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On generalized cyclotomic derivations

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Abstract. In this article, we study the field of rational constants and Darboux polynomials of a generalized cyclotomic K-derivation d of $K[X]$. It is shown that d is without Darboux polynomials if and only if $K(X)^d = K$. The result is also studied in the tensor product of polynomial algebras.

Keywords. Darboux polynomial; Jouanolou derivation; cyclotomic derivation.

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1. Introduction

Throughout this article, K denotes a field of characteristic zero, $K[X] = K[x_1, x_2, \dots, x_n]$ is the polynomial algebra in n variables over K and $K(X)$ denotes the field of fractions of $K[X]$. Let d be a K-derivation of $K[X]$ and $K[X]^d$ denote the algebra of constants of d. The K-derivation d of $K[X]$ uniquely extends to a K-derivation of $K(X)$ and we continue to denote it by d. $K(X)^d$ represents the field of rational constants of d, that is, $K(X)^d = \{f \in K(X); d(f) = 0\}$ and $K[X]^d \subseteq K(X)^d$.

A non-constant polynomial $f \in K[X]$ is said to be a Darboux polynomial of d if $d(f) = \lambda f$, for some $\lambda \in K[X]$ and in this case λ is called the co-factor of f. We say d is without Darboux polynomials if d has no Darboux polynomials. It is easy to observe that if d is without Darboux polynomials, then $K(X)^d = K$ but the converse of the above statement is not true, in general. One can refer to [1,2] for counter examples. In this paper, we study a class of monomial derivations for which $K(X)^d = K$ if and only if d is without Darboux polynomials. Note that a derivation d of $K[X]$ is said to be a monomial derivation if $d(x_i)$ is a monomial for every $1 \leq i \leq n$.

In [4], Nowicki and Ollagnier studied the Darboux polynomials and field of rational constants of a monomial derivation over the polynomial algebra $K[X]$. For $1 \leq i \leq n$, let $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbb{N}^n$. Consider the monomial derivation d given by $d(x_i) = X^{\alpha_i}$, where X^{α_i} denotes the monomial $x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$. Then one can associate a matrix A given by $A = [\alpha_{ij}] - I$ with the monomial derivation d. Let w_d denote the determinant of the matrix A. A monomial derivation d is said to be normal if $w_d \neq 0$ and $\alpha_{ii} = 0\ \forall\ 1 \leq i \leq n$. In [4], Nowicki and Ollagnier proved that for a normal derivation d, $K(X)^d = K$ if and only

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if d is without Darboux polynomials. In this article, they also raised a similar question in the case of $w_d = 0$. For $n = 3$, an independent proof was given to show that the result is true even if $w_d = 0$. It was also observed that the idea used to prove the result for $n = 3$ case could not be extended further. At the end of the article, they mentioned the following example of monomial derivation d on $K[x, y, z, w]$;

$$d(x) = w^2, \;\; d(y) = zw, \;\; d(z) = y^2, \;\; d(w) = xy,$$

and raised the same question about the field of constants and Darboux polynomials. Note that for this derivation, $w_d = 0$. Our study of the field of constants of monomial derivations is motivated by the aforesaid example.

In this article, we study a large class of monomial derivations for which $K(X)^d = K$ if and only if d is without Darboux polynomial. The example mentioned above is a very particular case of our result. Further, our result is independent of the condition $w_d \neq 0$.

2. The main result

Let s be a non-negative integer. A derivation d on $K[X]$ is said to be homogeneous of degree s, if

$$d(A^m) \subseteq d(A^{m+s}) \qquad \forall\, m \geq 1,$$

where A^m is the K-subspace of all the homogeneous polynomials of degree m. In particular, a monomial derivation d on $K[X]$ is homogeneous of degree s if for each $1 \leq i \leq n$, $d(x_i)'s$ are monomials of total degree $s + 1$.

For a homogeneous K-derivation d of $K[X]$, the following is a well known result ([5] Lemma 2.1).

