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Numerical radius inequalities for tensor product of operators

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Abstract. The two well-known numerical radius inequalities for the tensor product $A \otimes B$ acting on $\mathbb{H} \otimes \mathbb{K}$, where A and B are bounded linear operators defined on complex Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$, respectively are $\frac{1}{2}\|A\|\|B\| \leq w(A \otimes B) \leq \|A\|\|B\|$ and $w(A)w(B) \leq w(A \otimes B) \leq \min\{w(A)\|B\|, w(B)\|A\|\}$. In this article, we develop new lower and upper bounds for the numerical radius $w(A \otimes B)$ of the tensor product $A \otimes B$ and study the equality conditions for those bounds.

Keywords. Numerical radius; operator norm; tensor product; Cartesian decomposition; bounded linear operator.

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1. Introduction

The numerical range of a bounded linear operator on a complex Hilbert space has been an active area of research over a long period of time due to its application in different areas of pure and applied sciences. Development of bounds for the numerical radius (an important numerical constant associated with the numerical range) has attracted many mathematicians [1–6,13,14,18] in recent years. The same for the tensor product of two operators acting on Hilbert spaces has been done by a few mathematicians [7–10]. In this article, we focus on the development of bounds of the numerical radius of tensor product of two operators defined on complex Hilbert spaces. Before proceeding further, we introduce the notation and terminologies to be used throughout the paper.

Let A be a bounded linear operator on a complex Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $\mathbb{B}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators defined on $\mathbb{H}$. Let A^* denote the adjoint of A and $|A|$ denote the positive operator $(A^*A)^{1/2}$. The Cartesian decomposition of A is $A = \Re(A) + i\Im(A)$, where $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$ are known as real and imaginary part of A, respectively. The numerical range or the field of values of A is defined as the range of the mapping $x \mapsto \langle Tx, x\rangle$ on the unit sphere of $\mathbb{H}$ and is denoted by $W(A)$. From the famous Toeplitz–Hausdorff theorem,

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it follows that the numerical range is always a convex set. The numerical radius and Crawford number of A, denoted as $w(A)$ and $c(A)$ respectively, are defined as $w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$ and $c(A) = \inf\{|\lambda| : \lambda \in W(A)\}$. The operator norm of A is defined as $\|A\| = \{\|Ax\| : x \in \mathbb{H}, \|x\| = 1\}$. It is well known (see [18]) that $w(A) = \sup_{\theta\in\mathbb{R}} \|\mathfrak{R}(e^{i\theta}A)\| = \sup_{\theta\in\mathbb{R}} \|\mathfrak{I}(e^{i\theta}A)\|$. It is easy to check that the numerical radius $w(\cdot)$ is a norm on $\mathbb{B}(\mathbb{H})$, and is equivalent to the operator norm that satisfies the inequality $\frac{1}{2}\|A\| \le w(A) \le \|A\|$. The first inequality becomes equality if $A^2 = 0$ and the second inequality becomes equality if $A^*A = AA^*$. In recent times, this inequality has been improved using different techniques. Interested readers can refer to [1,3–5,13,14] and the references therein.

Next, we focus our attention to the tensor product of two complex Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$, which is defined as the completion of the inner product space consisting of all elements of the form $\sum_{i=1}^{n} x_i \otimes y_i$ for $x_i \in \mathbb{H}$ and $y_i \in \mathbb{K}$ for $n \ge 1$, under the inner product $\langle x \otimes y, z \otimes w\rangle = \langle x, z\rangle\langle y, w\rangle$. The tensor product of the spaces $\mathbb{H}$ and $\mathbb{K}$ is denoted by $\mathbb{H} \otimes \mathbb{K}$. Here the expression $x \otimes y$ is defined algebraically so as to be bilinear in the two arguments x and y. The tensor product of two operators A on $\mathbb{H}$ and B on $\mathbb{K}$, denoted by $A \otimes B$ is defined as $(A \otimes B)(x \otimes y) = Ax \otimes By$ for $x \otimes y \in \mathbb{H} \otimes \mathbb{K}$. For any $A \in \mathbb{B}(\mathbb{H})$ and $B \in \mathbb{B}(\mathbb{K})$, $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$ as it satisfies $\|A \otimes B\| = \|A\|\|B\|$. Therefore, any $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$ satisfies the following inequality:

