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Rota–Baxter operators on groups

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Abstract. The theory of Rota–Baxter operators on rings and algebras has been developed since 1960. In 2020, the notion of Rota–Baxter operator on a group was defined. Further, it was proved that one may define a skew left brace on any group endowed with a Rota–Baxter operator. Thus, a group endowed with a Rota–Baxter operator gives rise to a set-theoretical solution to the Yang–Baxter equation. We provide some general constructions of Rota–Baxter operators on a group. Given a map on a group, we study its extensions to a Rota–Baxter operator. We state the connection between Rota–Baxter operators on a group and Rota–Baxter operators on an associated Lie ring. We describe Rota–Baxter operators on sporadic simple groups.

Keywords. Rota–Baxter operator; Rota–Baxter group; simple group; sporadic group; factorization.

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1. Introduction

Rota–Baxter operators for commutative algebras firstly appeared in the paper of Baxter [4] in 1960. Since that time the theory of Rota–Baxter operators has been developed by different authors including J-C Rota, L Guo, C Bai and K Ebrahimi-Fard, see details in the monograph [18]. Let us emphasize the important connections of Rota–Baxter operators to Yang–Baxter equation [5,26], Loday algebras [2,17], double Poisson algebras [14,25] and others.

In 2020, Guo *et al.* [19] defined the notion of Rota–Baxter operator (RB-operator, in short) on a group. A map $B\colon G \to G$ is called a Rota–Baxter operator (of weight 1) if $B(g)B(h) = B(gB(g)hB(g)^{-1})$ for all $g, h \in G$. A group G with a Rota–Baxter operator B is called a Rota–Baxter group.

The main motivation of [19] to introduce this new notion is the following: Given a Rota–Baxter Lie group (G, B), the tangent map of B at the identity is a Rota–Baxter operator

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of weight 1 on the Lie algebra of the Lie group G. If a crossed homomorphism ϕ from a group G to itself is invertible, then ϕ^{-1} is an RB-operator on G [19]. Some basic examples and properties of Rota–Baxter groups were also stated in the pioneering work [19].

Goncharov [13] introduced the notion of a Rota–Baxter operator on a cocommutative Hopf algebra as a combination of both Rota–Baxter operators on a group and on a Lie algebra. In particular, every such Rota–Baxter operator on the Hopf algebra $\Bbbk[G]$ arises as a linear extension of an RB-operator defined on a group G.

Skew left braces were introduced in the paper of Guarnieri and Vendramin [15] for studying set-theoretical solutions to the Yang–Baxter equation. Skew left braces are connected with regular subgroups of the group holomorphs, with bijective 1-cocycles. In [3], it was proved that any group with a Rota–Baxter operator defines a skew left brace. Also, the connections of Rota–Baxter groups with regular subgroups of the group holomorphs, with bijective 1-cocycles, and with set-theoretical solutions to the Yang–Baxter equation were found.

The *main goal* of the current work is to give different constructions of Rota–Baxter operators on a group. Let us highlight three key results of the work directly.

In [1,10,16], Rota–Baxter operators on the algebra $\Bbbk^n = \Bbbk \oplus \cdots \oplus \Bbbk$ were studied. Given a group G and an RB-operator R on $\Bbbk^n$, we obtain an RB-operator B on $G^n = G \times G \times \cdots \times G$ in a canonical way.

Theorem 32. *Let $G^n = G \times G \times \cdots \times G$ and let $\Bbbk$ be a field. Let R be a Rota–Baxter operator of weight 1 on $\Bbbk e_1 \oplus \Bbbk e_2 \oplus \cdots \oplus \Bbbk e_n$, $R(e_i) = \sum_{k=1}^{n} r_{ik} e_k$, and the matrix of R is upper-triangular. Define a map $B\colon G^n \to G^n$ as follows*:

$$B((g_1, \ldots, g_n)) = (t_1, \ldots, t_n), \quad t_i = g_i^{r_{ii}} g_{i-1}^{r_{i-1\,i}} \ldots g_1^{r_{1i}}.$$

Then B is a Rota–Baxter operator on G^n.

Let G be a group. Consider its lower central series G_i, $i = 1, 2, \ldots$, where $G_1 = G$ and $G_{i+1} = [G, G_i]$. The associated graded abelian group $L(G) = \bigoplus_{n \geq 1} G_n/G_{n+1}$ is a Lie ring under the product $[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1}$, where $[x, y]$ is the group commutator.

Theorem 46. *Let G be a group and let B be a Rota–Baxter operator on G such that $B(G_n) \subset G_n$ for all $n \geq 1$. Then R defined as $R(xG_{i+1}) = B(x)G_{i+1}$ is a Rota–Baxter operator of weight 1 on the Lie ring $L(G)$.*

By elementary Rota–Baxter operators, we mean maps B_0 and B_{-1}, which act on a group G as follows: $B_0(g) = e$ and $B_{-1}(g) = g^{-1}$.

It is known that given a group G, every exact factorization $G = HL$, where H, L are subgroups of G such that $H \cap L = \{e\}$, defines a so-called splitting Rota–Baxter operator [19], which acts by the formula $B(hl) = l^{-1}$.

Theorem 53. *Let G be a sporadic simple group. If $G = M_{22}$ or G is not a Mathieu group, then G has only elementary RB-operators. If $G \in \{M_{11}, M_{12}, M_{23}, M_{24}\}$, then all RB-operators on G are splitting* (*see Table* 1).

Let us give a short outline of the paper. In §2, we give the required preliminaries on Rota–Baxter operators on groups. Here, we also consider the new group product $g \circ h = gB(g)hB(g)^{-1}$ on a group G with an RB-operator B. The group $G_B = \langle G, \circ \rangle$ was defined in [19]. In §3, we study constructions of RB-operators appeared due to exact factorizations (double and triple) of groups and properties of such constructions. In §4,

Table 1. Exact factorizations of sporadic groups [12].

G	H	L
M_{11}	M_{10}	$\mathbb{Z}_{11}$
	$M_9.2$	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$
M_{12}	M_{11}	A_4, D_{12}, $\mathbb{Z}_6 \times \mathbb{Z}_2$
	$M_9.2$	$\mathrm{PSL}_2(11)$
M_{23}	M_{22}	$\mathbb{Z}_{23}$
	$\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$	$\mathrm{P\Sigma L}_3(4)$, $2^4 \rtimes A_7$
M_{24}	M_{23}	$A_4 \times \mathbb{Z}_2$, D_{24}, $D_8 \rtimes \mathbb{Z}_3$
	M_{23}	S_4
	M_{23}	B_{24}
	$\mathrm{PSL}_2(23)$	$\mathrm{P\Sigma L}_3(4)$, $2^4 \rtimes A_7$

we give constructions of RB-operators involved homomorphisms, projections and some exact formulas. We find RB-operators on n-abelian groups (Proposition 22) and on 2-step nilpotent groups (Corollary 25). In §5, we study RB-operators on direct products of groups. In §6, we study extensions of a given map β on a group G to an RB-operator B on G. Applying the necessary condition, we construct and study the auxiliary group $\bar{G}_\beta$. In §7, the problems concerning extensions of Rota–Baxter groups are posted. In §8, we find a connection between RB-operators on a group and RB-operators on its associated Lie ring. In §9, we state that on a finite non-abelian simple group G every Rota–Baxter operator with a trivial kernel has the form $B(g) = g^{-1}$ (Theorem 50). Applying this result and the list of all factorizations of sporadic groups [12], we describe all RB-operators on all sporadic simple groups in Theorem 53.

2. Preliminaries

Let A be an algebra over a field $\Bbbk$. A linear operator R on A is called a Rota–Baxter operator of weight $\lambda \in \Bbbk$ if

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy). \tag{1}$$

for all $x, y \in A$. An algebra endowed with a Rota–Baxter operator is called a Rota–Baxter algebra.

Let us consider an analogue of Rota–Baxter operator of weight ± 1 on a group.

DEFINITION 1 [19]

Let G be a group.

(a) A map $B\colon G \to G$ is called a *Rota–Baxter operator* of weight 1 if

$$B(g)B(h) = B(gB(g)hB(g)^{-1}) \tag{2}$$

for all $g, h \in G$.

(b) A map $C\colon G \to G$ is called a *Rota–Baxter operator* of weight -1 if

$$C(g)C(h) = C(C(g)hC(g)^{-1}g) \tag{3}$$

for all $g, h \in G$.

A group endowed with a Rota–Baxter operator is called a *Rota–Baxter group* (RB-group). Definition 1(b) implies the following result directly.

Lemma 2. *If $C\colon G \to G$ is a Rota–Baxter operator of weight -1 on a group G, then the operator $\widetilde{C}(g) = gC(g^{-1})$, $g \in G$, is also a Rota–Baxter operator of weight -1.*

In [19], it was proved that if B is a Rota–Baxter operator of weight 1 on a group G, then the map $C(g) = B(g^{-1})$ is a Rota–Baxter operator of weight -1. Another connection between Rota–Baxter operators of weights 1 and -1 gives the following.

PROPOSITION 3

If $B\colon G \to G$ is a Rota–Baxter operator of weight 1 on a group G, then the operator $C(g) = gB(g)$, $g \in G$, is a Rota–Baxter operator of weight -1 on G.

Proof. One needs to check (2). By the definition of C, it is equivalent to

$$gB(g)hB(h) = C(gB(g)hB(g)^{-1}),$$

i. e.,

$$B(g)B(h) = B(gB(g)hB(g)^{-1}).$$

□

Hence, we have a bijection between Rota–Baxter operators of weight 1 and -1 on a group G. At the end of the paper, we will consider only Rota–Baxter operators of weight 1 and will call them simply Rota–Baxter operators (RB-operators).

Example 4. Let G be a group. Then

(a) the map $B_0(g) = e$ is an RB-operator on G,
(b) the map $B_{-1}(g) = g^{-1}$ is an RB-operator on G.

We will call the Rota–Baxter operators B_0 and B_{-1}, *the elementary RB-operators*. A group G is called *RB-elementary*, if any RB-operator on G is elementary. It is evident that every cyclic group of prime order is RB-elementary. Other examples of RB-elementary groups will be given in §9.

Given a group G, we use the usual notation of the commutator $[g, h] = g^{-1}h^{-1}gh$ for $g, h \in G$; the commutator subgroup $[H, L]$ for subgroups H, L of G is a subgroup that is generated by the set of commutators $\{[h, l] \mid h \in H,\ l \in L\}$.

Let us state some elementary facts about RB-operators.

Lemma 5. *If* $B : G \to G$ *is a Rota–Baxter operator on a group* G*, then*

(*a*) $B(e) = e$;
(*b*) $B(g)B(g^{-1}) = B([g^{-1}, B(g)^{-1}])$;
(*c*) $B(g)B(B(g)) = B(gB(g))$;
(*d*) $B(g)^{-1} = B(B(g)^{-1}g^{-1}B(g))$ *for any* $g \in G$.