Lemma 2.1. *Let d be a homogeneous K-derivation of $K[X]$ of degree s. If $f \in K[X]$ is a Darboux polynomial of d with the co-factor λ, then λ is a homogeneous polynomial of degree s and all the homogeneous components of f are also Darboux polynomials with the same co-factor λ.*

Before proceeding further, we fix some notations and give some more definitions. Let $n \geq 2$ be a positive integer. A derivation d of $K[X]$ is called cyclotomic if, for $1 \leq i \leq n-1$, $d(x_i) = x_{i+1}$ and $d(x_n) = x_1$. In the same line we have defined the generalized cyclotomic derivation. Let $S = \{x_1, \ldots, x_n\}$ denote the set of n-variables, $K[S]$ be the K-algebra generated by S and $k > 1$ be a positive integer. A monomial derivation d of $K[S]$ is said to be a generalized cyclotomic derivation if we can split S into mutually disjoint k parts (say $S_i, \;\; 1 \leq i \leq k, \;\; k \in \mathbb{N}$ and $2 \leq k \leq n$) such that $d(S_i) \subseteq K[S_{i+1}]$ for $1 \leq i \leq k - 1$ and $d(S_k) \subseteq K[S_1]$, where $K[S_i]$ denotes the K-algebra generated by S_i.

Let us redefine the variables as $S_i = \{x_{i,1}, \ldots, x_{i,t_i}\}$ for $1 \leq i \leq k$ and take $S = \cup S_i$. Then we have the following.

DEFINITION 2.2

A derivation d of $K[S]$ is said to be generalized cyclotomic derivation if

$$d(x_{i,j}) = x_{i+1,1}^{\alpha_{i_j,1}} x_{i+1,2}^{\alpha_{i_j,2}} \cdots x_{i+1,t_{i+1}}^{\alpha_{i_j,t_{i+1}}} \quad \forall\, 1 \leq j \leq t_i, \;\; 1 \leq i \leq k-1$$

and

$$d(x_{k,j}) = x_{1,1}^{\alpha_{k_j,1}} x_{1,2}^{\alpha_{k_j,2}} \cdots x_{1,t_1}^{\alpha_{k_j,t_1}} \quad \forall\, 1 \le j \le t_k,$$

where $\alpha_{i_j,l}$ for every $1 \le j \le t_i$, $1 \le l \le t_{i+1}$, $1 \le i \le k-1$ and $\alpha_{k_j,l}$ for every $1 \le j \le t_k$, $1 \le l \le t_1$ are non negative integers.

Let s be a positive integer. We say d is homogeneous generalized cyclotomic derivation of degree $s-1$ if $d(x_{ij})$ are monomials of total degree s. Now we state our main result.

Theorem 2.3. *Let d be a homogeneous generalized cyclotomic derivation of $K[S]$ of degree $s-1$ as defined above. Then d is without Darboux polynomials if and only if $K(S)^d = K$.*

Proof.

($\Rightarrow$) Easy to prove.
($\Leftarrow$) Assume that $K(S)^d = K$. Suppose d has a Darboux polynomial f such that $d(f) = \lambda f$. Then by Lemma 2.1, we may assume that f is homogeneous and λ is a homogeneous of degree $s-1$. Write λ as

$$\lambda = \sum_{\sum_{1\le i\le k} |\beta_i| = s-1} a_\beta X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k},$$

where $X_i^{\beta_i}$ denotes the monomial $x_{i,1}^{\beta_{i,1}} \cdots x_{i,t_i}^{\beta_{i,t_i}}$ and $|\beta_i| = \sum_{j=1}^{j=t_i} \beta_{ij}$ for $1 \le i \le k$.

Let $N = (1 + s + s^2 + \cdots + s^{k-1})$ and let ξ be the primitive N-th root of unity. For $0 \le i \le k-1$, define $q_i = \sum_{j=0}^{i} s^j$. Observe that $q_{k-1} = N$. Consider a K-automorphism σ of $K[S]$ given by

$$\sigma(x_{k-i,j}) = \xi^{q_i} x_{k-i,j} \quad \forall\, 1 \le j \le t_{k-i}.$$

Moreover,

$$\begin{aligned}
\sigma^{-1} d\sigma(x_{k-i,j}) &= \sigma^{-1} d(\xi^{q_i} x_{k-i,j}) \\
&= \xi^{q_i} \sigma^{-1} d(x_{k-i,j}) \\
&= \xi^{q_i} \sigma^{-1} \left(x_{k-i+1,1}^{\alpha_{(k-i)_j,1}} x_{k-i+1,2}^{\alpha_{(k-i)_j,2}} \cdots x_{k-i+1,t_{k-i+1}}^{\alpha_{(k-i)_j,t_{k-i+1}}} \right) \\
&= \xi^{q_i} \xi^{-q_{i-1}\left(\alpha_{(k-i)_j,1} + \cdots + \alpha_{(k-i)_j,t_{k-i+1}}\right)} d(x_{k-i,j}) \\
&= \xi^{q_i} \xi^{-s(q_{i-1})} d(x_{k-i,j}) \\
&= \xi d(x_{k-i,j}).
\end{aligned}$$