$$\frac{1}{2}\|A\|\|B\| \le w(A \otimes B) \le \|A\|\|B\|. \tag{1}$$

Observe that the constants $\frac{1}{2}$ and 1 are best possible. Then $\frac{1}{2}\|A\|\|B\| = w(A \otimes B)$ if $A^2 = 0$, B is normal and $w(A \otimes B) = \|A\|\|B\|$ if A, B are normal. We also note the following well-known inequality

$$w(A)w(B) \le w(A \otimes B) \le \min\{w(A)\|B\|, w(B)\|A\|\}. \tag{2}$$

The first inequality follows from the fact that $w(A \otimes B) \ge |\langle (A \otimes B)x \otimes y, x \otimes y\rangle| = |\langle Ax, x\rangle|\,|\langle By, y\rangle|$ with $\|x\| = \|y\| = 1$. Following [12, Theorem 3.4], the second inequality follows from the fact that $A \otimes B = (A \otimes I_{\mathbb{K}})(I_{\mathbb{H}} \otimes B)$ ($I_{\mathbb{H}}$ and $I_{\mathbb{K}}$ are the identity operators on $\mathbb{H}$ and $\mathbb{K}$, respectively) and the two operators $A \otimes I_{\mathbb{K}}$ and $I_{\mathbb{H}} \otimes B$ double commute, that is, $A \otimes I_{\mathbb{K}}$ commutes with both $I_{\mathbb{H}} \otimes B$ and its adjoint $I_{\mathbb{H}} \otimes B^*$. Gau *et al.* [10] studied various equality conditions of the above inequalities in (2). We see that the first inequality in (1) is not comparable, in general, with the first inequality in (2). In this paper, we obtain many improvements of the first inequality in (1). We also give a complete characterization for the equality of $w(A \otimes B) = \frac{1}{2}\|A\|\|B\|$. Other equality conditions are also studied. Further, we obtain various refinements of the second inequality in (1).

2. Main results

We begin this section by noting that for the tensor product of two operators $A \otimes B$, the numerical radius of $A \otimes B$ satisfies the following inequality:

$$w(A)w(B) \le w(A \otimes B) \le 2w(A)w(B). \tag{3}$$

Observe that the scalars 1 and 2 are the best possible constants. If A or B is normal then $w(A)w(B) = w(A \otimes B)$ and if $A^2 = B^2 = 0$ then $w(A \otimes B) = 2w(A)w(B)$. We first obtain an upper bound for the numerical radius of $A \otimes B$ which improves the second inequality in (3). For this purpose, we define the numerical radius distance $d(A)$ of the operator A from the scalar operators, which is defined as $d(A) = \inf\{w(A-\lambda I) : \lambda \in \mathbb{C}\}$. Now, we are in a position to prove the following improvement.

Theorem 1. *Let* $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. *Then*

$$w(A \otimes B) \leq \min\{w(A)(w(B)+d(B)), w(B)(w(A)+d(A))\} \leq 2w(A)w(B).$$

Proof. First we consider the function $f : \mathbb{C} \to \mathbb{R}$ defined by $f(\lambda) = w(B - \lambda I)$ for all $\lambda \in \mathbb{C}$. Clearly, f is continuous. Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 2w(B)\}$. It is easy to show that if $\lambda \in \mathbb{C}\backslash\mathbb{D}$, then $f(\lambda) > w(B)$. Since $f(0) = w(B)$ and $0 \in \mathbb{D}$, f attains its minimum value $d(B)$ at some $\lambda_0 \in \mathbb{D}$. If $\lambda_0 = 0$, then $w(A \otimes B) \leq w(A)\,(w(B) + d(B)) = 2w(A)w(B)$. Now, let $\lambda_0 \neq 0$ and consider $\mu = \frac{\overline{\lambda_0}}{|\lambda_0|}$. Then we get

