Proc. Indian Acad. Sci. (Math. Sci.) (2023) 133:7
https://doi.org/10.1007/s12044-023-00724-8

Some properties of forced maps with a non-autonomous dynamical system as the base

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MS received 6 May 2022; revised 13 September 2022; accepted 25 October 2022

Abstract. For a class of forced non-autonomous dynamical systems, we prove the existence of invariant graphs and classify invariant graphs by supposing that the fibre maps have negative Schwarzian derivative. Also, we investigate the existence of attracting basins that are intermingled.

Keywords. Invariant graph; negative Schwarzian derivative; intermingled basins.

2010 Mathematics Subject Classification. 37B20, 37B55, 37C60, 37C70.

1. Introduction

This work is devoted to introducing a mathematical framework for the investigation of dynamical systems forced by a non-autonomous dynamical system and explaining some of its properties. Non-autonomous dynamical systems yield very flexible models than autonomous cases for the study and description of real world processes and may be used to describe the evolution of a wider class of phenomena, including systems which are forced or driven [5,8,10–12]. A forced dynamical system by a non-autonomous dynamical system is modelled, as a skew product

$$\begin{aligned} T : X \times Y &\to X \times Y \\ (x, y) &\mapsto (\theta(x), g(x, y)), \end{aligned}$$

where X and Y are compact metric spaces, $g : X \times Y \to Y$, and the dynamics of the forcing process are described by the base transformation $\theta : X \to X$, which is defined by iterations of a non-autonomous dynamical system on an arbitrary point $x \in X$ and we will describe it in Section 2.

Quasiperiodically forced systems with one dimensional base are studied by several authors [2,6,9,14]. In [14], Sturman and Stark proved that a compact invariant set of a $C^{1+\alpha}$ quasiperiodically forced map, whose maximal Liapunov exponent is strictly negative, is the union of a finite number of smooth periodic curves (invariant graphs). In Section 2, we introduce invariant graphs for the system which are the most simple invariant objects. Indeed invariant graphs occur generically as boundary subsets of invariant compact sets. Here, we prove that a compact invariant set $K = X \times Y'$ with $Y' \subset Y$ of a dynamical system

Published online: 03 April 2023

forced by a non-autonomous dynamical system, whose maximal Liapunov exponent is strictly negative, is the union of a finite number of invariant graphs.

The classification of quasiperiodically forced interval maps for the existence invariant graphs is done by Jager [6], Keller [9] and Bezhaeva and Oseledets [2]. For a system of quasiperiodically forced interval maps, in which all the fibre maps have strictly negative Schwarzian derivative, Jager [6] showed that there are at most three invariant graphs. In this work, Section 3 is devoted to this kind of classification.

In Section 4, we focus on attracting basins which are intermingled. A simple example of a skew product map on $S^1 \times I$ such that the two boundary circles are measure theoretic attractors whose attracting basins are intermingled has been described in [7]. In a slightly more generality, by emphasizing the importance of negative Schwarzian derivative, the result of intermingled basins for skew product map on cylinder $S^1 \times [0, 1]$, is obtained in [3]. Here, our results of intermingled basins is devoted to the skew product map on $X \times [0, 1]$ forced by a non-autonomous dynamical system. Moreover we provide an example which fits in our situation.

2. Invariant subsets and invariant graphs

Following [10], a *non-autonomous dynamical system*, or an NDS for short, is a pair $(X_{1,\infty}, f_{1,\infty})$, where $X_{1,\infty} = (X_n)_{n=1}^{\infty}$ is a sequence of sets and $f_{1,\infty} = (f_n)_{n=1}^{\infty}$ is a sequence of maps $f_n : X_n \to X_{n+1}$. If all the sets X_n are compact metric spaces and all the f_n are continuous, we say that $(X_{1,\infty}, f_{1,\infty})$ is a *topological NDS*. Whenever the sets X_n are smooth manifolds and the f_n are C^r maps, we speak of a C^r NDS. Here, we assume that all the sets X_n are equal to smooth manifold X, all the maps $f_n : X \to X$ are $C^{1+\alpha}$, d is the metric defined on X, and we abbreviate $(X_{1,\infty}, f_{1,\infty})$ by $(X, f_{1,\infty})$. We define

$$f_k^n := f_{k+n-1} \circ \cdots \circ f_k \text{ for } k \in \mathbb{N},\ n \in \mathbb{N}, \text{ and } f_k^0 := id_X. \tag{1}$$

We also take $f_k^{-n}(A) := (f_k^n)^{-1}(A)$ for every subset $A \subset X$. The *trajectory* of a point $x \in X$ is the sequence $\{f_1^n(x)\}_{n=0}^{\infty}$.

An NDS $(X, f_{1,\infty})$ given by a sequence of local diffeomorphisms f_n is *expanding* if for every $n \in \mathbb{N}$, the mapping f_n is an expanding map with *expansion factor* $\sigma_n > 1$. This means that

$$\|Df_n(x)v\| \geq \sigma_n \|v\| \tag{2}$$

for every $x \in X$ and every vector v tangent to X at the point x.

Let f be an expanding local diffeomorphism of class C^r, $r \geq 1$ on X with expansion factor $\sigma > 1$. Then, there exists $\rho > 0$ such that, for any pre-image x of any point $y \in X$, there exists a map $h : B(y, \rho) \to X$ of class C^r such that $f \circ h = id$, $h(y) = x$ and

$$d(h(y_1), h(y_2)) \leq \sigma^{-1} d(y_1, y_2) \text{ for every } y_1, y_2 \in B(y, \rho). \tag{3}$$

Mappings h as in the statement are called *inverse branches* of the local diffeomorphism f; moreover, the conclusion (3) implies that the inverse branches are contractions, with uniform contraction rate. The constant ρ is called *injectivity constant*. An NDS (X_∞, f_∞) is called *uniformly expanding* if it satisfies the following conditions:

(1) $(X, f_{1,\infty})$ is an expanding NDS given by a sequence of expanding maps f_n with expansion factor σ_n and injectivity constant ρ_n;
(2) there exists a uniform bound $\sigma > 1$ on expansion factors σ_n, i.e., $\sigma_n \geq \sigma$ for each $n \in \mathbb{N}$;

(3) there exists $\rho > 0$ such that $\rho_n \geq \rho$, for each $n \in \mathbb{N}$.