Proof.

(a) It follows from (2), considered with $g = h = e$.
(b) It follows from (2), considered with $h = g^{-1}$.
(c) It follows from (2), considered with $h = B(g)$.
(d) It follows from the equalities that

$$B(g)B(B(g)^{-1}g^{-1}B(g)) = B(g\underline{B(g)B(g)^{-1}}g^{-1}\underline{B(g)B(g)^{-1}}) \\ = B(gg^{-1}) = B(e) = e.$$

Thus the lemma is proved.

□

In [19], it was noted (without proof) that given a group G and a Rota–Baxter operator B on G, the sets $\ker(B) = \{g \in B \mid B(g) = e\}$ and $\mathrm{Im}\,(B) = \{B(g) \mid g \in G\}$ are subgroups of G. Let us write down the proof, for convenience.

Lemma 6 [19]. *Let* B *be a Rota–Baxter operator on a group* G*. Then*

(*a*) $\ker(B)$ *is a subgroup of* G,
(*b*) $\mathrm{Im}\,(B)$ *is a subgroup of* G.

Proof.

(a) We have $B(e) = e$, by Lemma 5(a). Let $g, h \in \ker(B)$. Then by (2),

$$B(g^{-1}) = B(g)B(g^{-1}) = B(gB(g)g^{-1}B(g)^{-1}) = B(e) = e, \\ e = B(g)B(h) = B(gB(g)hB(g)^{-1}) = B(gh).$$

(b) Directly by (2) we have that if $g, h \in \mathrm{Im}\,(B)$, then $gh \in \mathrm{Im}\,(B)$. Suppose $g \in \mathrm{Im}\,(B)$, then $g^{-1} \in \mathrm{Im}\,(B)$, by Lemma 5(d).

□

If a Rota–Baxter operator B on a group G is invertible, i.e., the map $B^{-1} : G \to G$ exists, then B^{-1} is a crossed homomorphism from G to itself [19]. The following result can be useful in attempting to construct RB-operators B on a given group with $\ker(B) \neq \{e\}$.

Lemma 7. *Let* B *be a Rota–Baxter operator on a group* G*. If* $B(g) = e$ *for some* $g \in G$*, then* $B(h) = B(gh)$ *for any* $h \in G$*. In particular, if*

$$G = \coprod_{i \in I} \ker(B) g_i$$

is the decomposition of G in the disjoint union of cosets, then $B(x)=B(y)$ if x and y lie in the same coset.

Proof. It follows from (2), considered with such g and h. □

The following claim gives a possibility to construct new RB-operators from the known ones.

Lemma 8 [19]. *Let B be a Rota–Baxter operator on a group G. Then the operator $\tilde{B}(g)=g^{-1}B(g^{-1})$ is also a Rota–Baxter operator on a group G.*

In particular, $\widetilde{B_0}=B_{-1}$. It is easy to see that $\tilde{\tilde{B}}=B$.

The following observation is very important when one is interested in the classification of all RB-operators on a given group.

Lemma 9. *Let B be a Rota–Baxter operator on a group G. Let φ be an automorphism of G. Then $B^{(\varphi)}=\varphi^{-1}B\varphi$ is a Rota–Baxter operator on a group G.*

Proof. The following equalities imply the statement

$$\begin{aligned}\varphi(B^{(\varphi)}(g)B^{(\varphi)}(h)) &= \varphi(\varphi^{-1}(B(\varphi(g)))\varphi^{-1}(B(\varphi(h))))\\ &= B(\varphi(g))B(\varphi(h))\\ &= B(\varphi(g)B(\varphi(g))\varphi(h)B(\varphi(g))^{-1})\\ &= \varphi\varphi^{-1}B\varphi(g\varphi^{-1}B(\varphi(g))h\varphi^{-1}(B(\varphi(g))^{-1}))\\ &= \varphi(B^{(\varphi)}(gB^{(\varphi)}(g)h(B^{(\varphi)}(g))^{-1})).\end{aligned}$$

□

Thus, we may describe Rota–Baxter operators on a group up to conjugation by its automorphism.

Lemma 10. *Let G be a group, φ be an automorphism of G, and B be a Rota–Baxter operator on G. Then $(\tilde{B})^{(\varphi)}=\widetilde{B^{(\varphi)}}$.*

Proof. We have

$$\begin{aligned}(\tilde{B})^{(\varphi)}(g) &= \varphi^{-1}(\tilde{B}(\varphi(g)))) = \varphi^{-1}((\varphi(g))^{-1}B((\varphi(g))^{-1}))\\ &= g^{-1}B^{(\varphi)}(g^{-1}) = \widetilde{B^{(\varphi)}}(g).\end{aligned}$$

□

Let (G,B) and (G',B') be Rota–Baxter groups. A map $\Phi\colon G\to G'$ is a Rota–Baxter group homomorphism if Φ is a group homomorphism such that $\Phi B=B'\Phi$.

In [19], a new binary operation $\circ\colon G\to G$ on a Rota–Baxter group (G,B) was defined.

PROPOSITION 11 [19]

Let $(G, \cdot, B)$ be a Rota–Baxter group. Then

(*a*) *The pair $(G, \circ)$ with the product*

$$g \circ h = gB(g)hB(g)^{-1}, \tag{4}$$

where $g, h \in G$, is also a group.

(*b*) *The operator B is a Rota–Baxter operator on the group $(G, \circ)$.*

(*c*) *The map $B\colon (G, \circ) \to (G, \cdot)$ is a homomorphism of Rota–Baxter groups.*

We will denote the group $(G, \circ)$ as G_B.

Given a group G and a Rota–Baxter operator B on G, consider a map B_+ on G defined as follows: $B_+(g) = gB(g)$ [19]. Note that B_+ is a group homomorphism from G_B to G. Lemma 5(c) mentions that $B_+B = BB_+$.

Example 12. Let G be a group with a Rota–Baxter operator B.

(a) If $B = B_0$, then

$$g \circ h = gB_0(g)hB_0(g)^{-1} = gh.$$

Hence, both products $\circ$ and $\cdot$ coincide.

(b) If $B = B_{-1}$, then

$$g \circ h = gB_{-1}(g)hB_{-1}(g)^{-1} = hg.$$

Hence, $(G, \circ)$ is a group with the opposite product.

(c) If G is abelian, then $g \circ h = g \cdot h$ for all $g, h \in G$.

A group G is called factorizable if $G = HL$ for subgroups H and L. The expression $G = HL$ is called a factorization of G. If additionally $H \cap L = \{e\}$, then such a factorization is called exact. If H and L are proper subgroups of G, we call a factorization $G = HL$ as a proper one.

In [19], Guo *et al.* proved the results analogous to the ones stated by Semenov-Tyan-Shanskii [26] for Lie algebras.

PROPOSITION 13 [19]

If (G, B) is an RB-group, then

(*a*) $\ker(B)$ *and* $\ker(B_+)$ *are normal in* G_B;

(*b*) $\ker(B) \trianglelefteq \mathrm{Im}\,(B_+)$ *and* $\ker(B_+) \trianglelefteq \mathrm{Im}\,(B)$ *in* G;

(*c*) *We have the isomorphism of quotient groups*

$$\mathrm{Im}\,(B_+)/\ker(B) \cong \mathrm{Im}\,(B)/\ker(B_+); \tag{5}$$

(*d*) *We have the factorization*

$$G = \mathrm{Im}\,(B_+)\mathrm{Im}\,(B). \tag{6}$$

Let us find the exact formulas for words written in terms of the operation $\circ$ in G_B. Given an integer k, denote by $a^{\circ(k)}$ the k-th power of a under the product $\circ$.

PROPOSITION 14

Let (G, B) be a Rota–Baxter group. If A is a subset of G and

$$w = a_{i_1}^{\circ(k_1)} \circ a_{i_2}^{\circ(k_2)} \circ \cdots \circ a_{i_s}^{\circ(k_s)}, \quad a_{i_j} \in A,\ k_j \in \mathbb{Z}$$

is presented by a word under the operation $\circ$, then

$$w = (a_{i_1}B(a_{i_1}))^{k_1}(a_{i_2}B(a_{i_2}))^{k_2}(a_{i_s}B(a_{i_s}))^{k_s}B(a_{i_s})^{-k_s}B(a_{i_{s-1}})^{-k_{s-1}}\cdots B(a_{i_1})^{-k_1}. \tag{7}$$

In particular, $a^{\circ(k)} = (B_+(a))^k(B(a))^{-k}$ for every integer k and $a^{\circ(-1)} = B(a)^{-1}a^{-1}B(a)$.

Proof. Firstly, it is easy to show by induction on $n \in \mathbb{N}$ that $a^{\circ(n)} = (B_+(a))^n(B(a))^{-n}$. Indeed, for $n = 0$, we have $e = e$. The induction step follows from the equalities

$$a^{\circ(n)} = a \circ (a^{\circ(n-1)}) = aB(a)a^{\circ(n-1)}B(a)^{-1} = (aB(a))^n B(a)^{-n}.$$

One can check directly that $B(a)^{-1}a^{-1}B(a)$ is the inverse element to a in the group G_B. Now, again, by induction on $n \in \mathbb{N}$ and by Lemma 5(d), we compute

$$\begin{aligned} a^{\circ(-n)} &= a^{\circ(-1)} \circ (a^{\circ(-(n-1))}) \\ &= B(a)^{-1}a^{-1}\underline{B(a)B(B(a)^{-1}a^{-1}B(a))}(aB(a))^{-(n-1)} \\ &\quad \times B(a)^{n-1}(B(B(a)^{-1}a^{-1}B(a)))^{-1} \\ &= (aB(a))^{-1}(aB(a))^{-(n-1)}B(a)^{n-1}B(a)^1 \\ &= (aB(a))^{-n}B(a)^n. \end{aligned}$$

Denote by $w' = a_{i_2}^{\circ(k_2)} \circ \cdots \circ a_{i_s}^{\circ(k_s)}$. Applying that B is a homomorphism from G_B to G, we get

$$\begin{aligned} w = a_{i_1}^{\circ(k_1)} \circ w' &= (B_+(a_{i_1}))^{k_1}\underline{(B(a_{i_1}))^{-k_1}B(a_{i_1}^{\circ(k_1)})}w'B(a_{i_1}^{\circ(k_1)})^{-1} \\ &= (B_+(a_{i_1}))^{k_1}w'(B(a_{i_1}))^{-k_1}. \end{aligned}$$

Thus, by induction on s we derive formula (7). □

3. Constructions via factorizations

The next example allows us to construct non-elementary RB-operators.

Example 15 [19]. Let G be a group. Given an exact factorization $G = HL$, a map $B\colon G \to G$ is defined as

$$B(hl) = l^{-1},$$

which is a Rota–Baxter operator on G.