$$\begin{aligned} w\left(A \otimes B\right) &= w\left(A \otimes (\mu B)\right) \\ &= w\left(A \otimes \mathfrak{R}\left(\mu B\right) + iA \otimes \mathfrak{I}\left(\mu B\right)\right) \\ &\leq w\left(A \otimes \mathfrak{R}\left(\mu B\right)\right) + w\left(A \otimes \mathfrak{I}\left(\mu B\right)\right) \\ &= w\left(A\right)\left(\|\mathfrak{R}\left(\mu B\right)\| + \|\mathfrak{I}\left(\mu B\right)\|\right) \\ &\qquad \left(\mathfrak{R}\left(\mu B\right), \mathfrak{I}\left(\mu B\right) \text{ are self-adjoint}\right) \\ &= w\left(A\right)\left(\|\mathfrak{R}\left(\mu B\right)\| + \|\mathfrak{I}\left(\mu\left(B - \lambda_0 I\right)\right)\|\right) \\ &\leq w\left(A\right)\left(w\left(\mu B\right) + w\left(\mu\left(B - \lambda_0 I\right)\right)\right) \\ &= w\left(A\right)\left(w\left(B\right) + w\left(B - \lambda_0 I\right)\right). \end{aligned}$$

Hence

$$w(A \otimes B) \leq w(A)\left(w(B) + d(B)\right). \tag{4}$$

Similarly, we can prove that

$$w(A \otimes B) \leq w(B)\left(w(A) + d(A)\right). \tag{5}$$

Now, combining (4) and (5) we get the desired first inequality. The second inequality follows from $d(A) \leq w(A)$ and $d(B) \leq w(B)$. □

Remark 2. For any operator $A \in \mathbb{B}(\mathbb{H})$, clearly $d(A) = w(A)$ if and only if A is a numerical radius orthogonal to I in the sense of Birkhoff–James. Recall that A is a numerical radius orthogonal to B in the sense of Birkhoff–James [16] if $w(A + \lambda B) \geq w(A)$ for all $\lambda \in \mathbb{C}$. Therefore, it follows from Theorem 1 that if $w(A \otimes B) = 2w(A)w(B)$, then both A and B are numerical radius orthogonal to I in the sense of Birkhoff–James. For more on numerical radius orthogonality, we refer to [16].

To prove next result, we need the following sequence of lemmas.

Lemma 3 [17, p. 20]. *Let* $A \in \mathbb{B}(\mathbb{H})$ *be positive, and let* $x \in \mathbb{H}$ *with* $\|x\| = 1$. *Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$$

for all $r \geq 1$.

Lemma 4 [11, *pp*. 75–76]. *Let* $A \in \mathbb{B}(\mathbb{H})$, *and let* $x \in \mathbb{H}$. *Then*

$$|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle^{1/2} \langle |A^*|x, x\rangle^{1/2}.$$

Lemma 5 [15, *Corollary* 2]. *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *be positive. Then*

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|A^{1/2}B^{1/2}\|.$$

Now, we obtain the following upper bounds for $w(A \otimes B)$.

Theorem 6. *Let* $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. *Then*

$$\begin{aligned} w^2(A \otimes B) &\leq \frac{1}{4}\| \, |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \, \| \\ &\quad + \frac{1}{2}\| \, \Re\left(|A||A^*| \otimes |B||B^*|\right) \, \| \\ &\leq \frac{1}{4}\left(\|A\|\|B\| + \|A^2\|^{1/2}\|B^2\|^{1/2}\right)^2 \\ &\leq \|A\|^2\|B\|^2. \end{aligned}$$