An NDS $(X, f_{1,\infty})$ is called *backward minimal* if for every $x \in X$, the set $\{f_1^{-n}(x) : n \in \mathbb{N}\}$ is dense in X.

Lemma 1. *Let $(X, f_{1,\infty})$ be a uniformly expanding NDS. Then for each open ball B of radius ε around x, there exists a positive integer n so that $f_1^n(B) = X$ and therefore it is backward minimal.*

Proof. Given any $n \geq 1$, suppose that $f_1^n(B)$ does not cover the whole manifold. Then, there exists some curve γ connecting $f_1^n(x)$ to a point $y \in X \backslash f_1^n(B)$, and that curve may be taken with a length smaller than $\mathrm{diam}(X) + 1$. Lifting γ by the local diffeomorphism f_1^n, we obtain a curve γ_n connecting x to some point $y_n \in X \backslash B$. Then, $\varepsilon \leq \mathrm{length}(\gamma_n) \leq \sigma^{-n}(\mathrm{diam}(X) + 1)$ provides a lower bound on the length γ_n on the possible value of n. This contradicts the fact that $\lim_{n\to+\infty} \sigma^{-n} = 0$.

Hence, $f_1^n(B) = X$ for every n sufficiently large, as claimed. □

In the following, we suppose that $(X, f_{1,\infty})$ is a *uniformly expanding* NDS, given by a sequence of expanding $C^{1+\alpha}$ local diffeomorphisms f_n on a compact connected Riemannian manifold X. Take the mapping $\theta : X \to X$ which is defined as follows:

$$\theta(x) = f_1(x) \quad \text{and} \quad \theta^n(x) = f_1^n(x). \tag{4}$$

Suppose (Y, d') is a compact manifold and $g : X \times Y \to Y$ is $C^{1+\eta}$. For any $x \in X$, by g_x, we mean the mapping $g(x, .) : Y \to Y$. Now we define the skew product

$$\begin{aligned} T : X \times Y &\to X \times Y \\ (x, y) &\mapsto (\theta(x), g(x, y)). \end{aligned} \tag{5}$$

For any $n \geq 2$, we define $T^n(x, y) := (\theta^n(x), g^n(x, y))$, where $g^n(x, y) = g(\theta^{n-1}(x), g^{n-1}(x, y))$. The natural projection from $X \times Y$ to X and to Y is denoted by π_1 and π_2, respectively.

The mapping $\varphi : X \to Y$ is called an invariant graph if $T(x, \varphi(x)) = (\theta(x), \varphi(\theta(x)))$, indeed for all $x \in X$ and $n \in \mathbb{N}$, we have $g^n(x, \varphi(x)) = \varphi(\theta^n(x))$. Also, the mapping φ is called a periodic graph if there exists some $q \in \mathbb{N}$ such that φ is an invariant graph for T^q, it means that for all $x \in X$ and $n \in \mathbb{N}$, we have $g^{nq}(x, \varphi(x)) = \varphi(\theta^{nq}(x))$. We call q period of graph φ. By Φ, we mean the graph of φ. We write the Lebesgue measure on X by Leb, and the Lebesgue measure on $X \times Y$ by m.

Remark 2. Notice Leb is an ergodic measure for (X, f_1), i.e., for any measurable set $A \subseteq X$ with $f_1(A) \subseteq A$, we have $\mathrm{Leb}(A) = 0$ or 1, see ([1], Corollary 2.13).

DEFINITION 3

Let T be the skew product of the form (5) and $(x, y) \in X \times Y$, if the limit

$$\lambda(x, y) := \lim_{n\to\infty} \frac{1}{n} \log \|D_y g^n(x, y)\|$$

exists, it is called the normal Liapunov exponent in (x, y). Also, for an invariant graph φ with $\log \|D_y g(x, \varphi(x))\| \in \mathcal{L}^1_m$, its Liapunov exponent is defined as

$$\lambda(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \|D_y g^n(x, \varphi(x))\|.$$

DEFINITION 4

Let M be a metric space and $\mathcal{B}$ is a σ-algebra on M. A measure ν on $\mathcal{B}$ is half-invariant by measurable transformation $F : M \to M$ if

$$\forall B \in \mathcal{B} \;\; \nu(F^{-1}(B)) \leq \nu(B).$$

We observe that the Lebesgue measure on X, Leb, is a half-invariant measure by f_n, for any $n \in \mathbb{N}$. So it has been easily seen that Leb is a half-invariant measure by θ^n. The following theorem is a generalization of Birkhoff's ergodic theorem to non-invariant measures.

Theorem 5 ([4], Theorem 1.2). *Let $(M, \mathcal{B}, \nu)$ be a measure space, $F : M \to M$ a measurable transformation, and assume that ν is a σ-finite measure half-invariant by F. Then, for any non-negative $g \in \mathcal{L}^1(M, \nu)$, we have the following*:

(1) *The limit $g_*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(F^j(x))$ exists for ν almost every point x*;
(2) *The function g_* is ν-integrable and F-invariant*;
(3) *If $\nu(M) < +\infty$, then $\int g_* d\nu = \int g d\nu$.*

Let $F : M \to M$ be a measurable map on a compact metrizable space M. We shall say that a sequence $\{\zeta_n\}$ of functions $\zeta_n : M \to \mathbb{R}$ is sub-additive if $\zeta_{n+m}(x) \leq \zeta_n(x)\zeta_m(F^n(x))$ for all $x \in M$. A sequence $\{\xi_n\}$ of non-negative functions $\xi_n : M \to \mathbb{R}^+$ is sub-multiplicative if $\xi_{n+m}(x) \leq \xi_n(x)\xi_m(F^n(x))$ for all $x \in M$.