Let us call such Rota–Baxter operator on G a *splitting Rota–Baxter operator*. Note that in this case $B_+(hl) = h$ and $G_B \cong H \times L^{\mathrm{op}}$, where L^{op} is the set L with the opposite product $l * l' = l'l$.

PROPOSITION 16

Let G be a group and let $B\colon G \to G$ be an RB-operator on G. Then B is a splitting Rota–Baxter operator on G if and only if $B(gB(g)) = e$ for all $g \in G$.

Proof. Suppose that B is a splitting RB-operator on G with subgroups H and L. Take $g = hl$, where $h \in H$ and $l \in L$. Then

$$B(gB(g)) = B(hlB(hl)) = B(hll^{-1}) = B(h) = e.$$

Conversely, suppose that $B(gB(g)) = e$ for all $g \in G$. Let us prove that $G = \ker(B)\mathrm{Im}\,(B)$ and this factorization is exact. Firstly, let $a \in \ker(B) \cap \mathrm{Im}\,(B)$. Then $B(a) = e$ and there exists $g \in G$ such that $a = B(g)$. By the assumption, we have $e = B(gB(g)) = B(ga)$. Since $ga, a \in \ker(B)$, by Lemma 6(a), we conclude that $g \in \ker(B)$. So, $a = e$.

Secondly, let $g \in G$. The equality $B(gB(g)) = e$ implies that $gB(g) = a \in \ker(B)$. Thus, $g = ab$, where $b = B(g)^{-1} \in \mathrm{Im}\,(B)$, by Lemma 6(b). □

Remark 17. By Lemma 5, the condition $B(gB(g)) = e$ is equivalent to the relation $B(g)B^2(g) = e$ or $B^2(g) = B(g)^{-1}$. It means that if B is a splitting RB-operator, then B inverts all elements of $\mathrm{Im}\,(B)$.

Let us extend Example 15 on some triple factorizations.

PROPOSITION 18

Let G be a group such that $G = HLM$, where H, L and M are subgroups of G with pairwise trivial intersections. Let C be a Rota–Baxter operator on L. Moreover, $[H, L] = [C(L), M] = e$. Then the map $B\colon G \to G$ defined by the formula

$$B(hlm) = C(l)m^{-1}$$

is a Rota–Baxter operator on G.

Proof. Let $h, h' \in H$, $l, l' \in L$, and $m, m' \in M$. Then

$$\begin{aligned} C(l)C(l')m^{-1}m'^{-1} &= C(l)m^{-1}C(l')m'^{-1} \\ &= B(hlm)B(h'l'm') \\ &= B(hlmB(hlm)h'l'm'B(hlm)^{-1}) \\ &= B(hlmC(l)m^{-1}h'l'm'mC(l)^{-1}) \\ &= B(hlC(l)\underline{mm^{-1}}h'l'C(l)^{-1}m'm) \\ &= B(hh'lC(l)l'C(l)^{-1}m'm) \\ &= C(lC(l)l'C(l)^{-1})m^{-1}m'^{-1}, \end{aligned}$$

which is fulfilled by (2). □

Let us call a Rota–Baxter operator on $G = HLM$ defined by Proposition 18 with the help of C as a *triangular-splitting* one. Note that $G_B \cong H \times L_C \times M^{\mathrm{op}}$.

COROLLARY 19

Let $G = \langle a, b\rangle$ be two generated 2-step nilpotent group. Put $c = [b, a]$. Then $G = \langle a\rangle\langle c\rangle\langle b\rangle$ and for any integer k, the map

$$B(a^\alpha c^\beta b^\gamma) = c^{k\beta} b^{-\gamma}, \quad \alpha, \beta, \gamma \in \mathbb{Z},$$

is an RB-operator on G.

Given a splitting RB-operator B on a group G, we have an exact factorization

$$G = \ker(B)\mathrm{Im}\,(B). \tag{8}$$

It is easy to see that if G is abelian, then there exists a non-splitting RB-operator B on G such that (8) is fulfilled. Indeed, if G is a direct sum of abelian subgroups: $G = H \times L$, then any map $hl \mapsto \varphi(l)$, where φ is an endomorphism of L, defines an RB-operator on G satisfying (8).

The next claim generalizes this observation.

PROPOSITION 20

Given a semidirect product $G = H \rtimes L$, let C be a Rota–Baxter operator on L. Then a map $B\colon G \to G$ defined by the formula $B(hl) = C(l)$, where $h \in H$ and $l \in L$ is a Rota–Baxter operator.

Proof. We state the claim by the following computations for $h, h' \in H$ and $l, l' \in L$,

$$\begin{aligned} B(hl)B(h'l') &= C(l)C(l'),\\ B(hlB(hl)h'l'B(hl)^{-1}) &= B(hlC(l)h'l'C(l)^{-1})\\ &= B(hh'^{C(l)^{-1}l^{-1}}lC(l)l'C(l)^{-1})\\ &= C(lC(l)l'C(l)^{-1})\\ &= C(l)C(l'). \end{aligned}$$

□

Note that $G_B \cong H \rtimes L_C$.

4. Constructions via homomorphisms, projections, etc.

When G is an abelian group, then B is a Rota–Baxter operator on G if and only if B is an endomorphism of G.

PROPOSITION 21

(*a*) *Let* (G, B) *be a Rota–Baxter group and* B *be an automorphism of* G. *Then* G *is abelian.*
(*b*) *If* G *is a group and* H *is its abelian subgroup, then any homomorphism* (*or antihomomorphism*) $B: G \to H$ *is a Rota–Baxter operator.*

Proof. Suppose that a Rota–Baxter operator $B: G \to G$ is an endomorphism, then

$$\underline{B(g)}B(h) = B(gB(g)hB(g)^{-1}) = \underline{B(g)}B^2(g)B(h)(B^2(g))^{-1}, \quad g, h \in G.$$

It is equivalent to the relation

$$[B^2(g), B(h)] = e. \tag{9}$$

(a) If B is an automorphism, then G is an abelian by (9).
(b) If $\mathrm{Im}\,(B) \subset H$, where H is abelian, then B is an RB-operator by the stated above equivalence of (2) and (9).

When B is antihomomorphism, the proof is similar. □

The following constructions of Rota–Baxter operators work only for groups very close to abelian ones.

Given an integer k, a group G is called k-abelian [23] if $(gh)^k = g^k h^k$ for all $g, h \in G$.

PROPOSITION 22

Let G *be a group, and fix a natural* n. *A map* $B: G \to G$ *defined as* $B(g) = g^n$ *is a Rota–Baxter operator on* G *if and only if* G *is* $(n+1)$-*abelian.*

Proof. Given $g, h \in G$, the identity (2) for the map $B(x) = x^n$ has the form

$$g^n h^n = (g^{n+1} h g^{-n})^n = g^{n+1} h g^{-n} g^{n+1} h g^{-n} g^{n+1} \cdots h g^{-n} = g^n (gh)^n g^{-n}.$$

So, it is equivalent to the equality $(gh)^n = h^n g^n$ or to the following one:

$$(hg)^{n+1} = h(gh)^n g = h^{n+1} g^{n+1}.$$

It means that B is a Rota–Baxter operator on G if and only if G is $(n+1)$-abelian. □

Since every k-abelian group is also $(1-k)$-abelian, we may represent the induced product in G_B as follows:

$$g \circ h = g g^n h g^{-n} = g^{n+1} h^{n+1} h^{-n} g^{-n} = (gh)^{n+1} (hg)^{-n}.$$

Remark 23. The statement of Proposition 22 in "if" direction follows from Proposition 21, since the n-th power of a $(n+1)$-abelian group G lies in $Z(G)$ [20].

Given a group G and $g \in G$, we use the following notation, $[g, G] = \{[g, h] \mid h \in G\}$.

PROPOSITION 24

Let G be a group. Then a map $B(x) = g^{-1}x^{-1}g$ is a Rota–Baxter operator on G if and only if $[g, G] \subset Z(G)$.

Proof. Suppose that $B(x) = g^{-1}x^{-1}g$ is a Rota–Baxter operator on G. It means that

$$g^{-1}x^{-1}y^{-1}g = g^{-1}(xg^{-1}x^{-1}gyg^{-1}xg)^{-1}g = g^{-1}g^{-1}x^{-1}gy^{-1}g^{-1}xgx^{-1}g$$

for all $x, y \in G$. This equality is equivalent to the following one:

$$x^{-1}y^{-1} = [g, x]x^{-1}y^{-1}[g, x^{-1}].$$

Denote by $s = x^{-1}y^{-1}$, then we have $[x, g]^s = [g, x^{-1}]$ for all $x, s \in G$. Fixing $x \in G$, we conclude that $[g, x] \in Z(G)$ for every $x \in G$.

Conversely, suppose that $g \in G$ is such an element that $[g, x] \in Z(G)$ for every $x \in G$. By the previous part, it is enough to state that $[x, g] = [g, x^{-1}]$ for all $x, g \in G$. Since

$$[g, x^{-1}] = g^{-1}xgx^{-1} = x^{-1}g^{-1}xg[g^{-1}xg, x^{-1}] = [x, g] \cdot [x^g, x^{-1}],$$

we have to prove that $[x^g, x^{-1}] = e$. Denote $c = [g, x^{-1}]$. We finish the proof by the formulas

$$[x^g, x^{-1}] = [[g, x^{-1}]x, x^{-1}] = [cx, x^{-1}] = x^{-1}c^{-1}xcxx^{-1} = e.$$

□

Given a group G with an element g such that $[g, G] \subset Z(G)$, we may compute the product of G_B, where $B(x) = g^{-1}x^{-1}g$ is

$$x \circ y = xB(x)yB(x)^{-1} = xg^{-1}x^{-1}gyg^{-1}xg = yxg^{-1}x^{-1}gg^{-1}xg = yx,$$

so this product coincides with the product induced by the RB-operator B_{-1}.

COROLLARY 25

Let G be a 2-step nilpotent group, $g \in G$, then the map $B_g(x) = g^{-1}x^{-1}g$ is a Rota–Baxter operator.

It is evident that this map B_g is a bijection. Denote by $\mathcal{RB}(G)$ the set of all Rota–Baxter operators on G. Hence, when G is a 2-step nilpotent group, we have a map $\varphi\colon G/[G, G] \to \mathcal{RB}(G)$ defined as follows: $\varphi(g) = B_g$.

Let us try to construct a Rota–Baxter operator by the formula $B(g) = agb$ with the help of some fixed a and b. In comparison to Proposition 24, we allow a and b be not necessarily inverse to each other, however, we do not inverse g.

COROLLARY 26

Let G be a group, and fix $a, b \in G$. A map $B\colon G \to G$ defined as $B(g) = agb$, is a Rota–Baxter operator on G if and only if G is abelian and $b = a^{-1}$, i.e., $B = \mathrm{id}$.

Proof. If G is abelian, then every endomorphism of G is a Rota–Baxter operator on G, including the identity map.