Proof. Let $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$. Now, using Lemma 4, we get

$$\begin{aligned} |\langle (A \otimes B)f, f\rangle| &\leq \langle |A \otimes B|f, f\rangle^{1/2} \left\langle |A^* \otimes B^*|f, f\right\rangle^{1/2} \\ &\leq \frac{1}{2}\left\langle \left(|A \otimes B| + |A^* \otimes B^*|\right) f, f\right\rangle. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |\langle (A \otimes B)f, f\rangle|^2 &\leq \frac{1}{4}\left\langle \left(|A \otimes B| + |A^* \otimes B^*|\right) f, f\right\rangle^2 \\ &\leq \frac{1}{4}\langle \left(|A \otimes B| + |A^* \otimes B^*|\right)^2 f, f\rangle \ \text{(by Lemma 3)} \\ &= \frac{1}{4}\langle \left(|A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2\right) f, f\rangle \\ &\quad + \frac{1}{2}\left\langle \Re(|A||A^*| \otimes |B||B^*|) f, f\right\rangle \\ &\leq \frac{1}{4}\| \, |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \| \\ &\quad + \frac{1}{2}\|\Re(|A||A^*| \otimes |B||B^*|)\|. \end{aligned}$$

Taking supremum over all $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$, we get the first inequality. Now, let $A = U|A|$ and $A^* = V|A^*|$ be the polar decomposition of A and A^*, respectively. Then $\||A||A^*|\| = \|U^*A^2V\| = \|A^2\|$. Similarly, $\||B||B^*|\| = \|B^2\|$. Therefore, $\| \, |A||A^*| \otimes |B||B^*| \, \| = \|A^2\|\|B^2\|$ and by using Lemma 5, we have

$$\frac{1}{4}\| \, |A|^2 \otimes |B|^2 + |A^*|^2 \otimes |B^*|^2 \|$$

$$\leq \frac{1}{4}\big(\|\,|A|^2 \otimes |B|^2\| + \||A||A^*| \otimes |B||B^*|\,\|\big)$$
$$= \frac{1}{4}\big(\|A\|^2\|B\|^2 + \|A^2\|\|B^2\|\big). \tag{6}$$

Also,

$$\frac{1}{2}\|\mathfrak{R}(|A||A^*| \otimes |B||B^*|)\| \leq \frac{1}{2}\|A^2\|\|B^2\| \leq \frac{1}{2}\|A\|\|A^2\|^{1/2}\|B\|\|B^2\|^{1/2}. \tag{7}$$

Combining (6) and (7), we get the second inequality. The third inequality follows trivially. □

From Theorem 6 and inequality (1), it follows that $w(A \otimes B) = \frac{1}{2}\|A\|\|B\|$ if $A^2 = B^2 = 0$. Next bound reads as follows.

Theorem 7. *Let* $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. *Then*

$$w^2(A \otimes B) \leq \frac{1}{2}\|A^*A \otimes B^*B + AA^* \otimes BB^*\| \leq \|A\|^2\|B\|^2.$$

Proof. Let $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$. Then by the Cartesian decomposition of $A \otimes B$, we get

$$\begin{aligned}
|\langle (A \otimes B)f, f\rangle|^2 &= \langle \mathfrak{R}(A \otimes B)f, f\rangle^2 + \langle \mathfrak{I}(A \otimes B)f, f\rangle^2 \\
&\leq \|\mathfrak{R}(A \otimes B)f\|^2 + \|\mathfrak{I}(A \otimes B)f\|^2 \\
&= \langle \mathfrak{R}^2(A \otimes B)f, f\rangle + \langle \mathfrak{I}^2(A \otimes B)f, f\rangle \\
&= \langle (\mathfrak{R}^2(A \otimes B) + \mathfrak{I}^2(A \otimes B))f, f\rangle \\
&\leq \|\mathfrak{R}^2(A \otimes B) + \mathfrak{I}^2(A \otimes B)\| \\
&= \frac{\|A^*A \otimes B^*B + AA^* \otimes BB^*\|}{2}.
\end{aligned}$$

Therefore, taking supremum over all $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$, we get the first inequality. The second inequality follows from the triangle inequality of the operator norm $\|\cdot\|$. □

Next, we obtain a lower bound for $w(A \otimes B)$.