By using this generalization of Birkhoff's ergodic theorem, we can obtain Kingman's subadditive ergodic theorem (see [13]) for half-invariant measures, straightforwardly.

Theorem 6. *Let $F : M \to M$ be a measurable map on a compact metrizable space M, ν an F-half-invariant measure and $\{\varphi_n\}$ an integrable sub-additive sequence. Then the limit*

$$\lim_{n\to\infty} \frac{1}{n} \varphi_n(x) = \bar{\varphi}(x)$$

exists for ν-almost every x.

The essential supremum φ_μ of $\bar{\varphi}$ with respect to ν is defined by

$$\varphi_\nu = \inf\{D : \bar{\varphi}(x) \leq D \text{ for } \mu \text{ almost every } x \in M\}.$$

Now by using Theorem 6, we have the following theorem, the details of its proof which are similar to ([14], Theorem 1.12), and so we omit it.

Theorem 7. *Suppose that $F : M \to M$ is a continuous map on a compact metrizable space M, and $\{\xi_n\}$ a sub-multiplicative sequence of continuous non-negative functions*

$\xi_n : M \to \mathbb{R}^+$. Let a be a constant such that $\varphi_\nu < a$ for every F-half-invariant measure ν, where $\varphi_n = \log \xi_n$ and φ_ν is as above. Then there exists some $\delta > 0$ and an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have for all $x \in M$,

$$\frac{1}{n}\varphi_n(x) \leq a - \delta.$$

Theorem 8. *Let $K = X \times Y' \subset X \times Y$ be a compact T-invariant set. Suppose that for any half-invariant measure μ supported on K, the essential supremum of the maximal normal Liapunov exponent, $\lambda_\mu := \inf\{A; \lambda(x, y) \leq A$ for μ a.e. $(x, y)\}$ is strictly negative. Then there exists some $\lambda < 0$ and a constant C such that*

$$\|D_y g_x^n\| \leq Ce^{\lambda n}$$

for all $(x, y) \in K$ and $n \in \mathbb{N}$.

Proof. By the chain rule, $\xi_n(x, y) = \| D_y g_x^n(y) \|$ is sub-multiplicative, and hence by Theorem 7, there exists a $\delta > 0$ and an $n_0 \in \mathbb{N}$ such that $\frac{1}{n}\varphi_n(x, y) < -\delta$ for all $n \geq n_0$, where $\varphi_n = \log \| D_y g_x^n(y) \|$. So $\| D_y g_x^n(y) \| \leq e^{-\delta n}$ for all $(x, y) \in K$ and all $n > n_0$. Let

$$c' = \max_{i=1,\ldots,n_0} \sup_{(x,y)\in K} \{e^{-\delta i} \| D_y g_x^i(y) \|\}.$$

Thus $\| D_y g_x^n(y) \| \leq c' e^{-\delta n}$ for all $(x, y) \in K$ and all $n \leq n_0$. It is enough to take $C = \max\{c', 1\}$.

□

Theorem 9. *Let $T : X \times Y \to X \times Y$ be a skew product defined as (5) and $K = X \times Y'$ be an invariant subset of $X \times Y$, in which Y' is a closed subset of Y. Also, suppose that there exist constants $\lambda < 0$ and $c > 0$ such that $\|D_y g_x^n\| \leq ce^{\lambda n}$, for all $n \in \mathbb{N}$ and $(x, y) \in K$. Then K is the union of a finite number of continuous periodic graphs.*

Remark 10. Let $\psi_1, \ldots, \psi_l$ be periodic graphs in Theorem 9 and $q_1, \ldots, q_l$ be their periods, respectively. Take $q = \text{lcm}(q_1, \ldots, q_l)$, and observe that $\psi_1, \ldots, \psi_l$ are invariant graphs of T^q. Without loss of generality, we can suppose that $q = 1$ and $\psi_1, \ldots, \psi_l$ are invariant graphs of T.

Now we express some preliminaries and lemmas which are necessary for the proof of Theorem 9. Denote that, without loss of generality, we can suppose that $c = 1$. We define some boxes as follows:

$B(y, \varepsilon) = \{y' \in Y; d(y, y') \leq \varepsilon\}$,
$\mathcal{B}(y, \varepsilon) = X \times B(y, \varepsilon)$ and
$\mathcal{B}(\varepsilon) = \bigcup_{y\in Y'} \mathcal{B}(y, \varepsilon)$.

We can suppose that there exists some $\varepsilon_1 > 0$ such that for all $(x, y) \in \mathcal{B}(\varepsilon_1)$, $\|D_y g_x\| \leq e^\lambda + \frac{1}{2}(1 - e^\lambda) < 1$. For the convenience of notation, we write $\gamma' = e^\lambda + \frac{1}{2}(1 - e^\lambda)$ and $\gamma = \gamma' + \frac{1}{2}(1 - \gamma')$, note that $e^\lambda < \gamma' < \gamma < 1$.

Since for all $x \in X$, g_x is $C^{1+\eta}$, and there exists some constant $\tilde{C}$ such that

$$\begin{aligned}\|D_y g_x - D_{y'} g_{x'}\| &\leq \|D_y g_x - D_y g_{x'}\| + \|D_y g' x - D_y g_{x'}\| \\ &\leq \tilde{C}(d(x,x')^\eta + d'(y,y')^\eta),\end{aligned}$$

for any $(x, y), (x', y') \in \mathcal{B}(\varepsilon_1)$.

