\lambda_{\mathrm{sym}^4 f}(p)) \\ &\quad +(1+\lambda_{\mathrm{sym}^2 f}(p)+\lambda_{\mathrm{sym}^4 f}(p)+\lambda_{\mathrm{sym}^6 f}(p)) \\ &= 2+4\lambda_{\mathrm{sym}^2 f}(p)+3\lambda_{\mathrm{sym}^4 f}(p)+\lambda_{\mathrm{sym}^6 f}(p).\end{aligned}$$

Then this lemma follows from standard arguments. □

For $j = 1, 2, 3, 4$, the works of Gelbert and Jacquet [6], Kim [12] and Kim and Shahidi [13,14] show that $L(s, \mathrm{sym}^j f)$, $j = 1, 2, 3, 4$ are general L-functions in the sense of Perelli [29]. The recent result of Newton and Thorne [27] implies that $\mathrm{sym}^j f$, $j \in \mathbb{N}^+$ is an automorphic cuspidal representation of $GL(j + 1)$. This means that $L(s, \mathrm{sym}^j f)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $j + 1$. Thus for $j \in \mathbb{N}^+$, $L(s, \mathrm{sym}^j f)$ are also general L-functions in the sense of Perelli [29]. For general L-functions, we have the following averaged and individual convexity bounds, which is an immediate result of [29, Theorem 4] and [25, Proposition 1].

Lemma 7. *Assume that $\mathfrak{L}(s)$ is a general L-function of degree m. Then for any $\varepsilon > 0$, we have*

$$\mathfrak{L}(\sigma+it) \ll (1+|t|)^{(m/2)(1-\sigma)+\varepsilon} \tag{13}$$

uniformly for $1/2 \le \sigma \le 1+\varepsilon$ and $|t| \ge 1$; and

$$\int_T^{2T} |\mathfrak{L}(\sigma+it)|^2 dt \ll T^{\max\{m(1-\sigma),1\}+\varepsilon} \tag{14}$$

uniformly for $1/2 \le \sigma \le 1$ and $T > 1$.

To prove Theorems 1 and 2, we also need the following individual and average subconvexity bounds.

Lemma 8. *For any $\varepsilon > 0$, we have*

$$\int_0^T |\zeta\,(5/7+it)|^{12}\, dt \ll_\varepsilon T^{1+\varepsilon}$$

uniformly for $T \ge 1$, and

$$\zeta(\sigma+it) \ll_\varepsilon (1+|t|)^{(13/42)(1-\sigma)+\varepsilon}$$

uniformly for $1/2 \le \sigma \le 1+\varepsilon$ and $|t| \ge 1$.

Proof. See [10, Theorem 8.4 and (8.87)] and [1, Theorem 5], respectively. □

Lemma 9. *For any* $\varepsilon > 0$, *we have*

$$L(\sigma + it, \mathrm{sym}^2 f) \ll_{f,\varepsilon} (1 + |t|)^{(6/5)(1-\sigma)+\varepsilon}$$

uniformly for $1/2 \le \sigma \le 1 + \varepsilon$ *and* $|t| \ge 1$.

Proof. See [18, Corollary 1.2]. □

Lemma 10. *For* $U \ge U_0$, *where* U_0 *is sufficiently large, there exists a* $T^* \in (U, 2U)$ *such that*

$$\max_{\sigma \ge 1/2} \left|\zeta\left(\sigma + iT^*\right)\right| \le \exp\left(C(\log\log U)^2\right).$$

Proof. See [30, Lemma 2]. □

3. Proof of Theorem 1

In this section, we shall give the proof of Theorem 1. Recall that for $\Re\mathrm{e}(s) > 1$,

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda_{f\otimes f\otimes f}(n)^2}{n^s}.$$

By Lemma 5, we have

$$L(s) = \zeta(s)^5 L(s, \mathrm{sym}^2 f)^9 L(s, \mathrm{sym}^4 f)^5 L(s, \mathrm{sym}^6 f) U(s),$$

where the function $U(s)$ is a Dirichlet series absolutely convergent in $\Re\mathrm{e}(s) > 1/2$.

Applying the Perron's formula (see [11, Proposition 5.54]), we have

$$\sum_{n \le x} \lambda_{f\otimes f\otimes f}(n)^2 = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} L(s)\frac{x^s}{s}ds + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{15}$$

where $b = 1 + \varepsilon$ and $1 \le T \le x$ is a parameter which will be determined later.