Theorem 8. *Let* $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. *Then*

$$w(A \otimes B) \geq \frac{1}{2}\|A\|\|B\| + \frac{1}{4}\left|\|A \otimes B + A^* \otimes B^*\| - \|A \otimes B - A^* \otimes B^*\|\right|.$$

Proof. Let $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$. Then by the Cartesian decomposition of $A \otimes B$, we get

$$|\langle (A \otimes B)f, f\rangle|^2 = \langle \mathfrak{R}\,(A \otimes B)\,f, f\rangle^2 + \langle \mathfrak{I}(A \otimes B)f, f\rangle^2.$$

Therefore, we have

$$|\langle (A \otimes B)f, f\rangle| \geq |\langle \mathfrak{R}(A \otimes B)f, f\rangle|.$$

Taking supermum over all $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$ we get,

$$w(A \otimes B) \geq \|\mathfrak{R}(A \otimes B)\|. \tag{8}$$

Also,

$$|\langle (A \otimes B) f, f \rangle| \geq |\langle \mathfrak{J}(A \otimes B) f, f \rangle|.$$

Taking supermum over all $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$, we get

$$w(A \otimes B) \geq \|\mathfrak{J}(A \otimes B)\|. \tag{9}$$

Combining (8) and (9) we get,

$$\begin{aligned} w(A \otimes B) &\geq \max\{\|\mathfrak{R}(A \otimes B)\|, \|\mathfrak{J}(A \otimes B)\|\} \\ &= \frac{\|\mathfrak{R}(A \otimes B)\| + \|\mathfrak{J}(A \otimes B)\|}{2} \\ &\quad + \frac{|\|\mathfrak{R}(A \otimes B)\| - \|\mathfrak{J}(A \otimes B)\||}{2} \\ &\geq \frac{\|\mathfrak{R}(A \otimes B) + i\mathfrak{J}(A \otimes B)\|}{2} \\ &\quad + \frac{|\|\mathfrak{R}(A \otimes B)\| - \|\mathfrak{J}(A \otimes B)\||}{2} \\ &= \frac{\|A\|\|B\|}{2} + \frac{|\|A \otimes B + A^* \otimes B^*\| - \|A \otimes B - A^* \otimes B^*\||}{4}. \end{aligned}$$

Therefore, we get the desired inequality. □

As a consequence of the above theorem, we have the following corollary.

COROLLARY 9

Let $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. If the equality $w(A \otimes B) = \frac{\|A\|\|B\|}{2}$ holds, then $\|A \otimes B + A^* \otimes B^*\| = \|A \otimes B - A^* \otimes B^*\| = \|A\|\|B\|$.

Proof. If $w(A \otimes B) = \frac{\|A\|\|B\|}{2}$, then from Theorem 8 we get $\|A \otimes B + A^* \otimes B^*\| = \|A \otimes B - A^* \otimes B^*\|$. Now

$$\begin{aligned} \|A \otimes B + A^* \otimes B^*\| &\leq 2w(A \otimes B) \\ &= \|A\|\|B\| \\ &\leq \frac{\|A \otimes B + A^* \otimes B^*\| + \|A \otimes B - A^* \otimes B^*\|}{2} \\ &= \|A \otimes B + A^* \otimes B^*\|. \end{aligned}$$

So, we get the desired equalities $\|A \otimes B + A^* \otimes B^*\| = \|A \otimes B - A^* \otimes B^*\| = \|A\|\|B\|$. □

Next, we obtain a complete characterization for the equality of $w(A \otimes B) = \frac{\|A\|\|B\|}{2}$.

PROPOSITION 10

Let $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. Then the equality $w(A \otimes B) = \frac{\|A\|\|B\|}{2}$ holds if and only if $\|e^{i\theta} A \otimes B + e^{-i\theta} A^* \otimes B^*\| = \|e^{i\theta} A \otimes B - e^{-i\theta} A^* \otimes B^*\| = \|A\|\|B\|$, for all $\theta \in \mathbb{R}$.