Lemma 11. *$T(\mathcal{B}(\varepsilon_1)) \subseteq \mathcal{B}(\varepsilon_1)$ and for any $(x, y) \in \mathcal{B}(\varepsilon_1)$, we have $\|Dg_x^n\| \leq \gamma^n$.*

Proof. Let $(x, y) \in \mathcal{B}(y_0, \varepsilon_1)$ for some $y_0 \in Y'$. Since $d'(g_x(y), g_x(y_0)) \leq \gamma' d(y, y_0) \leq \gamma'\varepsilon_1 \leq \varepsilon_1$, clearly $T(x, y) = (\theta(x), g(x, y)) \in \mathcal{B}(\varepsilon_1)$.

The second part of the lemma is obtained by the chain rule, straightforwardly. □

By compactness of K, we can find $y_1, \ldots, y_s \in Y'$ such that $\mathcal{B}(\varepsilon_1) = \bigcup_{i=1}^s \mathcal{B}(y_i, \varepsilon_1)$ and since the sets $\mathcal{B}(y_i, \varepsilon_1)$ may overlap, we work with the connected components and denote them by $\tilde{B}_1, \ldots, \tilde{B}_r$. By Δ, we mean the minimal distance between any two of $\tilde{B}_i$ and note that the diameter of any $\tilde{B}_i$ in Y-direction is at most $2s\varepsilon_1$.

Remark 12. Consider any two points $(x, y), (x, y') \in T^n(\tilde{B}_i)$ for some $i \in \{1, \ldots, r\}$ and $n \in \mathbb{N}$. There are points $(\hat{x}, \hat{y}), (\hat{x}, \hat{y}') \in \tilde{B}_i$ such that $(x, y) = T^n(\hat{x}, \hat{y})$ and $(x, y') = T^n(\hat{x}, \hat{y}')$. So $d(y, y') = d(g^n(\hat{x}, \hat{y}), g^n(\hat{x}, \hat{y}')) \leq \|D_y g_{\hat{x}}^n\| d(\hat{y}, \hat{y}') \leq \gamma^n d(\hat{y}, \hat{y}') \leq \gamma^n 2s\varepsilon_1$.

Take $N \in \mathbb{N}$ so that for any $n \geq N$, $\gamma^n 2s\varepsilon_1 < \Delta$.

Lemma 13. *For every $n \geq N$ and any $l \in \{1, \ldots, r\}$, there exists a unique $l' \in \{1, \ldots, r\}$ such that $T^n(\tilde{B}_l) \cap \tilde{B}_{l'} \neq \emptyset$.*

Proof. Let $(x, y_i) \in \tilde{B}_l \cap K$, for some $i \in \{1, \ldots, s\}$. Since K is invariant there exists $l' \in \{1, \ldots, r\}$ such that $T^n(x, y_i) \in \tilde{B}_{l'}$. Therefore, $T^n(\tilde{B}_l) \cap \tilde{B}_{l'} \neq \emptyset$. Now suppose $(x, y) \in \tilde{B}_l$ is arbitrary. So $d(g^n(x, y), g^n(x, y_i)) \leq \gamma^n d(y, y_i) \leq 2s\varepsilon_1\gamma^n < \Delta$ and since Δ is the minimal Y-direction distance, the uniqueness is proved. □

DEFINITION 14

For any $n \geq N$, we define $\sigma_n : \{1, \ldots, r\} \to \{1, \ldots, r\}$ as $\sigma_n(l) = l'$, where l' is the unique number and $T^n(\tilde{B}_l) \cap \tilde{B}_{l'} \neq \emptyset$.

Lemma 15. *The mapping σ_n is one-to-one and onto.*

Proof. For any $l' \in \{1, \ldots, r\}$, $\tilde{B}_{l'} \cap K \neq \emptyset$ and since K is invariant and $K = T^n(K) \subset T^n(\tilde{B}_1) \cup \cdots \cup T^n(\tilde{B}_r)$, there exists some $l \in \{1, \ldots, r\}$ such that $T^n(\tilde{B}_l) \cap \tilde{B}_{l'} \neq \emptyset$, thus σ_n is onto. Clearly, this map is one-to-one. □

Now we are ready to prove Theorem 9. Define $W_l := \bigcap_{n \geq N} T^n(\tilde{B}_{\vartheta_n(l)})$, where ϑ_n is the inverse of σ_n. Since K is invariant, for any $n \geq N$, $K \cap \tilde{B}_l \subseteq T^n(\tilde{B}_1) \cup \ldots \cup T^n(\tilde{B}_r)$, but by Lemma 15, $\tilde{B}_l$ intersect just one of $T^n(\tilde{B}_i)$ for $i = 1, \ldots, r$, so $K \cap \tilde{B}_l \subset W_l$. Also, notice that for any $x \in X$, $\{x\} \times Y' \cap K \cap \tilde{B}_l \neq \emptyset$ and this intersection is exactly one point. Indeed, if $(x, y), (x, y') \in \{x\} \times Y' \cap K \cap \tilde{B}_l$, then $(x, y), (x, y') \in W_l$, by using Remark 12, and we have $d(y, y') \leq \gamma^n 2s\varepsilon_1$ for any $n \geq N$. Thus $y = y'$.

Now, we define the mapping $\psi_l : X \to Y$, where $(x, \psi_l(x))$ is the unique point of $\{x\} \times Y' \cap K \cap \tilde{B}_l$. Clearly, $K \subseteq \text{graph}\,\psi_1 \cup \cdots \cup \text{graph}\,\psi_r$. Let $\mathcal{C}(Y)$ be the set of compact subsets of Y. The mapping $\alpha : X \to \mathcal{C}(Y)$, in which $\alpha(x) = \{\psi_1(x), \ldots, \psi_r(x)\}$ is continuous. Since K is invariant for any $l \in \{1, \ldots, r\}$, there exists $l' \in \{1, \ldots, r\}$ such that $T(\text{graph}\,\psi_l) = \text{graph}\,\psi_{l'}$. We denote l' by $\sigma'(l)$. Now the invariance of K implies that σ' is onto and hence is also one-to-one. So for any $l \in \{1, \ldots, r\}$, there exists q_l such that $T^{q_l}(\text{graph}\,\psi_l) = (\text{graph}\,\psi_l)$ and the proof of Theorem 9 is completed.