Conversely, suppose that a map B on G defined by the formula $B(g) = agb$ is a Rota–Baxter operator on G. Since $B(e) = e$, we have $b = a^{-1}$. Thus, B is an automorphism of G and by Proposition 21(a), we have that G is abelian, and so $B = \mathrm{id}$. □

DEFINITION 27

A Rota–Baxter group (G, B) has the *deep* m (m is a natural number or ω) if m is a minimal such that

$$G > B(G) > B^2(G) > \cdots > B^m(G) = B^{m+1}(G).$$

Note that the RB-group (G, B_{-1}) has deep 0 and the RB-group (G, B), where $|G| > 1$, $B = B_0$ or B is splitting, has deep 1.

The following example shows that there are RB-groups of infinite deep.

Example 28. Let $(\mathbb{Z}, +)$ be the infinite cyclic group and $B\colon \mathbb{Z} \to \mathbb{Z}$, $B(x) = 2x$ for any $x \in \mathbb{Z}$. Then $(\mathbb{Z}, B)$ is a Rota–Baxter group and it has the deep ω since

$$\mathbb{Z} > B(\mathbb{Z}) = 2\mathbb{Z} > B^2(\mathbb{Z}) = 4\mathbb{Z} > \cdots > B^{\omega}(\mathbb{Z}) = \bigcap_{k=1}^{\infty} B^k(\mathbb{Z}) = 0.$$

If $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is a free non-abelian group, then we can define its projection $\pi_1\colon F_n \to \langle x_1 \rangle$ by

$$\pi_1(x_1) = x_1, \quad \pi_1(x_i) = e, \quad i \neq 1.$$

Then $F_n = \ker(\pi_1)\langle x_1 \rangle$ and for any integer k the map $B(g) = \pi_1(g)^k$ is a Rota–Baxter operator on F_n. The following proposition generalizes this observation.

PROPOSITION 29

Let $G = K * L$ be a free product with abelian L and $\pi_1\colon G \to L$ be a projection on the second component. Then, for any endomorphism χ of L, the map $B(g) = \chi(\pi_1(g))$ is a Rota–Baxter operator on G.

Question 1. Let G be a group and H be its proper subgroup. Under which conditions there is a Rota–Baxter operator B such that $B(G) = H$? In particular, if F is a free non-abelian group, is there an RB-operator on F such that its image is the commutator subgroup F' of F?

5. Constructions via direct products

Using RB-operators on some groups, we can define RB-operators on direct products of these groups as follows.

PROPOSITION 30

Let G be a group such that $G = H \times L$.

(*a*) *Let B_H be a Rota–Baxter operator on H and let B_L be a Rota–Baxter operator on L. Then a map $B\colon G \to G$ defined by the formula $B(hl) = B_H(h)B_L(l)$, where $h \in H$ and $l \in L$ is a Rota–Baxter operator.*

(*b*) *If L is abelian and $|L| > 2$, then there exists a non-splitting Rota–Baxter operator on G.*

Proof.

(a) It is straightforward.

(b) On the contrary, suppose that every Rota–Baxter operator on G is splitting. Define a Rota–Baxter operator $B(\psi)$ on G with the help of an automorphism ψ of L as $B(\psi)((h,l)) = \psi(l)$. By Proposition 21, it is a Rota–Baxter operator on G. By Proposition 16, we should have $B(lB(l)) = \psi(l)\psi^2(l) = e$ for all $l \in L$. Take $\psi = \mathrm{id}$, so we have $l^2 = e$ for all $l \in L$. Since L is abelian, we get that L is a direct sum of copies of $\mathbb{Z}_2$. By the condition $|L| > 2$, we may find nonzero and different $l_1, l_2 \in L$. Let us consider an automorphism ψ of L which interchanges l_1 and l_2 and does not change all other elements. It is easy to see that $B(\psi)$ is non-splitting. We arrive at a contradiction. □

It is easy to check that the following example gives Rota–Baxter operators on G^n, and we will generalize it.

Example 31. Let $G^n = G \times G \times \ldots \times G$. Then the following maps from G^n to G^n:

$$B((g_1,\ldots,g_n)) = (e, g_1, g_2g_1, g_3g_2g_1, \ldots, g_{n-1}g_{n-2}\cdots g_1),$$
$$\widetilde{B}((g_1,\ldots,g_n)) = \left(g_1^{-1}, g_2^{-1}g_1^{-1}, g_3^{-1}g_2^{-1}g_1^{-1}, \ldots, g_n^{-1}g_{n-1}^{-1}g_{n-2}^{-1}\cdots g_1^{-1}\right)$$

are Rota–Baxter operators.

These Rota–Baxter operators may be extended to the ones on the infinite Cartesian product of G.

Let R be a Rota–Baxter operator of weight 1 on the direct sum of fields (considered as associative algebra) $\Bbbk^n = \Bbbk e_1 \oplus \Bbbk e_2 \oplus \cdots \oplus \Bbbk e_n$, where $e_ie_j = \delta_{ij}e_i$. A linear operator $R(e_i) = \sum_{k=1}^n r_{ik}e_k$, $r_{ik} \in \Bbbk$ is an RB-operator of weight 1 on $\Bbbk^n$ if and only if the following conditions are satisfied [1,10]:

(1) $r_{ii} = 0$ and $r_{ik} \in \{0,1\}$ or $r_{ii} = -1$ and $r_{ik} \in \{0,-1\}$ for all $k \neq i$;
(2) if $r_{ik} = r_{ki} = 0$ for $i \neq k$, then $r_{il}r_{kl} = 0$ for all $l \notin \{i,k\}$;
(3) if $r_{ik} \neq 0$ for $i \neq k$, then $r_{ki} = 0$ and $r_{kl} = 0$ or $r_{il} = r_{ik}$ for all $l \notin \{i,k\}$.

Moreover, we may suppose that the matrix of R is upper-triangular [10,16].

Theorem 32. *Let $G^n = G \times G \times \cdots \times G$ and let $\Bbbk$ be a field. Let R be a Rota–Baxter operator of weight 1 on $\Bbbk e_1 \oplus \Bbbk e_2 \oplus \cdots \oplus \Bbbk e_n$, $R(e_i) = \sum_{k=1}^n r_{ik}e_k$, and the matrix of R is upper-triangular. Define a map $B\colon G^n \to G^n$ as follows*:

$$B((g_1,\ldots,g_n)) = (t_1,\ldots,t_n), \quad t_i = g_i^{r_{ii}}g_{i-1}^{r_{i-1\,i}}\cdots g_1^{r_{1i}}. \tag{10}$$

Then B is a Rota–Baxter operator on G^n.

Proof. Write down the left-hand and right-hand sides of (2) for B, where $g=(g_1,\dots,g_n)$ and $h=(h_1,\dots,h_n)$:

$$B(g)B(h)=\big(g_1^{r_{11}}h_1^{r_{11}},\, g_2^{r_{22}}g_1^{r_{12}}h_2^{r_{22}}h_1^{r_{12}},\dots,\, g_n^{r_{nn}}g_{i-1}^{r_{n-1\,n}}\dots g_1^{r_{1n}}h_n^{r_{nn}}h_{i-1}^{r_{n-1\,n}}\dots h_1^{r_{1n}}\big), \tag{11}$$

$$B(gB(g)hB(g)^{-1})=\big(t_1^{r_{11}},\, t_2^{r_{22}}t_1^{r_{12}},\dots,\, t_n^{r_{nn}}t_{n-1}^{r_{n-1\,n}}\dots t_1^{r_{1n}}\big), \tag{12}$$

where

$$\begin{aligned}&t_1=g_1g_1^{r_{11}}h_1g_1^{-r_{11}},\quad t_2=g_2g_2^{r_{22}}g_1^{r_{12}}h_2g_1^{-r_{12}}g_2^{r_{22}},\dots,\\ &t_n=g_ng_n^{r_{nn}}g_{n-1}^{r_{n-1\,n}}\dots g_1^{r_{1n}}h_ng_1^{-r_{1n}}\dots g_{n-1}^{-r_{n-1\,n}}g_n^{-r_{nn}}.\end{aligned}$$

Let us prove by induction on $i=1,\dots,n$ that the i-th coordinates of (11) and (12) are equal. For $i=1$, we have

$$g_1^{r_{11}}h_1^{r_{11}}=\big(g_1g_1^{r_{11}}h_1g_1^{-r_{11}}\big)^{r_{11}},$$

since when $r_{11}=0$, it is trivial equality $e=e$, and when $r_{11}=-1$, we get $g_1^{-1}h_1^{-1}=(h_1g_1)^{-1}$.

Suppose we have proved that i-th coordinates of (11) and (12) are equal for all $i<k$, where $2\le k\le n$. Let us prove the equality

$$g_i^{r_{ii}}g_{i-1}^{r_{i-1\,i}}\dots g_1^{r_{1i}}h_i^{r_{ii}}h_{i-1}^{r_{i-1\,i}}\dots h_1^{r_{1i}}=t_i^{r_{ii}}t_{i-1}^{r_{i-1\,i}}\dots t_1^{r_{1i}},$$

for $i=k$. If $r_{ii}=0$, then we may apply the induction hypothesis to $i-1$ and to the coefficients $r'_{s\,i-1}=r_{si}$. If $r_{ii}=-1$, then we again apply the induction hypothesis to $i-1$ and to the coefficients $r'_{s\,i-1}=r_{si}$ to derive that

$$\begin{aligned}t_i^{r_{ii}}t_{i-1}^{r_{i-1\,i}}\dots t_1^{r_{1i}}&=(\underline{g_ig_i^{-1}}g_{i-1}^{r_{i-1\,i}}\dots g_1^{r_{1i}}h_ig_1^{-r_{1i}}\dots g_{i-1}^{-r_{i-1\,i}}g_i)^{-1}t_{i-1}^{r_{i-1\,i}}\dots t_1^{r_{1i}}\\ &=g_i^{-1}g_{i-1}^{r_{i-1\,i}}\dots g_1^{r_{1i}}h_i^{-1}\underline{g_1^{-r_{1i}}\dots g_{i-1}^{-r_{i-1\,i}}g_{i-1}^{r_{i-1\,i}}\dots g_1^{r_{1i}}}h_{i-1}^{r_{i-1\,i}}\dots h_1^{r_{1i}}\\ &=g_i^{r_{ii}}g_{i-1}^{r_{i-1\,i}}\dots g_1^{r_{1i}}h_i^{r_{ii}}h_{i-1}^{r_{i-1\,i}}\dots h_1^{r_{1i}}.\end{aligned}$$ □

Remark 33. Analogously to [16], we may state that $G_B^n\cong G^n$.