Therefore, we have $\sigma^{-1} d\sigma = \xi d$. Let $F = \prod_{i=0}^{N-1} \sigma^i(f)$. Clearly, F is not a constant polynomial. Furthermore,

$$d(F) = d\left(\prod_{i=0}^{N-1} \sigma^i(f)\right)$$

$$
\begin{aligned}
&= \sum_{i=0}^{N-1} \sigma^0(f) \cdots d(\sigma^i(f)) \cdots \sigma^{N-1}(f) \\
&= \sum_{i=0}^{N-1} \sigma^0(f) \cdots \xi^i \sigma^i d(f) \cdots \sigma^{N-1}(f) \\
&= \sum_{i=0}^{N-1} \sigma^0(f) \cdots \xi^i \sigma^i (\lambda f) \cdots \sigma^{N-1}(f) \\
&= \left(\sum_{i=0}^{N-1} \xi^i \sigma^i(\lambda) \right) \sigma^0(f) \cdots \sigma^{N-1}(f), \\
&= \Lambda F,
\end{aligned}
$$

where $\Lambda = \sum_{i=0}^{N-1} \xi^i \sigma^i(\lambda)$.

Now, let us do the precise calculation for Λ. If we look at the m-th term in the sum, we have

$$
\begin{aligned}
\xi^m \sigma^m(\lambda) &= \xi^m \sigma^m \left[\sum_{\sum_{1 \le i \le k} |\beta_i| = s-1} a_\beta X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k} \right] \\
&= \xi^m \left[\sum_{\sum_{1 \le i \le k} |\beta_i| = s-1} a_\beta \, \sigma^m(X_1^{\beta_1}) \ldots \sigma^m(X_k^{\beta_k}) \right] \\
&= \xi^m \left[\sum_{\sum_{1 \le i \le k} |\beta_i| = s-1} a_\beta \left(\prod_{l=1}^{l=k} \sigma^m(x_{l1}^{\beta_{l1}}) \cdots \sigma^m(x_{lt_l}^{\beta_{lt_l}}) \right) \right] \\
&= \xi^m \left[\sum_{\sum_{1 \le i \le k} |\beta_i| = s-1} a_\beta \prod_{l=1}^{l=k} \xi^{m p_l q_{k-l}} X_l^{\beta_l} \right],
\end{aligned}
$$

where $p_l = \beta_{l1} + \cdots + \beta_{lt_l}$ for all $1 \le l \le k$. Therefore,

$$
\xi^m \sigma^m(\lambda) = \xi^m \left[\sum_{\sum_{1 \le i \le k} |\beta_i| = s-1} a_\beta \xi^{m \left(\sum_{l=1}^{l=k} p_l q_{k-l} \right)} X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k} \right].
$$

Observe that $q_{k-1} = N$ and $\xi^N = 1$, therefore the above equation reduces to

$$\xi^m \sigma^m(\lambda) = \xi^m \left[\sum_{\sum\limits_{1\le i\le k} |\beta_i| = s-1} a_\beta \xi^{m\left(\sum\limits_{l=2}^{l=k} p_l q_{k-l}\right)} X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k} \right]$$
$$= \sum_{\sum\limits_{1\le i\le k} |\beta_i| = s-1} a_\beta \xi^{m(\delta+1)} X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k},$$

where $\delta = \sum_{l=2}^{l=k} p_l q_{k-l}$. More precisely,

$$\begin{aligned}
\delta &= \sum_{l=2}^{l=k} q_{k-l} p_l \\
&= \sum_{l=2}^{l=k} \left(\sum_{j=0}^{k-l} s^j \right) \left(\beta_{l,1} + \cdots + \beta_{l,t_l} \right) \\
&= \sum_{l=0}^{l=k-2} s^l (\beta_{2,1} + \cdots + \beta_{2,t_2} + \cdots + \beta_{k,1} + \cdots + \beta_{k-l,t_{k-l}}) \\
&\le \sum_{l=0}^{l=k-2} s^l (s-1) = s^{k-1} - 1.
\end{aligned}$$