Proof. The sufficient part of the proposition follows easily, we only prove the necessary part. If the equality $w(A \otimes B) = \frac{\|A\|\|B\|}{2}$ holds, then from Corollary 9, we get $\|A \otimes B + A^* \otimes B^*\| = \|A \otimes B - A^* \otimes B^*\| = \|A\|\|B\|$. As for all $\theta \in \mathbb{R}$, $e^{i\theta} A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$ and $w(A \otimes B) = w(e^{i\theta} A \otimes B)$. Therefore we have for all $\theta \in \mathbb{R}$, $\|e^{i\theta} A \otimes B + e^{-i\theta} A^* \otimes B^*\| = \|e^{i\theta} A \otimes B - e^{-i\theta} A^* \otimes B^*\| = \|A\|\|B\|$, as desired. □

In the following theorem, we obtain another lower bound for $w(A \otimes B)$ which is incomparable with the bound obtained in Theorem 8.

Theorem 11. *Let $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. Then*

$$w(A \otimes B) \geq \frac{\|A\|\|B\|}{2} + \frac{|\,\|A \otimes B + iA^* \otimes B^*\| - \|A \otimes B - iA^* \otimes B^*\|\,|}{4}.$$

Proof. Let $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$. Then by the Cartesian decomposition of $A \otimes B$, we get

$$\begin{aligned} |\langle (A \otimes B) f, f\rangle|^2 &= \langle \Re(A \otimes B) f, f\rangle^2 + \langle \Im(A \otimes B) f, f\rangle^2 \\ &\geq \frac{1}{2}\left(|\langle \Re(A \otimes B) f, f\rangle| + |\langle \Im(A \otimes B) f, f\rangle|\right)^2 \\ &\geq \frac{1}{2}|\langle (\Re(A \otimes B) \pm \Im(A \otimes B))\, f, f\rangle|^2. \end{aligned}$$

Therefore,

$$|\langle (A \otimes B) f, f\rangle| \geq \frac{1}{\sqrt{2}}|\langle (\Re(A \otimes B) \pm \Im(A \otimes B)) f, f\rangle|.$$

Taking supermum over all $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$, we get

$$w(A \otimes B) \geq \frac{\|\Re(A \otimes B) \pm \Im(A \otimes B)\|}{\sqrt{2}}. \tag{10}$$

Now, it follows from the inequalities in (10) that

$$\begin{aligned} &w(A \otimes B) \\ &\geq \frac{1}{\sqrt{2}} \max\{\|\Re(A \otimes B) + \Im(A \otimes B)\|, \|\Re(A \otimes B) - \Im(A \otimes B)\|\} \\ &= \frac{\|\Re(A \otimes B) + \Im(A \otimes B)\| + \|\Re(A \otimes B) - \Im(A \otimes B)\|}{2\sqrt{2}} \\ &\quad + \frac{|\|\Re(A \otimes B) + \Im(A \otimes B)\| - \|\Re(A \otimes B) - \Im(A \otimes B)\||}{2\sqrt{2}} \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\|(\mathfrak{R}(A\otimes B)+\mathfrak{J}(A\otimes B))+i(\mathfrak{R}(A\otimes B)-\mathfrak{J}(A\otimes B))\|}{2\sqrt{2}}\\
&\quad+\frac{|\|\mathfrak{R}(A\otimes B)+\mathfrak{J}(A\otimes B)\|-\|\mathfrak{R}(A\otimes B)-\mathfrak{J}(A\otimes B)\||}{2\sqrt{2}}\\
&=\frac{\|(1+i)A^*\otimes B^*\|}{2\sqrt{2}}\\
&\quad+\frac{|\|\mathfrak{R}(A\otimes B)+\mathfrak{J}(A\otimes B)\|-\|\mathfrak{R}(A\otimes B)-\mathfrak{J}(A\otimes B)\||}{2\sqrt{2}}\\
&=\frac{\|A\|\|B\|}{2}+\frac{|\,\|A\otimes B+iA^*\otimes B^*\|-\|A\otimes B-iA^*\otimes B^*\|\,|}{4}.
\end{aligned}
$$

Therefore, we get the desired inequality. □

Remark 12.