COROLLARY 16

Let $K = X \times Y'$ be a compact T-invariant set. Suppose that for any measure μ supported on K, the essential supremum of the maximal normal Liapunov exponent $\lambda_\mu < 0$. Then K is the union of a finite number of continuous invariant graphs.

3. The classification of existence of invariant graphs

In this section, we consider the skew product T as follows:

$$\begin{aligned} T : X \times [a, b] &\to X \times [a, b] \\ (x, y) &\mapsto (\theta(x), g(x, y)). \end{aligned} \tag{6}$$

Here, we put some extra hypotheses on the system.

Hyp (1). $\{f_n\}_{n=1}^{\infty}$ are locally diffeomorphisms which are onto on X.
Hyp (2). For any $y \in [0, 1]$ and $n \in \mathbb{N}$, the mapping $g(., y)$ is continuous, $g(x, .)$ is continuously differentiable on (x, y) and for all $x \in X$, $g(x, y_1) \leq g(x, y_2)$ if $y_1 \leq y_2$, $g(x, a) > a$, $g(x, b) < b$.

We denote the set of skew products defined as (5), satisfying hypothesis Hyp (1) and Hyp (2), by $\mathbf{T}$.

Now consider a compact T-invariant set $K = X \times Y' \subseteq X \times [a, b]$. It is clear that the mapping $\varphi^+(x) := \sup\{y \in [a, b]; (x, y) \in K\}$ is an invariant graph, analogously $\varphi^-(x)$ which is defined via infimum is an invariant graph, too.

DEFINITION 17

Consider the skew product $T \in \mathbf{T}$. The global attractor of T is the set $K_{\max} := \cap_{n \in \mathbb{N}} T^n(X \times [a, b])$. The upper and lower bounding graphs of the system $(T, X \times [a, b])$ is defined as

$$\begin{aligned} \varphi_T^+(x) &:= \sup\{y \in [a, b]; (x, y) \in K_{\max}\}, \\ \varphi_T^-(x) &:= \inf\{y \in [a, b]; (x, y) \in K_{\max}\}. \end{aligned}$$

For an invariant graph φ, the measure $\mu_\varphi(A)$ is defined as $\mu_\varphi(A) := \text{Leb}(\pi_1(A \cap \Phi))$ for all $A \in \mathcal{B}(T, X \times [a, b])$.

DEFINITION 18

Let $T \in \mathbf{T}$ and φ be an invariant graph. The essential closure of φ with respect to μ_φ is

$$\bar{\Phi}^{\text{ess}} := \{(x, y);\ \mu_\varphi(U \cap \Phi) > 0 \text{ for all open neighbourhood } U \text{ of } (x, y)\},$$

is a compact T-invariant set. Also $\bar{\Phi}^{\text{ess}} = \text{supp}(\mu_\varphi)$ and is contained in every other compact set which contains μ_φ-a.e. point of Φ.

If the Liapunov exponent of a continuous invariant graph φ, $\lambda(\varphi)$ is negative, then there exists a constant $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\frac{1}{n_0} \log |Dg_x^{n_0}(y)| < \lambda \tag{7}$$

for any $(x, y) \in U_\varepsilon(\varphi) := \{(x, y);\ y \in B_\varepsilon(\varphi(x)\}$. It shows that continuous invariant graphs with negative Liapunov exponent are attracting in the very strong sense i.e., an iterate of the skew product T acts contracting on a neighbourhood of the graph. But if the graph φ is non-continuous, the following lemma which is just a slightly adapted version of Corollary 16 has been obtained.

Lemma 19. *Let $T \in \mathbf{T}$ and φ be a non-continuous invariant graph with a negative Liapunov exponent. Then $\bar{\Phi}^{\text{ess}}$ contains at least one invariant graph with a non-negative Liapunov exponent.*

PROPOSITION 20

Suppose $T \in \mathbf{T}$ and φ is an invariant graph with $\lambda(\varphi) < 0$. Then for Leb-*a.e.* $x \in X$, *there exists a constant $\delta_x > 0$, such that for any $y \in B_{\delta_x}(\varphi(x))$, we have*

$$|g_x^n(y) - \varphi(\theta^n(x))| \to 0 \ \ as \ \ (n \to \infty).$$

Proof. Let the function $F : X \times [a, b] \to X \times [a, b]$ be bounded Lipschitz-continuous with the following properties:

- there exists some $c > o : F(x, y) \geq \max\{c, |Dg_x(y)| \ \ \forall (x, y) \in X \times [a, b]\}$;
- $\int_X \log F(x, \varphi(x))\, dx < \frac{3}{4}\lambda$.

Then $\log F$ will be Lipschitz as well. Thus there exists some $L > 0$ such that for any $y, z \in [a, b]$ and $x \in X$,

$$F(x, y) \leq e^{L|y-z|} F(x, z). \tag{8}$$

Consider the mapping $G : x \mapsto \log F(x, \varphi(x))$. Applying Theorem 5 to G implies that

$$\int_X \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log F(\theta^j(x), \varphi(\theta^j(x)))\, dx = \int_X \log F(x, \varphi(x))\, dx.$$

Therefore for Leb-a.e. $x \in X$, there is a constant $K = K_x > 1$ such that for any $n \in \mathbb{N}$,

$$\prod_{i=0}^{n-1} F(\theta^i(x), \varphi(\theta^i(x))) \leq K e^{\frac{n\lambda}{2}}. \tag{9}$$