COROLLARY 34

We may slightly modify (10) *as follows*: *Let* $\psi_2, \ldots, \psi_n \in \mathrm{Aut}(G)$. *Then a map* $P\colon G^n \to G^n$ *defined by the formula*

$$P((g_1, \ldots, g_n)) = (t_1, \ldots, t_n),$$
$$t_1 = g_1^{r_{11}}, \quad t_i = g_i^{r_{ii}} \psi_i\big(g_{i-1}^{r_{i-1\,i}} \psi_{i-1}\big(g_{i-2}^{r_{i-2\,i}} \ldots \psi_2\big(g_1^{r_{1i}}\big)\big)\big), \ i \geq 2,$$

is a Rota–Baxter operator on G^n.

Proof. Define an automorphism φ of G^n as follows:

$$\varphi((g_1, g_2, g_3, \ldots, g_n)) = \big(g_1, \psi_2^{-1}(g_2), \psi_2^{-1}\psi_3^{-1}(g_3), \ldots, \psi_2^{-1} \ldots \psi_n^{-1}(g_n)\big).$$

By Lemma 9, $B^{(\varphi)}$ is a Rota–Baxter operator on G^n, where B is defined by (10). Note that $P = B^{(\varphi)}$. □

COROLLARY 35

Given a group $G = H \times H \times L$ *for some* L *and a non-trivial group* H, *the map* $B\colon G \to G$ *defined as* $B((h_1, h_2, l)) = (e, h_1, e)$ *is a non-splitting Rota–Baxter operator.*

Proof. Applying Example 31 and Proposition 21, we conclude that B is a Rota–Baxter operator on G. Denote $g = (h_1, h_2, l)$. Since $B(gB(g)) = (e, h_1, e_L)$ is not the identity of G when $h_1 \neq e$, the operator B is not splitting by Proposition 16. □

6. Extensions of maps to RB-operators

In the previous sections, we have considered some possibilities to construct RB-operators on groups. In general case, we formulate as follows.

Question 2. Let $G = \langle A \rangle$ be a group generated by a set $A = \{a_i \mid i \in I\}$. Suppose that $U = \{u_i\}_{i \in I}$ is a subset of G with the same cardinality as A. Under which conditions the map $\beta\colon A \to U$, $a_i \mapsto u_i$, $i \in I$ can be (uniquely) extended to an RB-operator $B\colon G \to G$?

Suppose at first that we know that (G, B) is an RB-group. What is a minimal subset $M \subseteq G$ such that the values $B(m)$ for all $m \in M$ uniquely define the values $B(g)$ for all elements of G?

The next theorem gives the answer to this question.

Theorem 36. *Let* G *be a group, let* $A = \{a_i\}_{i \in I}$ *be a subset of* G, *and let* B *be an RB-operator on* G. *Then*

(*a*) *We can find the values of* B *on all elements of the subgroup* $\langle A \rangle_B \leq (G, \circ)$ *by the values* $B(a_i)$, $a_i \in A$. *The image* $B(\langle A \rangle_B)$ *is generated by elements* $B(a_i)$, $i \in G$.
(*b*) *The values of* B *on* A *uniquely define the values of* B *on all elements of* G, *if* A *is a generating set of* G_B.

Proof.

(a) By assumption, (G, B) is a Rota–Baxter group and $B(a_i) = u_i$, $i \in I$. Suppose that an element g of $\langle A\rangle_B$ is presented by some word

$$w = a_{i_1}^{\circ(k_1)} \circ a_{i_2}^{\circ(k_2)} \circ \cdots \circ a_{i_s}^{\circ(k_s)}, \quad a_{i_j} \in A,\ k_{i_j} \in \mathbb{Z}.$$

Since B is a homomorphism from G_B to G, we conclude that

$$B(w) = B\big(a_{i_1}^{\circ(k_1)}\big) B\big(a_{i_2}^{\circ(k_2)}\big) \dots B\big(a_{i_s}^{\circ(k_s)}\big) = u_{i_1}^{k_1} u_{i_2}^{k_2} \dots u_{i_s}^{k_s}.$$

Hence, we can find the values of B on the group that is generated by the set A under the product $\circ$.

(b) If the set $\langle A\rangle_B$ is equal to G, then we can find all values $B(g)$, $g \in G$ by the values $B(a_i)$, $a_i \in A$.

□

Now we consider a question on construction of RB-operators on an arbitrary group G. In this case, we do not know generating set of G_B. Anyway, we can take a generating set of G and define a map β on this set. Let $G = \langle A\rangle$ be a group generated by a set $A = \{a_i \mid i \in I\}$. Suppose that $U = \{u_i\}_{i\in I}$ is a subset of G with the same cardinality as A and the map $\beta\colon A \to U$ is defined by $\beta(a_i) = u_i$, $i \in I$.

Let us construct a group $\bar{G}_\beta$. To do this, we take a set of letters $\bar{A} = \{\bar{a}_i \mid i \in I\}$ which does not intersect with A and has the same cardinality as A. Let $\langle \bar{A}\rangle$ be a free group with basis $\bar{A}$. We denote the operation in this group by $\circ$. Any element of $\langle \bar{A}\rangle$ can be uniquely presented by a reduced word of the form

$$w = \bar{a}_{i_1}^{\circ(k_1)} \circ \bar{a}_{i_2}^{\circ(k_2)} \circ \cdots \circ \bar{a}_{i_s}^{\circ(k_s)},$$

where $\bar{a}_{i_j} \in \bar{A}$, $k_{i_j} \in \mathbb{Z} \setminus \{0\}$. Define a map $\bar{\beta}\colon \langle \bar{A}\rangle \to G$ by the rule

$$\bar{\beta}(w) = \beta(a_{i_1})^{k_1} \beta(a_{i_2})^{k_2} \dots \beta(a_{i_s})^{k_s} = u_{i_1}^{k_1} u_{i_2}^{k_2} \dots u_{i_s}^{k_s}.$$

It is easy to see that this map is a homomorphism of $\langle \bar{A}\rangle$ to $\langle U\rangle$.

Define a map $\pi : \langle \bar{A}\rangle \to G$ by the formula

$$\pi(w) = (a_{i_1}\beta(a_{i_1}))^{k_1} (a_{i_2}\beta(a_{i_2}))^{k_2} \dots (a_{i_s}\beta(a_{i_s}))^{k_s} \beta(a_{i_s})^{-k_s} \\ \times \ \beta(a_{i_{s-1}})^{-k_{s-1}} \dots \beta(a_{i_1})^{-k_1}.$$

The following lemma clarifies the connection between these two maps.

Lemma 37. *Let $w, w' \in \langle \bar{A}\rangle$, then*

(*a*) $\pi(w^{\circ(-1)}) = \bar{\beta}(w)^{-1}\pi(w)^{-1}\bar{\beta}(w)$,
(*b*) $\pi(w \circ w') = \pi(w)\bar{\beta}(w)\pi(w')\bar{\beta}(w)^{-1}$,
(*c*) $\pi((w')^{\circ(-1)} \circ w \circ w') = \bar{\beta}(w')^{-1}\pi(w')^{-1}\pi(w)\bar{\beta}(w)\pi(w')\bar{\beta}(w)^{-1}\bar{\beta}(w')$.

Proof.

(a) For the word

$$w = \bar{a}_{i_1}^{\circ(k_1)} \circ \bar{a}_{i_2}^{\circ(k_2)} \circ \cdots \circ \bar{a}_{i_s}^{\circ(k_s)},$$

its inverse equals to

$$w^{\circ(-1)} = \bar{a}_{i_s}^{\circ(-k_s)} \circ \bar{a}_{i_{s-1}}^{\circ(-k_{s-1})} \circ \cdots \circ \bar{a}_{i_1}^{\circ(-k_1)},$$

and

$$\pi(w^{\circ(-1)}) = (a_{i_s}\beta(a_{i_s}))^{-k_s}(a_{i_{s-1}}\beta(a_{i_{s-1}}))^{-k_{s-1}} \ldots \\ \times (a_{i_1}\beta(a_{i_1}))^{-k_1}\beta(a_{i_1})^{k_1}\beta(a_{i_2})^{k_2} \ldots \beta(a_{i_s})^{k_s}.$$

Hence

$$\pi(w^{\circ(-1)}) = \beta(a_{i_s})^{-k_s} \ldots \beta(a_{i_1})^{-k_1}\pi(w)^{-1}\beta(a_{i_1})^{k_1} \ldots \beta(a_{i_s})^{k_s}.$$

(b) Given a word

$$w' = \bar{a}_{j_1}^{\circ(l_1)} \circ \bar{a}_{j_2}^{\circ(l_2)} \circ \cdots \circ \bar{a}_{j_t}^{\circ(l_t)},$$

we compute

$$w \circ w' = \bar{a}_{i_1}^{\circ(k_1)} \circ \cdots \circ \bar{a}_{i_s}^{\circ(k_s)} \circ \bar{a}_{j_1}^{\circ(l_1)} \circ \cdots \circ \bar{a}_{j_t}^{\circ(l_t)}$$

and

$$\pi(w \circ w') = (a_{i_1}\beta(a_{i_1}))^{k_1} \ldots (a_{i_s}\beta(a_{i_s}))^{k_s} \cdot (a_{j_1}\beta(a_{j_1}))^{l_1} \ldots (a_{j_t}\beta(a_{j_t}))^{l_t} \\ \cdot \beta(a_{j_l})^{-l_t} \ldots \beta(a_{j_1})^{-l_1} \cdot \beta(a_{i_s})^{-k_s} \ldots \beta(a_{i_1})^{-k_1}.$$

Hence

$$\pi(w \circ w') = \pi(w)\beta(a_{i_1})^{k_1} \ldots \beta(a_{i_s})^{k_s}\pi(w') \cdot \beta(a_{i_s})^{-k_s} \ldots \beta(a_{i_1})^{-k_1} \\ = \pi(w)\bar{\beta}(w)\pi(w')\bar{\beta}(w)^{-1}.$$

(c) Take the product $(w')^{\circ(-1)} \circ w \circ w'$ and find

$$\pi((w')^{\circ(-1)} \circ w \circ w') = \beta(a_{j_l})^{-l_t} \ldots \beta(a_{j_1})^{-l_1}(\pi(w')^{-1}\pi(w)\beta(a_{i_1})^{k_1} \ldots \\ \times \quad \beta(a_{i_s})^{k_s}\pi(w') \\ \cdot \beta(a_{i_s})^{-k_s} \ldots \beta(a_{i_1})^{-k_1})\beta(a_{j_1})^{l_1} \ldots \beta(a_{j_t})^{l_t} \\ = \bar{\beta}(w')^{-1}\pi(w')^{-1}\pi(w)\bar{\beta}(w)\pi(w')\bar{\beta}(w)^{-1}\bar{\beta}(w').$$

□

Let $R = \{w \in \langle \bar{A} \rangle \mid \pi(w) = e\}$. Then it follows from the previous lemma

Lemma 38. *The set R is a subgroup of $\langle \bar{A} \rangle$. Suppose that $\bar{\beta}$ satisfies the following implication*:

$$\pi(w) = \pi(w') \Rightarrow \bar{\beta}(w) = \bar{\beta}(w'), \tag{13}$$

in particular, $\ker(\pi) \subseteq \ker(\bar{\beta})$, *then R is normal in $\langle \bar{A} \rangle$.*