This implies that $0 < 1 + \delta \le s^{k-1} < N$. Therefore, $\xi^{1+\delta} \neq 1$. Hence

$$\begin{aligned}
\Lambda &= \sum_{m=0}^{N-1} \left[\sum_{\sum\limits_{1\le i\le k} |\beta_i| = s-1} \xi^{m(\delta+1)} a_\beta X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k} \right] \\
&= \sum_{\sum\limits_{1\le i\le k} |\beta_i| = s-1} \left(\sum_{m=0}^{N-1} \xi^{(\delta+1)m} \right) a_\beta X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k} \\
&= \sum_{\sum\limits_{1\le i\le k} |\beta_i| = s-1} \left(\frac{1 - \xi^{N(\delta+1)}}{1 - \xi^{\delta+1}} \right) a_\beta X_1^{\beta_1} X_2^{\beta_2} \cdots X_k^{\beta_k} \\
&= 0.
\end{aligned}$$

Therefore, $d(F) = \Lambda F = 0$. In other words, $F \in K(S)^d$, a contradiction. □

Remark 2.4. From Theorem 2.3, we can prove that the monomial derivation d of $K[x, y, z, w]$ defined by

$$d(x) = w^2, d(y) = zw, d(z) = y^2, d(w) = xy$$

in [4] has no Darboux polynomials if and only if $K(x, y, z, w)^d = K$.

COROLLARY 2.5

The Jouanolou derivation d of $K[x_1, x_2, \ldots, x_n]$ defined by

$$d(x_1) = x_2^s, d(x_2) = x_3^s, \ldots, d(x_{n-1}) = x_n^s, d(x_n) = x_1^s,$$

for $s \geq 1$ and $n \geq 2$ has no Darboux polynomials if and only if $K(x_1, x_2, \ldots, x_n)^d = K$.

3. Generalized cyclotomic derivation in tensor product

Let m and n be positive integers. Assume that $K[X] = K[x_1, \ldots, x_n]$ and $K[Y] = K[y_1, \ldots, y_m]$ are polynomial algebras. Then $K[X] \otimes_K K[Y] \cong K[X, Y] = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a polynomial algebra. If d_1 and d_2 are K-derivations of $K[X]$ and $K[Y]$ respectively, then $d = d_1 \otimes 1 + 1 \otimes d_2$, denoted by $d_1 \oplus d_2$ is the K-derivation of $K[X, Y]$ such that $d|_{K[X]} = d_1$ and $d|_{K[Y]} = d_2$.

In [3], Nowicki and Ollagnier studied the Darboux polynomial of the tensor product of polynomial algebras and have proved the following result.

Lemma 3.1 ([3], Corollary 3.2). *Let d_1 and d_2 be homogeneous K-derivations of $K[X]$ and $K[Y]$ of degree $s \geq 1$. If d_1 and d_2 are without Darboux polynomials, then $d_1 \oplus d_2$ is also without Darboux polynomials.*

Using Lemma 3.1 and our result on generalized cyclotomic derivations, we have the following result.

Theorem 3.2. *Let d_1 and d_2 be homogeneous generalized cyclotomic derivations of $K[X]$ and $K[Y]$ of degree $s \geq 1$. Then, $d_1 \oplus d_2$ is without Darboux polynomial if and only if $K(X, Y)^{d_1 \oplus d_2} = K$.*

Proof.

($\Rightarrow$) Trivial to prove.
($\Leftarrow$) Let $K(X, Y)^{d_1 \oplus d_2} = K$. As $K(X)^{d_1} \subseteq K(X, Y)^{d_1 \oplus d_2} = K$, we have $K(X)^{d_1} = K$. Similarly, $K(Y)^{d_2} = K$. Therefore, from Theorem 2.3, d_1 and d_2 are without Darboux polynomials. Then by Lemma 3.1, $d_1 \oplus d_2$ is also without Darboux polynomials. □

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