Now we can choose $\delta > 0$ with $\frac{\lambda}{2} + \delta L < 1$ and $n_0 \in \mathbb{N}$, such that $K e^{n(\frac{\lambda}{2}+\delta L)} < 1$ for any $n \geq n_0$. Assume $0 < \delta_x < \delta$ is chosen such that for any $y \in B_{\delta_x}(\varphi(x))$,

$$|g_x^i(y) - \varphi(\theta^i(x))| < \delta, \quad \text{for any } \ i = 0, \ldots, n_0, \tag{10}$$

where $B_{\delta_x}(\varphi(x))$ is a ball of radius δ_x and center $\varphi(x)$. A straightforward induction yields that for any $n \in \mathbb{N}$ and any $y \in B_{\delta_x}(\varphi(x))$,

$$|g_x^n(y) - \varphi(\theta^n(x))| < \min\{\delta, \delta \cdot K \cdot e^{n(\frac{\lambda}{2}+\delta L)}\}. \tag{11}$$

For $n = 0$, this is obvious. Now suppose (11) holds for $i = 0, \ldots, n-1$ and let $y \in B_{\delta_x}(\varphi(x))$. Then

$$\begin{aligned} |g_x^n(y) - \varphi(\theta^n(x))| &\leq |y - \varphi(x)| \sup_{z \in B_{\delta_x}(\varphi(x))} |Dg_x^n(z)| \\ &\leq \delta \sup_{z \in B_{\delta_x}(\varphi(x))} \prod_{i=0}^{n-1} F(\theta^i(x), g_x^i(z)) \\ &\leq \delta \sup_{z \in B_{\delta_x}(\varphi(x))} \prod_{i=0}^{n-1} F(\theta^i(x), \varphi(\theta^i(x))) e^{L|\varphi(\theta^i(x)) - g_x^i(z)|} \\ &\delta \cdot K \cdot e^{n(\frac{\lambda}{2}+\delta L)}. \end{aligned}$$

Together with (10) in the case $n < n_0$, this proves the induction hypothesis, and thus the proposition. □

Lemma 21. *Let $T \in \mathbf{T}$ and φ be an invariant graph with $\lambda(\varphi) < 0$. Then one of the following is true*:

(i) $\varphi = \varphi_T^+$ *and for* Leb-*a.e.* $x \in X$ *and all* $y > \varphi(x)$,

$$|g^n(x, y) - \varphi(\theta^n(x))| \to 0 \ \ (n \to \infty);$$

(ii) $\varphi < \varphi_T^+$ Leb-*a.e., and there exists another invariant graph* $\psi \geq \varphi$ *with* $\lambda(\psi) \geq 0$, *such that for* Leb-*a.e.* $x \in X$,

$$\varphi(x) \leq y < \psi(x) \Rightarrow |g^n(x, y) - \varphi(\theta^n(x))| \to 0 \ \ (n \to \infty).$$

If in case (ii), *φ is continuous, then ψ is lower semi-continuous.*

Proof. If $\varphi < \varphi_T^+$ Leb-a.e., then

$$\psi(x) := \sup\{y \geq \varphi(x);\ |g^n(x, y) - \varphi(\theta^n(x))| \to 0\} \leq \varphi_T^+(x)$$

defines an invariant graph. According to the last proposition, it is clear that $\psi(x)$ is Leb-a.e. not equal to φ. Also, $\lambda(\psi)$ is non-negative, since if not, the above proposition would yield a contradiction to this convergence behaviour.

Now suppose $\varphi = \varphi_T^+$ and let $B := \{x;\, \psi(x) = b\}$ and $A = B^c$. Take $x \in \theta^{-1}(B)$ and suppose $y' > \varphi(x)$. Then

$$\begin{aligned} |g^{n+1}(x, y') - g^{n+1}(x, \varphi(x))| &= |g^n(\theta(x), g(x, y')) - g^n(\theta(x), g(x, \varphi(x))| \\ &= |g^n(\theta(x), y) - g^n(\theta(x), \varphi(\theta(x)))| := \alpha_n \end{aligned}$$

and since $\alpha_n \to 0\,(n \to \infty)$, we have $\theta^{-1}(B) \subseteq B$, also $\theta(A) \subseteq A$. By Remark 2, Leb(A) must be either 0 or 1. Therefore, Leb(B) is either 0 or 1. As there is no invariant graph above φ_T^+ it must be 1, which proves the statement in (i).

In part (ii), assume that φ is continuous, $\varepsilon > 0$ and n_0 are as in (7). To obtain the lower semi-continuity of ψ, it is enough to show that $\{x;\, \psi(x) > z\}$ is open for all $z \in [a, b]$. Suppose $\psi(x) > z$. We can suppose $z \geq \varphi(x)$, otherwise, we can replace z with an arbitrary $w \in [\varphi(x), \psi(x)]$. Since $\varphi(x) \leq z < \psi(x)$, there exists $n \in \mathbb{N}$ so that $(\theta^n(x), g^n(x, z)) \in U_{\varepsilon/2}(\varphi)$. Since for any $i \in \mathbb{N}$, f_i is continuous, there is a small neighbourhood V of x such that for any $x' \in V$, $\theta^n(x)$ and $\theta^n(x')$ are close to each other and by Hyp (2), we have $(\theta^n(x'), g^n(x', z)) \in U_\varepsilon(\varphi)$. But an orbit that enters $U_\varepsilon(\varphi)$ is trapped in there and will finally converge to φ, thus $V \subset \{x;\, \psi(x) > z\}$.

