Proc. Indian Acad. Sci. (Math. Sci.) (2023) 133:6
https://doi.org/10.1007/s12044-023-00727-5

Characterization of the potential function of a Ricci soliton

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MS received 7 July 2022; revised 4 December 2022; accepted 19 December 2022

Abstract. In this article, we have studied the behavior of the potential function along some geodesic in a Ricci soliton under some curvature restriction. In particular, we have showed that under some curvature restriction, the potential function reduced to a parabola along some geodesic. Furthermore, we have investigated the change of intersecting angles between the potential vector field and a geodesic in a Ricci soliton. Further, we have deduced the condition when the potential function becomes convex in a shrinking Ricci soliton. Finally, we have concluded the paper by showing the non-existence of convex potential in an expanding Ricci soliton having non-negative Ricci curvature.

Keywords. Ricci soliton; Busemann function; Riemannian manifold.

2010 Mathematics Subject Classification. 53C20, 53C21, 53C44.

1. Introduction

A Riemannian manifold (M, g) is called the Ricci soliton [7] if its Ricci tensor Ric satisfies the following equation:

$$2\mathrm{Ric} + \pounds_X g = 2\lambda g, \tag{1}$$

where $\pounds_X g$ is the Lie derivative of the metric tensor g with respect to the vector field X. In tensors of local coordinates system, (1) can also be expressed in the following form:

$$2R_{ij} + \nabla_i X_j + \nabla_j X_i = 2\lambda g_{ij}, \tag{2}$$

where R_{ij} denotes the components of the Ricci tensor. The Ricci soliton is said to be expanding (resp., steady and shrinking) if $\lambda < 0$ (resp., $\lambda = 0$ and $\lambda > 0$). The vector field X is known as potential vector field of the Ricci soliton, and if X is the gradient of some smooth function $f \in C^\infty(M)$, then f is called the potential function, and in this case, the Ricci soliton is called the gradient Ricci soliton. The gradient Ricci soliton takes the form

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \tag{3}$$

Published online: 29 March 2023

A function $f : M \to \mathbb{R}$ is said to be convex if its restriction to every geodesic is convex, i.e.,

$$f(\gamma((1-t)a+tb)) \leq (1-t)f(\gamma(a)) + tf(\gamma(b)), \tag{4}$$

for every geodesic segment $\gamma : [a, b] \to M$. A convex function is always continuous. If f is smooth, then the condition (3) is equivalent to the semi-definiteness of f along every geodesic. If the inequality of (3) is strict, then f is said to be a strictly convex function. The existence of a non-constant convex function in a manifold reveals some important topological and geometric properties of a manifold. Yau [14] showed that if a Riemannian manifold possesses a non-trivial convex function, then it has infinite volume. Nato *et al.* [9] proved that if the geodesic flow is conservative with respect to a Liouville measure in a Riemannian manifold, then all convex functions become constant. Mondal and Shaikh [8] proved a splitting theorem for a Ricci soliton having a convex potential. Cao and Zhou [3] found a bound of a potential function in a non-compact shrinking Ricci soliton. Wu [13] estimated a bound for the potential function in case of steady Ricci soliton. Again, in case of expanding Ricci soliton, Cao *et al.* [2] investigated the bounds for the potential function. For more work on Ricci soliton, see [1,5,8,10–12].

In this article, we have estimated the potential function along some geodesic in a Ricci soliton under some curvature restriction. In particular, we proved that under some curvature restriction, the potential function takes the form of a parabola along some geodesic. Moreover, we investigate the behaviour of the gradient vector field of the potential function along some geodesic. Further, we deduce the condition when the potential function becomes convex in a shrinking Ricci soliton. The main theorem of this paper is the following.

Theorem 1. *Let M be a gradient Ricci soliton satisfying the condition*

$$\int_0^t \mathrm{Ric}(\beta'(s), \beta'(s))ds \geq 0, \quad \forall t \geq 0, \tag{5}$$

along any geodesic $\beta : [0, \infty) \to M$. If M contains a line, then there exists a geodesic $\sigma : [0, l] \to M$ for some $l > 0$ such that along σ the potential function f takes the following form:

$$(f \circ \sigma)(t) = \frac{\lambda}{2}t^2 + (f \circ \sigma)'(0)t + (f \circ \sigma)(0). \tag{6}$$

Also, the angle $\measuredangle\theta_t$ between $\sigma'(t)$ and ∇f at $\sigma(t)$ decreases or increases with the increase of $t > 0$ in case of shrinking or expanding respectively. And when the Ricci soliton is steady, then $\measuredangle\theta_t = \measuredangle\theta_0\ \forall t > 0$. Moreover, in case of shrinking Ricci soliton, the potential function becomes convex along some geodesics.