Lemma 39. *Suppose that for some elements $w, w' \in \langle \bar{A} \rangle$ such that $\pi(w) = \pi(w')$, we have $\bar{\beta}(w) \neq \bar{\beta}(w')$, then β has no an extension to an RB-operator on G.*

Proof. Suppose that B is an extension of β to an RB-operator B on G and

$$w = \bar{a}_{i_1}^{\circ(k_1)} \circ \bar{a}_{i_2}^{\circ(k_2)} \circ \cdots \circ \bar{a}_{i_s}^{\circ(k_s)}, \quad w' = \bar{a}_{j_1}^{\circ(l_1)} \circ \bar{a}_{j_2}^{\circ(l_2)} \circ \cdots \circ \bar{a}_{j_t}^{\circ(l_t)}.$$

Consider two elements in G_B:

$$u = a_{i_1}^{\circ(k_1)} \circ a_{i_2}^{\circ(k_2)} \circ \cdots \circ a_{i_s}^{\circ(k_s)}, \quad u' = a_{j_1}^{\circ(l_1)} \circ a_{j_2}^{\circ(l_2)} \circ \cdots \circ a_{j_t}^{\circ(l_t)}.$$

Then

$$B(u) = \beta(a_{i_1})^{k_1} \dots \beta(a_{i_s})^{k_s} = \bar{\beta}(w) \neq \bar{\beta}(w') = \beta(a_{j_1})^{l_1} \dots \beta(a_{j_t})^{l_t} = B(u').$$

On the other hand,

$$B(u) = B(\pi(w)) = B(\pi(w')) = B(u'),$$

a contradiction. □

If $R = \ker(\pi)$ is a normal subgroup of $\langle \bar{A} \rangle$, then we can define a group $\bar{G}_\beta = \langle \bar{A} \rangle / R$. Denote by $[w]$ the element of $\bar{G}_\beta$ that is presented by the word w. The product in $\bar{G}_\beta$ is defined by the rule $[w] \circ [w'] = [w \circ w']$. Then $\bar{G}_\beta$ is generated by $[a_i]$, $i \in I$, and is defined by the relations $[w] = [w']$ for all words

$$w = \bar{a}_{i_1}^{\circ(k_1)} \circ \bar{a}_{i_2}^{\circ(k_2)} \circ \cdots \circ \bar{a}_{i_s}^{\circ(k_s)}, \quad w' = \bar{a}_{j_1}^{\circ(l_1)} \circ \bar{a}_{j_2}^{\circ(l_2)} \circ \cdots \circ \bar{a}_{j_t}^{\circ(l_t)}$$

such that $\pi(w) = \pi(w')$, i.e., the following relation holds in G:

$$\begin{aligned}(a_{i_1}\beta(a_{i_1}))^{k_1}(a_{i_2}\beta(a_{i_2}))^{k_2} \dots (a_{i_s}\beta(a_{i_s}))^{k_s} \beta(a_{i_s})^{-k_s} \beta(a_{i_{s-1}})^{-k_{s-1}} \dots \beta(a_{i_1})^{-k_1} \\ = (a_{j_1}\beta(a_{j_1}))^{l_1}(a_{j_2}\beta(a_{j_2}))^{l_2} \dots (a_{j_t}\beta(a_{j_t}))^{l_t} \beta(a_{j_l})^{-l_t} \beta(a_{j_{t-1}})^{-l_{t-1}} \dots \beta(a_{j_1})^{-l_1}.\end{aligned}$$

If condition (13) is satisfied, then the homomorphism $\bar{\beta} \colon \langle \bar{A} \rangle \to \langle U \rangle \leq G$ induces the homomorphism $\bar{G}_\beta \to \langle U \rangle$, which we denote by the same symbol $\bar{\beta}$. Also, the map $\pi : \langle \bar{A} \rangle \to G$ induces the map $\bar{G}_\beta \to G$, which we denote by $\bar{\pi}$. It is easy to see that $\mathrm{Im}\,(\pi) = \mathrm{Im}\,(\bar{\pi})$.

In the following example, we apply Lemma 39.

Example 40. Let

$$G = \left\langle s_1, s_2 \mid s_1^2 = s_2^2 = e,\ s_1 s_2 s_1 = s_2 s_1 s_2 \right\rangle$$

be the symmetric group S_3 on 3 symbols.

(a) If we take the map $\beta \colon \{s_1, s_2\} \to \{s_1, s_2\}$ such that $s_1 \mapsto s_1$, $s_2 \mapsto s_2$, then it has an extension to the RB-operator $B_{-1}(g) = g^{-1}$ on S_3. Let us find the group $\overline{G}_\beta$. Let $\langle \bar{A} \rangle$ be the free group generated by $\bar{A} = \{t_1, t_2\}$. We have a map $\pi : \langle \bar{A} \rangle \to G$ that is defined by the formula

$$\pi\left(t_1^{\circ(k_1)} \circ t_2^{\circ(l_1)} \circ \cdots \circ t_1^{\circ(k_s)} \circ t_2^{\circ(l_s)}\right) = s_2^{-l_s} s_1^{-k_s} \dots s_2^{-l_1} s_1^{-k_1}.$$

In this case, $R = \{w \in \langle \bar{A} \rangle \mid \pi(w) = e\}$ is normal in $\langle \bar{A} \rangle$ and the product in $\overline{G}_\beta$ is opposite to the product in G.

(b) Note that the group S_3 can also be generated by the elements $\tau_1 = s_1$, $\tau_2 = s_2$ and $\tau_3 = s_1 s_2 s_1$. Put

$$\beta(\tau_1) = \tau_1, \quad \beta(\tau_2) = \tau_2, \quad \beta(\tau_3) = \tau_2\tau_1$$

and take $\langle \bar{A} \rangle$ be a free group with basis $\bar{A} = \{t_1, t_2, t_3\}$. Then

$$\begin{aligned} \pi(t_1 \circ t_1) &= \pi(t_2 \circ t_2) = \pi(t_3 \circ t_1) = e, \\ \pi(t_1 \circ t_2 \circ t_1) &= \pi(t_2 \circ t_1 \circ t_2) = \tau_1\tau_2\tau_1, \\ \pi(t_1 \circ t_2) &= \pi(t_2 \circ t_3) = \pi(t_3 \circ t_3) = \tau_2\tau_1, \\ \pi(t_2 \circ t_1) &= \pi(t_3 \circ t_2) = \pi(t_1 \circ t_3) = \tau_1\tau_2. \end{aligned}$$

Hence, $\pi(\langle \bar{A} \rangle) = S_3$. On the other hand,

$$e = \bar{\beta}(t_1 \circ t_1) \neq \bar{\beta}(t_3 \circ t_1) = t_2.$$

It means that the condition of Lemma 39 does not hold and we can not extend the map β to a Rota–Baxter operator on S_3.

Now we can formulate the main result of the present section.

Theorem 41. *Let $G = \langle A \rangle$ be a group generated by a set $A = \{a_i \mid i \in I\}$. Suppose that $U = \{u_i\}_{i\in I}$ is a subset of G with the same cardinality as A and $\beta\colon A \to U$, $\beta(a_i) = u_i$, $i \in I$. If condition (13) is satisfied and the map $\bar{\pi} : \bar{G}_\beta \to G$ is bijective, then β can be extended to an RB-operator on G and G_B is isomorphic to $\bar{G}_\beta$.*

Proof. Construct the RB-operator B on G. Let $g \in G$. Since $\bar{\pi}$ is invertible on G, there exists a unique element $\bar{g} \in \bar{G}_\beta$ such that $\bar{\pi}(\bar{g}) = g$. Suppose that $\bar{g} = [w]$, where

$$w = \bar{a}_{i_1}^{\circ(k_1)} \circ \bar{a}_{i_2}^{\circ(k_2)} \circ \cdots \circ \bar{a}_{i_s}^{\circ(k_s)}$$

is a word in the free group $\langle \bar{A} \rangle$. Then we put

$$B(g) = B(\bar{\pi}(\bar{g})) = B(\pi(w)) = \bar{\beta}(w) = u_{i_1}^{k_1} u_{i_2}^{k_2} \dots u_{i_s}^{k_s}.$$

By condition (13), this map is defined on $[w]$ and does not depend on the choice of w.

Let us prove that B is an RB-operator on G. For $g' \in G$, there exists a word

$$w' = \bar{a}_{j_1}^{\circ(l_1)} \circ \bar{a}_{j_2}^{\circ(l_2)} \circ \cdots \circ \bar{a}_{j_t}^{\circ(l_t)}$$

such that $\pi(w') = \bar{\pi}([w']) = g'$. Then $B(g') = u_{j_1}^{l_1} u_{j_2}^{l_2} \dots u_{j_t}^{l_t}$.

On the other hand, by Lemma 37(b),

$$\pi(w \circ w') = \pi(w)\bar{\beta}(w)\pi(w')\bar{\beta}(w)^{-1}$$

and we have

$$\begin{aligned} B(gB(g)g'B(g)^{-1}) &= B(\pi(w \circ w')) \\ &= \bar{\beta}(w \circ w') \\ &= \bar{\beta}(w)\bar{\beta}(w') \\ &= u_{i_1}^{k_1} u_{i_2}^{k_2} \ldots u_{i_s}^{k_s} \cdot u_{j_1}^{l_1} u_{j_2}^{l_2} \ldots u_{j_t}^{l_t} \\ &= B(g)B(g'). \end{aligned}$$

Hence, B is indeed an RB-operator on G. □

The next proposition shows that the condition (13) is not sufficient for the existence of an RB-operator extending a given map β.

PROPOSITION 42

Consider the symmetric group S_3. Then

(*a*) *the map $\beta\colon s_1 \mapsto s_1$, $s_2 \mapsto e$ defines the group $\bar{G}_\beta$ and this group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
(*b*) *There is no an RB-operator on S_3 extending β.*

Proof.

(a) Take a free group $\langle t_1, t_2 \rangle$. Then we can find

$$\pi(t_1 \circ t_1) = \pi(t_2 \circ t_2) = e, \quad \pi(t_1 \circ t_2) = \pi(t_2 \circ t_1) = s_2 s_1.$$

From equalities

$$\bar{\beta}(t_1 \circ t_1) = \bar{\beta}(t_2 \circ t_2) = e, \quad \bar{\beta}(t_1 \circ t_2) = \bar{\beta}(t_2 \circ t_1) = s_1,$$

it follows that (13) holds and we can define

$$\bar{G}_\beta = \langle [t_1], [t_2] \mid [t_1] \circ [t_1] = [t_2] \circ [t_2] = e,\ [t_1] \circ [t_2] = [t_2] \circ [t_1] \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Hence, the map β does not define a group operation on the set S_3.
(b) Suppose that such an RB-operator B exists. Since $B(s_2) = e$, then by Lemma 7,

$$B(s_1 s_2) = B(s_2 s_1 s_2) = B(s_1 s_2 s_1).$$

We arrive at a contradiction by the following equalities:

$$s_1 B(s_1 s_2) = B(s_1) B(s_1 s_2) = B(s_1 \cdot s_1 \cdot s_1 s_2 \cdot s_1) = B(s_1 s_2 s_1) = B(s_1 s_2).$$

□

Now we consider RB-operators on free non-abelian groups. In this case, we do not need to define groups $\langle \bar{A} \rangle$ and $\bar{G}_\beta$. We can define a new operation $\circ$ directly on the free group.