□

PROPOSITION 22

Let $T \in \mathbf{T}$. If $\varphi^- \leq \varphi^+$ are two distinct non-continuous invariant graphs with a negative Liapunov exponent, φ^- is lower semi-continuous, φ^+ is upper semi-continuous and if there exists no other semi-continuous invariant graph in between, then the following is true:

(*i*) $K := \bar{\Phi}^{-\text{ess}} = \bar{\Phi}^{+\text{ess}}$ *is minimal;*
(*ii*) *There exists another invariant graph ψ, with non-negative Liapunov exponent and $\bar{\Psi}^{\text{ess}} = K$ between φ^- and φ^+;*
(*iii*) $B := \{x;\, \varphi^-(x) = \varphi^+(x)\} = \{x;\, \varphi^-$ *and* φ^+ *are both continuous in* $x\}$ *is dense in* X *and* Leb$(B) = 0$;
(*iv*) K *is a perfect set and has empty interior.*

Proof. The proof of the proposition is similar to ([6], Proposition 3.7). Note that for the proof of the second part of (iii), we just use Remark 2. Indeed B is an f_1-invariant set and since the two graphs are distinct, it cannot have full measure. Thus by Remark 2, we have Leb$(B) = 0$. □

DEFINITION 23

For $f \in C^3([a, b])$ with $Df > 0$, the Schwarzian derivative of f is defined as follows:

$$Sf := \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2.$$

Now we consider the class of skew products

$$\mathbf{T}_s := \{T \in \mathbf{T};\, g_x \in C^3([a, b]),\, D_y g_x > 0,\, Sg_x < 0 \text{ for all } x \in X\}.$$

The following theorem is analogous to ([6], Theorem 4.2) and we omit its proof.

Theorem 24. *Let $T \in \mathbf{T}_s$. Then there exist three possible cases*:

- (*i*) *There exists one invariant graph φ with $\lambda(\varphi) \leq 0$;*
- (*ii*) *There are two invariant graphs φ and ψ with $\lambda(\varphi) < 0$ and $\lambda(\psi) = 0$;*
- (*iii*) *There are three invariant graphs $\varphi^- \leq \psi \leq \varphi^+$ with $\lambda(\varphi^-) < 0$, $\lambda(\psi) > 0$ and $\lambda(\varphi^+) < 0$.*

Example 25. Here we give an example and use Theorem 24 to check the existence of invariant graphs. Consider the skew product

$$\begin{aligned} T : S^1 \times [0,1] &\longrightarrow S^1 \times [0,1] \\ (x, y) &\mapsto \left(\theta(x), y + \frac{x+1}{8}(1-y) - \frac{1}{4}y^2\right), \end{aligned} \tag{12}$$

where for any $n \in \mathbb{N}$, $f_n(x) = 2x + \frac{n}{n+1}\varepsilon$, where $\varepsilon > 0$ belongs to $\mathbb{Q}^c$ is very small and $\theta(x)$ is defined by (4). We observe that for any $(x, y) \in S^1 \times [0,1]$,

$$\log|D_y g^n(x, y)| = \sum_{i=1}^{n} \log\left|1 - \frac{\theta^i(x)+1}{8} - \frac{1}{2}g^{i-1}(x, y)\right|$$

and since $0 \leq \theta^i(x) \leq 1$ and $0 \leq g^i(x, y) \leq 1$, we have $\frac{1}{4} \leq 1 - \frac{\theta^i(x)+1}{8} - \frac{1}{2}g^{i-1}(x, y) \leq \frac{7}{8}$. So

$$\lambda(x, y) = \lim_{n\to\infty} \frac{1}{n} \log|D_y g^n(x, y)| = \log\left(\frac{7}{8}\right) < 0.$$

Therefore, by Theorem 24, there exists exactly one invariant graph φ with $\lambda(\varphi) < 0$.

4. Intermingled basins

Let the skew product $T : X \times [0,1] \to X \times [0,1]$, defined as in (5), be a C^3 differential map and for any $x \in X$, we suppose that $g(x, .) : [0,1] \to [0,1]$ is a diffeomorphism with $g_x(0) = 0$ and $g_x(1) = 1$. We say that a compact set $K \times J \subset X \times [0,1]$ is an attractor if $\{(x, y) \in X \times [0,1]; \omega(x, y) = K \times J\}$ has positive Lebesgue measure and the basin of attraction of A is $B(K \times J) = \{(x, y) \in X \times [0,1]; \omega(x, y) \subset K \times J\}$. Take the subsets $\mathcal{A}_0 = X \times \{0\}$ and $\mathcal{A}_1 = X \times \{1\}$. We denote their attracting basins by $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively. The attracting basins $\mathcal{B}_0$ and $\mathcal{B}_1$ are intermingled if, for every non-empty open set $U \subset X \times [0,1]$, both intersections $\mathcal{B}_0 \cap U$ and $\mathcal{B}_1 \cap U$ have strictly positive Lebesgue measure.

Theorem 26. *If $Sg_x(y) < 0$ a.e. and if both basins have positive measure, then $\mathcal{B}_0 \cup \mathcal{B}_1$ has full measure. In fact, there exists an almost everywhere defined measurable function $\alpha : X \to [0,1]$ such that if $y < \alpha(x)$, then $(x, y) \in \mathcal{B}_0$ and if $y > \alpha(x)$, we have $(x, y) \in \mathcal{B}_1$. In addition, the function $\alpha(x)$ is an invariant graph.*

Proof. Since each g_x is an orientation preserving homeomorphism, there are unique numbers $0 \leq \alpha_0(x) \leq \alpha_1(x) \leq 1$ where the orbit of (x, y) has three cases:

(1) It converges to $\mathcal{A}_0$ if $y < \alpha_0(x)$;
(2) It converges to $\mathcal{A}_1$ if $y > \alpha_1(x)$;
(3) It does not converge to either of $\mathcal{A}_l$, $l = 0, 1$, if $\alpha_0(x) < y < \alpha_1(x)$.