If a Riemannian manifold possesses non-negative Ricci curvature, then the condition (5) is trivially satisfied. Hence, Theorem 1 concludes the following corollaries.

COROLLARY 2

In a gradient Ricci soliton with non-negative Ricci curvature and containing a line, the potential function f satisfies

$$\nabla^2 f(\sigma'(t), \sigma'(t)) = \lambda, \tag{7}$$

along a geodesic σ in M.

COROLLARY 3

Let M be a gradient expanding Ricci soliton with non-negative Ricci curvature and containing a line. Then the potential function is not convex in M.

According to Theorem 1, the potential function along a geodesic may be represented using the equation of a parabola. In [6], a condition somewhat similar to (5) was utilized. The main difference is that in [6], the condition must hold only for geodesic rays, while in our assumption, it should apply for every geodesic. The latter is somewhat more powerful than the preceding. The stronger condition is required to consider the non-negativity of Ricci curvature, which is a result of Theorem 1. Moreover, if the Ricci curvature is everywhere non-negative, then the condition (5) automatically follows. Therefore, the following is an easy consequence of Theorem 1.

COROLLARY 4

Let M be a gradient Ricci soliton with non-negative Ricci curvature. If M contains a line, then f reduces to the form (6) *along some geodesics.*

Here are some examples supporting Theorem 1.

Example 1.1. The flat metric on $\mathbb{R}^n$ together with the function $f(x) = \frac{\lambda}{2}|x|^2$ for $x \in \mathbb{R}^n$ forms a gradient Ricci soliton. Then along the geodesic $\gamma(t) = (\frac{t}{\sqrt{n}}, \dots, \frac{t}{\sqrt{n}})$ for $t \in [0, \infty)$, the potential function f takes the form

$$(f \circ \gamma)(t) = \frac{\lambda}{2} t^2.$$

Example 1.2. Consider the gradient shrinking Ricci soliton $M = \mathbb{R} \times \mathbb{S}^2$ with the product metric, $\lambda = 1$ and the potential function $f(x, y) = \frac{|x|^2}{2}$ for $(x, y) \in M$. Now take the geodesic $\gamma(t) = (t + 3, a)$ for $t \in [0, \infty)$, where $a \in \mathbb{S}^2$ is a fixed point. Then along γ, the function f reduces to the form

$$(f \circ \gamma)(t) = \frac{t^2}{2} + 3t + \frac{9}{2}.$$

2. Main proof

Proof of Theorem 1. Suppose $\gamma : (-\infty, +\infty) \to M$ is a line in M. Let b^+ and b^- be the Busemann functions corresponding to the rays $\gamma^+ = \gamma|_{[0,\infty)}$ and $\gamma^- = \gamma|_{(-\infty,0]}$ respectively. Specifically,

$$b^+(x) = \lim_{t\to\infty}(d(x, \gamma^+(t)) - t),$$
$$b^-(x) = \lim_{t\to\infty}(d(x, \gamma^-(-t)) - t).$$

Our assumption implies that the Ricci curvature is pointwise non-negative. Hence, the Cheeger–Gromoll splitting theorem, see [4, Theorem 1], implies that b^+ and b^- are super-harmonic functions. Now the calculation of Busemann function in Theorem 2 of [4] shows that for a given x, we can find a minimal geodesic σ_t from x to $\gamma(t)$. Let $\{t_n\}$ be a sequence such that $\sigma'_{t_n}(0) \to \sigma'(0)$, where $\sigma_t \to \sigma$ as $t \to \infty$. Then for all $y \in \sigma$, we get $|b^+(x) - b^+(y)| = d(x, y)$. It shows that $|\nabla b^+| = 1$ and σ is the integral curve of ∇b^+ through x. Take $\nabla b^+ = \eta$. Now construct an orthonormal frame $e_1, e_2, \ldots, e_{n-1}, \eta$ in a neighborhood of x which is parallel along σ. The construction shows that $\nabla_\eta \eta = 0$ at x. Now calculate

$$\begin{aligned}
\nabla^2 f(\eta, \eta) &= \lambda\langle \eta, \eta\rangle - \mathrm{Ric}(\eta, \eta)\\
&= \lambda - \sum_{i=1}^{n-1}\langle R(e_i, \eta)\eta, e_i\rangle\\
&= \lambda - \sum_{i=1}^{n-1}\langle \nabla_{e_i}\nabla_\eta \eta - \nabla_\eta\nabla_{e_i}\eta - \nabla_{[e_i,\eta]}\eta, e_i\rangle\\
&= \lambda + \sum_{i=1}^{n-1}\langle \nabla_\eta\nabla_{e_i}\eta, e_i\rangle + \sum_{i=1}^{n-1}\langle \nabla_{\nabla_{e_i}\eta}\eta, e_i\rangle\\
&= \lambda + \sum_{i=1}^{n-1}\eta\langle \nabla_{e_i}\eta, e_i\rangle + \sum_{i,j=1}^{n-1}\langle \nabla_{e_i}\eta, e_j\rangle\langle \nabla_{e_j}\eta, e_i\rangle\\
&= \lambda - \sum_{i=1}^{n-1}\eta\langle \eta, \nabla_{e_i}e_i\rangle + \|\nabla\eta\|^2\\
&= \lambda + \eta(\Delta b^+) + \|\nabla\eta\|^2 = \lambda + \|\nabla\eta\|^2.
\end{aligned}$$