Now we shift the line of integration in (15) to the line $\Re\mathrm{e}(s) = 5/7$. Note that $s = 1$ is the only pole of $L(s)$ of order 5 in the rectangle $5/7 \le \sigma \le 1 + \varepsilon$ and $|t| \le T$. The residue of $L(s)x^s/s$ at $s = 1$ is equal to $xP(\log x)$, where $P(t)$ is a polynomial in t of degree 4. The absolute convergence of $U(s)$ ensures that $U(s) \ll 1$ for $\Re\mathrm{e}(s) > 1/2$. From Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n \le x} \lambda_{f\otimes f\otimes f}(n)^2 &= xP(\log x) + \frac{1}{2\pi i}\left\{\int_{5/7-iT}^{5/7+iT} + \int_{5/7+iT}^{b+iT} + \int_{b-iT}^{5/7-iT}\right\} L(s)\frac{x^s}{s}ds \\ &\quad + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &= xP(\log x) + I_1 + I_2 + I_3 + O_{f,\varepsilon}\left(\frac{x^{1+\varepsilon}}{T}\right). \end{aligned} \tag{16}$$

For I_1, by Hölder's inequality, we have

$$I_1 \ll_{f,\varepsilon} x^{5/7+\varepsilon} \log T \max_{1 \le T_1 \le T} \max_{T_1 \le t \le 2T_1} T_1^{-1} |\tilde{L}(5/7 + it)| J_1(T_1)^{5/12} J_2(T_1)^{1/12} J_3(T_1)^{1/2},$$

where

$$\tilde{L}(5/7+it) = L(5/7+it, \mathrm{sym}^2 f)^9 L(5/7+it, \mathrm{sym}^4 f)^{29/6},$$
$$J_1(T_1) = \int_{T_1}^{2T_1} |\zeta(5/7+it)|^{12} dt,$$
$$J_2(T_1) = \int_{T_1}^{2T_1} \left|L\left(5/7+it, \mathrm{sym}^4 f\right)\right|^2 dt$$

and

$$J_3(T_1) = \int_{T_1}^{2T_1} \left|L\left(5/7+it, \mathrm{sym}^6 f\right)\right|^2 dt.$$

Then with the help of Lemmas 7, 8 and 9, we have

$$J_1(T_1) \ll_\varepsilon T_1^{1+\varepsilon}, \quad J_2(T_1) \ll_{f,\varepsilon} T_1^{10/7+\varepsilon}, \quad J_3(T_1) \ll_{f,\varepsilon} T_1^{2+\varepsilon},$$

and

$$|\tilde{L}(5/7+it)| \ll_{f,\varepsilon} (1+|t|)^{6/5\times(1-5/7)\times 9+5/2\times(1-5/7)\times 29/6+\varepsilon} \ll_{f,\varepsilon} (1+|t|)^{1373/210+\varepsilon}.$$

Therefore we obtain

$$I_1 \ll_{f,\varepsilon} x^{5/7+\varepsilon} \max_{1\le T_1\le T} T_1^{1373/210+5/12+10/7\times 1/12+2\times 1/2-1+\varepsilon} \ll_{f,\varepsilon} x^{5/7+\varepsilon} T^{2971/420+\varepsilon}. \tag{17}$$

For the integrals I_2 and I_3 over the horizontal segments, Lemmas 7, 9 and 10 with our choice of T imply that

$$I_2 + I_3 \ll_{f,\varepsilon} \max_{5/7\le\sigma\le b} x^\sigma T^{(10\varepsilon+6/5\times 9+5/2\times 5+7/2)(1-\sigma)-1+\varepsilon} \ll_{f,\varepsilon} \max_{5/7\le\sigma\le b} \left(\frac{x}{T^{134/5}}\right) T^{129/5+\varepsilon} \ll_{f,\varepsilon} x^{1+\varepsilon} T^{-1} + x^{5/7+\varepsilon} T^{233/35+\varepsilon}. \tag{18}$$

Inserting (17) and (18) into (16), we have

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n)^2 = xP(\log x) + O_{f,\varepsilon}(x^{1+\varepsilon}T^{-1} + x^{5/7+\varepsilon}T^{2971/420+\varepsilon}). \tag{19}$$

On taking $T = x^{120/3391}$, (19) turns into

$$\sum_{n\le x} \lambda_{f\otimes f\otimes f}(n)^2 = xP(\log x) + O_{f,\varepsilon}(x^{3271/3391+\varepsilon}).$$

Thus we complete the proof of Theorem 1.

4. Proof of Theorem 2

Since the proof of Theorem 2 is similar to that of Theorem 1, in this section, we sketch its proof. Unlike [24], we shift the contour of the corresponding integration to a different line.