Thus, we can define the area of $\mathcal{B}_0$ as $\int \alpha_0(x)dx$. Since this set has positive Lebesgue measure, it follows that the set $A := \{x \in X; \alpha_0(x) > 0\}$ must have positive Lebesgue measure. Also, it is evident that $\alpha_0(\theta^n(x)) = g_x^n(\alpha_0(x))$ and so $\theta^n(A) \subseteq A$ for any n, so by Remark 2, we have $\mathrm{Leb}(A) = 1$. Similarly, the set $B := \{x \in X; \alpha_1(x) < 1\}$ must have full Lebesgue measure. Suppose the inequalities $0 < \alpha_0(x) < \alpha_1(x) < 1$ are true for a set of positive Lebesgue measure. Since $Sg_x < 0$ for almost every $x \in M$, we can take $x_0 \in M$ such that $Sg_{x_0} < 0$ and $0 < \alpha_0(x_0) < \alpha_1(x_0) < 1$.

We define the skew product $\tilde{T} : X \times [0, 1] \to X \times [0, 1]$, $\tilde{T}(x, y) = (\theta(x), g_{x_0}(y))$. The mappings $\tilde{\alpha}_0(x)$ and $\tilde{\alpha}_1(x)$ are defined similarly to $\alpha_0(x)$ and $\alpha_1(x)$. It is clear that $\tilde{\alpha}_0(x) = \alpha_0(x_0)$ for all $x \in X$, and so $\tilde{\alpha}_0(\theta(x_0)) = \alpha_0(x_0)$. On the other hand, clearly we have $\tilde{\alpha}_0(\theta(x_0)) = g_{x_0}(\tilde{\alpha}_0(x_0))$. So, $\alpha_0(x_0) = g_{x_0}(\alpha_0(x_0))$. Similarly, $\alpha_1(x_0) = g_{x_0}(\alpha_1(x_0))$. So, $0, 1, \alpha_0(x_0)$ and $\alpha_1(x_0)$ are fixed points of g_{x_0}, but this is impossible since by ([3], Theorem B2), g_{x_0} has at most three fixed points. Thus the first part of the theorem is proved. Also, it is easily observed that the function $\alpha(x)$ is an invariant graph. □

Hyp (3). There are $x^-, x^+ \in X$ such that $g_x(y) < y$ for every $0 < y < 1$ and any x in a neighbourhood of x^- and $g_x(y) > y$ for any $0 < y < 1$ and x in a neighbourhood of x^+.

Theorem 27. *If the system* $(T, X \times [0, 1])$ *satisfying* Hyp (1) *and* Hyp (3), $Sg_x(y) < 0$ *a.e. and if* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *have positive measures, then they are intermingled.*

Proof. Define measures μ_0 and μ_1 on $X \times [0, 1]$ by $\mu_l(S) = m(\mathcal{B}_l \cap S)$ for $l = 0, 1$ and any measurable set $S \subseteq X \times [0, 1]$. Clearly, $T^{-1}(\mathrm{supp}(\mu_l)) \subseteq \mathrm{supp}(\mu_l))$, where $\mathrm{supp}(\mu_l)$ is the support of measure μ_l. It is enough to show that $\mathrm{supp}(\mu_l) = X \times [0, 1]$.

Select $(x_0, y_0) \in \mathrm{supp}(\mu_0)$, such that $0 < y_0 < 1$ and construct the sequence

$$\cdots \mapsto (x_{-2}, y_{-2}) \mapsto (x_{-1}, y_{-1}) \mapsto (x_0, y_0),$$

where $x_{-k} = \theta^{-k}(x_0), k \in \mathbb{N}$, closest to x^-. So $(x_{-n}, y_{-n}) \to (x^-, 1)$ as $n \to \infty$. Since $\mathrm{supp}(\mu_0)$ is closed and T-invariant, we have $(x^-, 1) \in \mathrm{supp}(\mu_0)$. Also since NDS $(X, f_{1,\infty})$ is backward minimal (Lemma 1), then pre-images $(x^-, 1)$ are dense in $\mathcal{A}_1$. So $\mathcal{A}_1$ is contained in $\mathrm{supp}(\mu_0)$. As we have seen in Theorem 26, basin $\mathcal{B}_0$ is a union of vertical segments $\{x\} \times [0, \alpha_0(x)]$ or $\{x\} \times [0, \alpha_0(x))$, so $\mathrm{supp}(\mu_0)$ is $X \times [0, 1]$. Similarly, $\mathrm{supp}(\mu_1)$ is equal to $X \times [0, 1]$. □

In the following, we use Kan's example (see [7]) and modify it to give an example satisfying the hypothesis of Theorem 27.

Example 28. Consider the skew product

$$\begin{aligned} T : S^1 \times [0, 1] &\longrightarrow S^1 \times [0, 1] \\ (x, y) &\mapsto (\theta(x), y + \cos(2\pi x)\frac{y}{32}(1 - y), \end{aligned} \tag{13}$$

where $\theta(x)$ is defined as in Example 25. For any $(x, 0) \in \mathcal{A}_0$, by Theorem 5,

$$\int_0^1 \lambda(x, 0)\, dx = \int_0^1 \log |D_y g^n(x, 0)|\, dx,$$

which is negative, see ([7], Lemma 2.1). Now, since $\int_0^1 \lambda(x, 0)\, dx < 0$, for a.e. $(x, 0) \in \mathcal{A}_0$, $\lambda(x, 0) < 0$. Similarly for a.e. $(x, 1) \in \mathcal{A}_1$, $\lambda(x, 1) < 0$. Therefore, basins $\mathcal{B}_0$ and $\mathcal{B}_1$ have positive measure. Also, we observe that Hyp (3) is correct for the system. Indeed, it is easily seen that for $x = 1/2$ and any point x in a neighbourhood of $1/2$, $g_x(y) < y$ and for $x = 0$ and any point x in a neighbourhood of $1/2$, $g_x(y) > y$. Moreover $Sg_x(y) < 0$ a.e., therefore by Theorem 27, $\mathcal{B}_0$ and $\mathcal{B}_1$ are intermingled.

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Communicating Editor: Anish Ghosh