It shows that along σ, $\nabla^2 f(\eta, \eta) \geq \lambda$. Again, the equation of Ricci soliton and our given assumption imply that along σ,

$$\int_0^t \nabla^2 f(\sigma'(t), \sigma'(t)) = \lambda t - \int_0^t \mathrm{Ric}(\sigma'(t), \sigma'(t))dt \leq \lambda t. \tag{8}$$

Hence, we obtain

$$\nabla^2 f(\sigma'(t), \sigma'(t)) = \lambda.$$

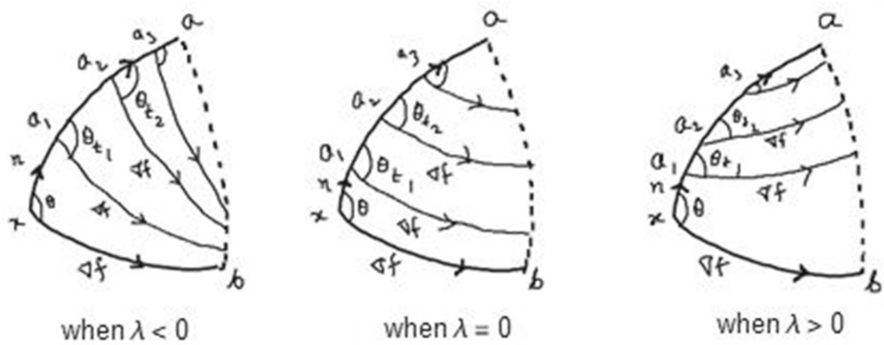

Figure 1. Scattering of the vector field ∇f along the geodesic σ.

Integrating the above equation twice yields

$$\int_0^t \int_0^r \frac{d^2}{ds^2}(f \circ \sigma)(s) ds dr = \lambda \frac{t^2}{2}. \tag{9}$$

It follows our first result, i.e.,

$$(f \circ \sigma)(t) = \frac{\lambda}{2} t^2 + (f \circ \sigma)'(0)t + (f \circ \sigma)(0). \tag{10}$$

Now from the above equation, we obtain

$$(f \circ \sigma)'(t) = \lambda t + (f \circ \sigma)'(0). \tag{11}$$

Therefore, if $\lambda > 0$, then $(f \circ \sigma)'(t) > (f \circ \sigma)'(0)$. Instead of $x = \sigma(0)$, if we choose the initial point as $\sigma(t_0)$ for some $t_0 > 0$, then following the same calculation, we have $(f \circ \sigma)'(t) > (f \circ \sigma)'(t_0)$ for $t > t_0$. Thus, in general, we can say that

$$(f \circ \sigma)'(t_2) > (f \circ \sigma)'(t_1) \ \text{ for } t_2 > t_1 > 0$$

which implies that

$$\measuredangle\theta_{t_1} > \measuredangle\theta_{t_2} \ \text{ for } t_2 > t_1 > 0,$$

where $\measuredangle\theta_{t_i}$ is the angle between $\sigma'(t_i)$ and ∇f at $\sigma(t_i)$, see Figure 1.

Similarly, for expanding (resp. steady) gradient Ricci soliton, we get, see first and second pictures of Figure 1,

$$\measuredangle\theta_{t_1} < \measuredangle\theta_{t_2} (\text{ resp. } \measuredangle\theta_{t_1} = \measuredangle\theta_{t_2}) \ \text{ for } t_2 > t_1 > 0.$$

If the Ricci soliton is non-expanding, then $\lambda \geq 0$. Therefore, differentiating two times both sides of (6), we get

$$\frac{d^2}{dt^2}(f \circ \sigma)(t) = \lambda \geq 0$$

which indicates that f is convex along the geodesic σ. □

Acknowledgements

The author would like to thank the unknown referee for carefully reading the paper and for thoughtful criticism of the manuscript, which helped improve the manuscript considerably. The author wants to express his gratitude to Prof. Seungsu Hwang for carefully reading the paper and for his valuable suggestions that have improved the paper. The author conveys sincere thanks to the Netaji Subhas Open University for financial assistance (Order No.: AC/140/2021-22 dated 22/11/2021).

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Communicating Editor: Mahuya Datta