The following proposition shows that in some cases we can reconstruct RB-operators by their values on the basis of a free group.

PROPOSITION 43

Let F be a free group with a basis $A = \{a_i\}_{i \in I}$. Then

(*a*) *the map $\beta\colon a_i \mapsto e$ for all $i \in I$ has a unique extension to an RB-operator. This operator is the trivial one, $B_0(g) = e$, $g \in F$.*
(*b*) *the map $\beta\colon a_i \mapsto a_i^{-1}$ for all $i \in I$ has a unique extension to an RB-operator. This operator is $B_{-1}(g) = g^{-1}$, $g \in F$.*

Proof.

(a) Suppose that B is an extension of β to an RB-operator on F. Below, we use the notation of the operation $\circ$ in the sense of (4). In this case, $a_i^{\circ(k)} = a_i^k$, and hence $B(a_i^{\circ(k)}) = e$ for all $i \in I$, $k \in \mathbb{Z}$. If

$$w = a_{i_1}^{\circ(k_1)} \circ a_{i_2}^{\circ(k_2)} \circ \cdots \circ a_{i_s}^{\circ(k_s)},$$

where $a_{i_j} \in A$, $k_{i_j} \in \mathbb{Z}$ is an arbitrary word. Then

$$B(w) = B\big(a_{i_1}^{\circ(k_1)}\big) B\big(a_{i_2}^{\circ(k_2)}\big) \dots B\big(a_{i_s}^{\circ(k_s)}\big) = e.$$

Let us show that in this case G_B is also generated by A. It follows from the equality

$$\begin{aligned} w &= a_{i_1}^{\circ(k_1)} \circ a_{i_2}^{\circ(k_2)} \circ \cdots \circ a_{i_s}^{\circ(k_s)} \\ &= (a_{i_1} B(a_{i_1}))^{k_1} (a_{i_2} B(a_{i_2}))^{k_2} \dots (a_{i_s} B(a_{i_s}))^{k_s} B(a_{i_s})^{-k_s} \\ &\times \quad B(a_{i_{s-1}})^{-k_{s-1}} \dots B(a_{i_1})^{-k_1} \\ &= a_{i_1}^{k_1} a_{i_2}^{k_2} \dots a_{i_s}^{k_s}. \end{aligned}$$

(b) The proof is similar.

□

Question 3. Find all Rota–Baxter operators on free non-abelian groups. In particular, is it true that any map β defined on the basis of a free group can be extended to an RB-operator on this group?

The following example shows that the values of β on the generators of F_n do not define values of β on all elements of F_n.

Example 44. Let $F_2 = \langle a, b\rangle$ be the free group of rank 2. Put $\beta(a) = a$, $\beta(b) = e$. By Proposition 29, there exists an RB-operator on F_2 which extends β. This operator is the homomorphism $B\colon F_2 \to \langle a\rangle$ that is defined on the generators as follows: $B(a) = \beta(a) = a$, $B(b) = \beta(b) = e$. Let us denote the product in F_2 as $\cdot$ and the product in $(F_2)_B$ as $\circ$. We try to define the operation $\circ$ on F_2 applying only map β. We have

$$\begin{aligned} &a^{\circ(k)} = a^k, \quad \beta(a^{\circ(k)}) = \beta(a)^k = a^k, \quad k \in \mathbb{Z}, \\ &b^{\circ(l)} = b^l, \quad \beta(b^{\circ(l)}) = \beta(b)^l = e, \quad l \in \mathbb{Z}. \end{aligned}$$

For words of syllable length two, we have

$$a^{\circ(k)} \circ b^{\circ(l)} = a^{2k} b^l a^{-k}, \quad b^{\circ(l)} \circ a^{\circ(k)} = b^l a^k, \quad k, l \in \mathbb{Z}.$$

By Proposition 14 applied for a word w of an arbitrary length, we have

$$\begin{aligned} w &= a^{\circ(k_1)} \circ b^{\circ(l_1)} \circ a^{\circ(k_2)} \circ b^{\circ(l_2)} \circ \cdots \circ a^{\circ(k_s)} \circ b^{\circ(l_s)} \\ &= a^{2k_1} b^{l_1} a^{2k_2} b^{l_2} \ldots a^{2k_s} b^{l_s} a^{-\sum k_i}. \end{aligned}$$

We can find

$$\beta(w) = a^{\sum k_i}.$$

Hence, we can construct an extension of β to the subgroup $S = \langle \{a, b\}, \circ \rangle$ generated by a and b in F_B. So, $\ker(\beta) = \{w \mid \sum k_i = 0\}$ is the normal closure of b in S.

Hence, S is not equal to F_2. In particular, ab does not lie in S.

Let us add a new generator ab to the set $\{a, b\}$ and put

$$\beta_1(a) = \beta_1(ab) = a, \quad \beta_1(b) = e.$$

Then in the group $T = \langle \{a, b, ab\}, \circ \rangle$, we have by Proposition 14,

$$(ab)^{\circ(m)} = (aba)^m a^{-m}, \quad m \in \mathbb{Z}.$$

In particular, $ab, a^{-1}b^{-1} \in T$, and T consists of elements

$$\begin{aligned} &a^{\circ(k_1)} \circ b^{\circ(l_1)} \circ (ab)^{\circ(m_1)} \circ a^{\circ(k_2)} \circ b^{\circ(l_2)} \circ (ab)^{\circ(m_2)} \circ \cdots \circ a^{\circ(k_s)} \circ b^{\circ(l_s)} \circ (ab)^{\circ(m_s)} \\ &\quad = a^{2k_1} b^{l_1} (aba)^{m_1} a^{2k_2} b^{l_2} (aba)^{m_2} \ldots a^{2k_s} b^{l_s} (aba)^{m_s} a^{-\sum(k_i + m_1)} \end{aligned}$$

for integers k_i, l_i, m_i. It is easy to check that $T \neq F_2$.

Question 4. Is it true that the group $(F_2)_B$ from this example is not finitely generated?

In Example 44, we have defined β on the group $\langle \{a, b\}, \circ \rangle$ that is a subset of F_2. We have seen that β has a non-trivial kernel and its image is not equal to F_2.

Question 5. Is there an RB-group (G, B) such that B is a surjective map with a non-trivial kernel?

7. Extensions of RB-groups

If (G, B) is an RB-group, then an RB-group (H, B_H) is called an RB-subgroup of (G, B), if H is a subgroup of G and B_H is the restriction of B to H. We can define a short exact sequence of RB-groups

$$1 \to (H, B_H) \to (G, B) \to (L, B_L) \to 1, \tag{14}$$

where (H, B_H) is an RB-subgroup of (G, B) and there is an epimorphism of RB-groups $(G, B) \to (L, B_L)$ such that B_L is induced by B. We say that (G, B) is an extension of (H, B_H) by (L, B_L).

As in the case of groups, we can formulate the following.

Question 6. Let (H, B_H) and (L, H_L) be two RB-groups. Can we find all RB-groups (G, B) for which there exists a short exact sequence (14)?

By Proposition 30, such RB-groups (G, B) exist.

Recall the construction of a wreath product (see, for example, [21]). Let H and L be groups, $\mathrm{Fun}(L, H)$ and $\mathrm{fun}(L, H)$ are Cartesian and direct sums of copies of H indexed by elements of L. This means that $\mathrm{Fun}(L, H)$ is the group of all functions $L \to H$ and $\mathrm{fun}(L, H)$ is its subgroup of functions with a finite support. For $f \in \mathrm{Fun}(L, H)$, $l \in L$, the function f^l is defined by the rule $f^l = f(lx)$, $x \in L$. The map

$$\hat{l}\colon \mathrm{Fun}(L, H) \to \mathrm{Fun}(L, H), \quad f \mapsto f^l,$$

is an automorphism of $\mathrm{Fun}(L, H)$, which sends $\mathrm{fun}(L, H)$ to itself, and the maps

$$L \to \mathrm{Aut}(\mathrm{Fun}(L, H)), \quad L \to \mathrm{Aut}(\mathrm{fun}(L, H)), \quad l \mapsto \hat{l},$$

are isomorphic embeddings. The Cartesian wreath product $H \bar{\wr} L = L \cdot \mathrm{Fun}(L, H)$ is a group with multiplication

$$lf \cdot l'f' = ll' \cdot f^{l'} f'.$$

The direct wreath product is the group $H \wr L = L \cdot \mathrm{fun}(L, H)$.

It is natural to formulate the following.

Question 7. Let $G = H \wr L$ be a direct wreath product or $G = H \bar{\wr} L$ be a Cartesian wreath product of H and L. What RB-operators can be defined on G?

The particular answer to this question gives the following.

PROPOSITION 45

Let G be a Cartesian or direct wreath product of groups H and L. Then

(*a*) *the map $B(lf) = f^{-1}$ defines an RB-operator on G;*
(*b*) *if L is abelian and φ is any endomorphism of L, then the map $B(lf) = l^{\varphi}$ defines an RB-operator on G;*
(*c*) *if action of L on* $\mathrm{Fun}(L, H)$ (*correspondingly, on* $\mathrm{fun}(L, H)$) *is trivial and B_L and B_H are RB-operators on L and on* $\mathrm{Fun}(L, H)$ (*on* $\mathrm{fun}(L, H)$ *respectively*), *then the map $B(lf) = B_L(l)B_H(f)$ defines an RB-operator on G.*

Proof.

(a) Since G has an exact factorization $G = L \cdot \mathrm{Fun}(L, H)$ or $G = L \cdot \mathrm{fun}(L, H)$, then the result follows from Example 15.
(b) Follows from Proposition 21(b).
(c) Follows from Proposition 30.

□

The non-trivial examples of RB-operators on $\mathrm{Fun}(L, H)$ and on $\mathrm{fun}(L, H)$ can be found by Theorem 32.

It is well-known that any extension of a group H by a group L can be embedded into the Cartesian wreath product $H \bar{\wr} L$, which is the Frobenius embedding.

For extensions of RB-groups we can formulate the following.

Question 8. Is it true that any RB-extension of (H, B_H) by (L, B_L) can be embedded into an RB-group $(H \bar{\wr} L, B)$ for some RB-operator B on $H \bar{\wr} L$?

By analogy with extension and lifting problems for automorphism groups of extensions, we can formulate the following.