Recalling Lemma 6 and applying Perron's formula again, we get

$$\sum_{n\leq x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}D(s)\frac{x^s}{s}ds+O_{f,\varepsilon}(x^{1+\varepsilon}T^{-1}), \tag{20}$$

where $b=1+\varepsilon$ and $1\leq T\leq x$ is a parameter to be chosen later. Shifting the contour of the integration in (20) to the line $\Re\mathrm{e}(s)=5/7$, we can obtain

$$\sum_{n\leq x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2=xQ(\log x)+\frac{1}{2\pi i}\int_{\mathfrak{H}}D(s)\frac{x^s}{s}ds+O_{f,\varepsilon}(x^{1+\varepsilon}T^{-1}), \tag{21}$$

where $\mathfrak{h}$ is the contour joining the points $b+iT, 5/7+iT, 5/7-iT, b-iT$ with straight lines, and $Q(t)$ is a polynomial in t of degree 1. The main term $xQ(\log x)$ derives from the residue of $D(s)x^s/s$ at the pole $s=1$ of order 2. The absolutely convergence of $V(s)$ guarantees that $V(s)\ll 1$ for $\Re\mathrm{e}(s)>1/2$. Thus we can rewrite (21) as

$$\sum_{n\leq x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2=xQ(\log x)+O_{f,\varepsilon}(\tilde{I}_1+\tilde{I}_2+x^{1+\varepsilon}T^{-1}), \tag{22}$$

where

$$\tilde{I}_1=x^{5/7+\varepsilon}\int_{-T}^{T}|D\,(5/7+it)|\,t^{-1}dt,$$

$$\tilde{I}_2=T^{-1}\int_{5/7}^{1+\varepsilon}x^{\sigma}\,|D\,(\sigma+iT)|\,d\sigma.$$

For I_1, by Hölder's inequality, Lemmas 7, 8 and 9 we have

$$\begin{aligned}\tilde{I}_1&\ll_{f,\varepsilon}x^{5/7+\varepsilon}\log T\max_{1\leq T_1\leq T}T_1^{(2/7)(6/5\times4+5/2\times7/3)+\varepsilon}J_1^{1/6}J_2^{1/3}J_3^{1/2}T_1^{-1}\\&\ll_{f,\varepsilon}x^{5/7+\varepsilon}\max_{1\leq T_1\leq T}T_1^{214/105+\varepsilon}J_1^{1/6}J_2^{1/3}J_3^{1/2},\end{aligned}$$

where

$$J_1=\int_{T_1}^{2T_1}|\zeta(5/7+it)|^{12}dt,$$

$$J_2=\int_{T_1}^{2T_1}|L(5/7+it,\mathrm{sym}^4 f)|^2dt,$$

$$J_3=\int_{T_1}^{2T_1}|L(5/7+it,\mathrm{sym}^6 f)|^2dt.$$

From Lemmas 7 and 8 it follows that

$$J_1\ll_{f,\varepsilon}T_1^{1+\varepsilon},\quad J_2\ll_{f,\varepsilon}T_1^{10/7+\varepsilon},\quad J_3\ll_{f,\varepsilon}T_1^{2+\varepsilon}.$$

Thus we get

$$\begin{aligned}\tilde{I}_1&\ll_{f,\varepsilon}x^{5/7+\varepsilon}\max_{1\leq T_1\leq T}T_1^{214/105+1/6+10/7\times1/3+2\times1/2+\varepsilon}\\&\ll_{f,\varepsilon}x^{5/7+\varepsilon}T^{773/210+\varepsilon}.\end{aligned} \tag{23}$$

For $\tilde{I}_2$, by Lemmas 7, 9 and 10 we have

$$\begin{aligned}\tilde{I}_2&\ll_{f,\varepsilon}\max_{5/7\leq\sigma\leq1+\varepsilon}x^{\sigma}T^{(10\varepsilon+6/5\times4+5/2\times3+7/2)(1-\sigma)-1+\varepsilon}\\&\ll_{f,\varepsilon}\max_{5/7\leq\sigma\leq1+\varepsilon}\left(\frac{x}{T^{79/5}}\right)^{\sigma}T^{74/5+\varepsilon}\\&\ll_{f,\varepsilon}x^{1+\varepsilon}T^{-1}+x^{5/7+\varepsilon}T^{123/35+\varepsilon}.\end{aligned} \tag{24}$$

Inserting (23) and (24) into (22), we get

$$\sum_{n\leq x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2 = xQ(\log x) + O_{f,\varepsilon}(x^{5/7+\varepsilon}T^{773/210+\varepsilon} + x^{1+\varepsilon}T^{-1}). \quad (25)$$

Taking $T = x^{60/983}$ in (25), we obtain

$$\sum_{n\leq x}\lambda_{\mathrm{sym}^2 f\otimes f}(n)^2 = xQ(\log x) + O_{f,\varepsilon}(x^{923/983+\varepsilon}),$$

which completes the proof of Theorem 2.

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