Question 9. Suppose that

$$1 \to H \to G \to L \to 1$$

is a short exact sequence of groups.

(a) *Extension problem*. Let (H, B_H) be an RB-group. Does there exist an RB-group (G, B) such that (H, B_H) is its RB-subgroup?
(b) *Lifting problem*. Let (L, B_L) be an RB-group. Do there exist an RB-group (G, B) and an epimorphism $(G, B) \to (L, B_L)$ such that B_L is induced by B?

8. Connection with associated Lie ring

The main motivation of Guo *et al.* [19] to introduce Rota–Baxter operators on groups was the connection with Rota–Baxter operators on Lie algebras, when the initial group is a Lie one. Let us consider another known construction of Lie ring by a given (not necessarily Lie) group.

Let G be a group, consider its lower central series G_i, $i = 1, 2, \ldots$, where $G_1 = G$ and $G_{i+1} = [G, G_i]$. The associated graded abelian group $L(G) = \bigoplus_{n \geq 1} G_n/G_{n+1}$ has the structure of a Lie ring under the product $[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1}$, where $[x, y]$ is the group commutator.

Theorem 46. *Let G be a group and let B be a Rota–Baxter operator on G such that $B(G_n) \subset G_n$ for all $n \geq 1$. Then R defined as $R(xG_{i+1}) = B(x)G_{i+1}$ is a Rota–Baxter operator of weight* 1 *on the Lie ring $L(G)$.*

Proof. Given $h \in G_i$ and $g \in G_{i+1}$, it is easy to check by Lemma 5(d) that $B(h)^{-1}B(hg) \in G_{i+1}$, so R is well-defined. Let $x \in G_i$ and $y \in G_j$. Due to (1), we have to state the equality

$$[R(xG_{i+1}), R(yG_{j+1})] = R([R(xG_{i+1}), yG_{j+1}] + [xG_{i+1}, R(yG_{j+1})] + [xG_{i+1}, yG_{j+1}]),$$

which equals in terms of B to the following one:

$$B(x)^{-1}B(y)^{-1}B(x)B(y)G_{i+j+1} = B([B(x), y] + [x, B(y)] + [x, y])G_{i+j+1}. \tag{15}$$

Applying the commutator properties and the fact that G_{i+j}/G_{i+j+1} is abelian, we may rewrite the inner expression inside the brackets from the right-hand side of (15) modulo G_{i+j+1} as follows:

$$\begin{aligned}
&[B(x), y] + [x, B(y)] + [x, y] \\
&= \underline{\underline{[B(x), y]^{B(y)}}} + \underline{[x, B(y)]^{B(x)} + [x, y]^{B(y)B(x)}} + \underline{\underline{[B(x), B(y)]}} + [B(y), B(x)] \\
&= \underline{[x, yB(y)]^{B(x)}} + \underline{\underline{[B(x), yB(y)]}} + [B(y), B(x)] \\
&= [xB(x), yB(y)] + [B(y), B(x)].
\end{aligned}$$

The formula (15) is a consequence of the equalities fulfilled in G,

$$\begin{aligned}
&B(y)B(x)B([xB(x), yB(y)] \cdot [B(y), B(x)])\\
&\quad = B(yB(y)xB(y)^{-1})B([xB(x), yB(y)] \cdot [B(y), B(x)])\\
&\quad = B\big(yB(y)x\underline{B(y)^{-1}B(y)}B(x)(xB(x))^{-1}(yB(y))^{-1}xB(x)y\underline{B(y)}\\
&\qquad \times \underline{B(y)^{-1}}B(x)^{-1}B(y)\underline{B(x)B(x)^{-1}}B(y)^{-1}\big)\\
&\quad = B(yB(y)x\underline{B(x)B(x)^{-1}x^{-1}}B(y)^{-1}y^{-1}xB(x)yB(x)^{-1}\underline{B(y)B(y)^{-1}})\\
&\quad = B(y\underline{B(y)B(y)^{-1}y^{-1}}xB(x)yB(x)^{-1})\\
&\quad = B(x)B(y).
\end{aligned}$$

□

Example 47. Let G be a 2-step nilpotent group and B is an RB-operator on G defined as in Proposition 24. Then, by Theorem 46, we get an RB-operator $R\colon L(G) \to L(G)$, where $L(G) = G/[G, G] \oplus [G, G]$. It is easy to verify that $R = -\mathrm{id}$.

In Corollary 25, we proved that if G is a 2-step nilpotent group and $g \in G$ is a fixed element, then the map $B_g(x) = g^{-1}x^{-1}g$, $x \in G$ is an RB-operator on G. Since G is 2-step nilpotent, then

$$B_g(x) = x^{-1}[g, x], \quad x \in G. \tag{16}$$

By analogy with (16), we can prove the following.

PROPOSITION 48

If L is a 2-step nilpotent ring or algebra, l is its fixed element, then the map $R : L \to L$,

$$R(x) = -x + [l, x], \quad x \in L \tag{17}$$

is a RB-operator of weight 1 on L.

Proof. We have to check that (1) with $\lambda = 1$ holds for all $x, y \in L$. Since $[R(x), y] + [x, R(y)] + [x, y] \in [L, L]$, we need to prove the equality

$$[R(x), R(y)] = -[R(x), y] - [x, R(y)] - [x, y]. \tag{18}$$

The left-hand side and the right-hand side of (18) respectively are equal to

$$\begin{aligned}
[R(x), R(y)] &= [[l, y], x] - [[l, x], y] + [x, y],\\
-[R(x), y] - [x, R(y)] - [x, y] &= [[l, y], x] - [[l, x], y] + [x, y],
\end{aligned}$$

and we get the required equality. □

Remark 49. It is known that given a Lie algebra $\langle L, \{,\}\rangle$ and given an RB-operator R of weight 1 on L, one may define the so-called post-Lie algebra $\langle L, \{,\}, [,]\rangle$ [6], where $[x, y] = \{R(x), y\} + \{x, R(y)\} + \{x, y\}$.

If we take the 3-dimensional Heisenberg Lie algebra $H_3 = L(e_1, e_2, e_3)$ with the multiplication table $[e_1, e_2] = e_3$ and define an RB-operator R of weight 1 on H_3 by the formula (17) with $l = e_1$. Then the corresponding post-Lie algebra appeared in Proposition 5.2(4) from [7] with the parameters $r_3 = -1$, $\alpha = \gamma = \delta = 0$, and $\beta = 1$.

In a more general context (not necessarily for 2-step nilpotent Lie algebras), post-Lie algebras arisen from the RB-operators defined by (17), were considered in [6, Proposition 6.2].

9. Rota–Baxter operators on finite simple groups

Given a group G, an automorphism ψ of G is called fixed-point-free, if $\psi(g) = g$ implies $g = e$. Let us state a group analogue of Proposition 2.21 from [8] proven for Lie algebras (see also [5]).

Theorem 50 [9]. *Let G be a finite non-abelian simple group. Let B be a Rota–Baxter operator on G such that* $\ker(B) = \{e\}$. *Then* $B(g) = g^{-1}$.

Proof. Since B is a homomorphism from G_B to G with a trivial kernel, we conclude that $G_B \cong G$, and G_B is also simple. Suppose that $\ker(B_+) \neq G_B$, where $B_+(g) = gB(g)$ is a homomorphism from G_B to G. By simplicity of G_B, we obtain that $\ker(B_+) = \{e\}$. So, we have an automorphism $\varphi = B^{-1}B_+$ of the simple group G_B. Note that φ is a fixed-point-free. Indeed, $g = \varphi(g) = B^{-1}(gB(g))$ would imply the equality $B(g) = gB(g)$, i.e., $g = e$. By [24], a finite group admitting a fixed-point-free automorphism is solvable. We have a contradiction. □

COROLLARY 51

Let G be a finite group and let B be an RB-operator on G. If the group G_B is non-abelian simple, then B is elementary and $G \cong G_B$.

Proof. Since $\ker(B)$ is a normal subgroup of G_B and G_B is simple, we have two cases: either $\ker(B) = G$ or $\ker(B) = \{e\}$. If $\ker(B) = G$, then B is elementary and $G \cong G_B$. If $\ker(B) = \{e\}$, then B is elementary, by Theorem 50 and $G \cong G_B$. □

COROLLARY 52

Let G be a finite non-abelian simple group. If G is not factorizable, then G is RB-elementary.

Proof. Let B be a non-elementary Rota–Baxter operator on such a group G. By Theorem 50, we have a proper factorization (6), and hence it is a contradiction. □

A list of finite simple non-factorizable groups is presented in [22, Table 4.1]. It includes some finite groups of Lie type (classical and exceptional) and 15 sporadic groups. We are going to describe all the Rota–Baxter operators on all sporadic groups.

Theorem 53. *Let G be a sporadic simple group. If $G = M_{22}$ or G is not a Mathieu group, then G has only elementary RB-operators. If $G \in \{M_{11}, M_{12}, M_{23}, M_{24}\}$, then all RB-operators on G are splitting (see Table 1).*

Proof. Our main tool in the proof is the list of all factorizations of sporadic groups [12]. Suppose that B is a non-elementary RB-operator on G. Thus, $\ker(B) \neq \{e\}$ and $\ker(B_+) \neq \{e\}$, by Corollary 52. We have a factorization (6). Suppose that one of the factors in (6) is simple. We may assume that $\mathrm{Im}(B)$ is simple, otherwise we consider $\widetilde{B}$ and $\mathrm{Im}(\widetilde{B})$. Since $\ker(B_+)$ is a normal subgroup of $\mathrm{Im}(B)$, we conclude that $\ker(B_+) = \mathrm{Im}(B)$. By (5), $\mathrm{Im}\,(B_+) = \ker(B)$. The last equality due to Proposition 16 implies that B is splitting.

Suppose initially that a factorization (6) is exact. Since $|G| = |\ker(B)| \cdot |\mathrm{Im}(B)|$, we again get the equality $\ker(B_+) = \mathrm{Im}(B)$, and so B is splitting.

Now, we look through all factorizations of sporadic groups. In Tables 1 and 2 [12], we have only one non-exact factorization with both non-simple factors, that is,

$$He = (\mathrm{Sp}(4, 4).2) \cdot (7^2 \rtimes SL_2(7)).$$

Let $\mathrm{Im}\,(B) = \mathrm{Sp}(4, 4) \cdot 2$ and $\mathrm{Im}(B_+) = 7^2 \rtimes SL_2(7)$, then $|\ker(B)| = |He|/|\mathrm{Im}(B)| = 2058$. Since $|\mathrm{Im}(B_+)| = 8 \cdot 2058$, we get by (5) that $\ker(B_+)$ is a subgroup of $\mathrm{Im}\,(B)$ of index 8. Analyzing maximal subgroups of $\mathrm{Sp}(4, 4)$ [11], we conclude that this case for a Rota–Baxter operator B is not possible. □

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