(i) If the equality $w(A\otimes B)=\frac{\|A\|\|B\|}{2}$ holds, then $\|A\otimes B+iA^*\otimes B^*\|=\|A\otimes B-iA^*\otimes B^*\|$. It should be mentioned here that the converse is not true. If we take $A=B=I$, then $\|A\otimes B+iA^*\otimes B^*\|=\|A\otimes B-iA^*\otimes B^*\|=\sqrt{2}$, but $w(A\otimes B)=1>1/2=\frac{\|A\|\|B\|}{2}$.
(ii) If we take $A=B=I$, then Theorem 8 gives $w(A\otimes B)\geq 1$, whereas Theorem 11 gives $w(A\otimes B)\geq 1/2$. Also, considering $A=\begin{pmatrix}2+2i & 0\\ 0 & 1\end{pmatrix}$ and $B=I$, we have that Theorem 8 gives $w(A\otimes B)\geq\sqrt{2}$, whereas Theorem 11 gives $w(A\otimes B)\geq\sqrt{2}+\frac{3}{2\sqrt{2}}$. Therefore, the bounds in Theorems 8 and 11 are not comparable, in general.

Next lower bound reads as follows.

Theorem 13. *Let $A\otimes B\in\mathbb{B}(\mathbb{H}\otimes\mathbb{K})$. Then*

$$
\begin{aligned}
w^2(A\otimes B)\geq{}&\frac{\|A^*A\otimes B^*B+AA^*\otimes BB^*\|}{4}\\
&+\frac{|\,\|A\otimes B+A^*\otimes B^*\|^2-\|A\otimes B-A^*\otimes B^*\|^2\,|}{8}.
\end{aligned}
$$

Proof. From the inequalities in (8) and (9), we obtain that

$$
\begin{aligned}
w^2(A\otimes B)&\geq\max\{\|\mathfrak{R}(A\otimes B)\|^2,\|\mathfrak{J}(A\otimes B)\|^2\}\\
&=\frac{\|\mathfrak{R}(A\otimes B)\|^2+\|\mathfrak{J}(A\otimes B)\|^2}{2}\\
&\quad+\frac{|\|\mathfrak{R}(A\otimes B)\|^2-\|\mathfrak{J}(A\otimes B)\|^2|}{2}\\
&\geq\frac{\|\mathfrak{R}^2(A\otimes B)+\mathfrak{J}^2(A\otimes B)\|}{2}\\
&\quad+\frac{|\|\mathfrak{R}(A\otimes B)\|^2-\|\mathfrak{J}(A\otimes B)\|^2|}{2}\\
&=\frac{\|A^*A\otimes B^*B+AA^*\otimes BB^*\|}{4}
\end{aligned}
$$

$$+\frac{|\,\|A\otimes B+A^*\otimes B^*\|^2-\|A\otimes B-A^*\otimes B^*\|^2\,|}{8}.$$

Therefore, we get the desired inequality. □

Remark 14.

(i) Considering $A=B=I$ in Theorem 11, we get $w(A\otimes B)\geq 1/2$, whereas Theorem 13 gives $w(A\otimes B)\geq 1$. On the other hand, considering $A=\begin{pmatrix}1+i & 0\\ 0 & 0\end{pmatrix}$ and $B=I$ in Theorem 11, we get $w(A\otimes B)\geq\sqrt{2}$, whereas Theorem 13 gives $w(A\otimes B)\geq 1$. Thus the bounds in Theorems 11 and 13 are not comparable, in general.
(ii) Let A be an operator satisfying $A^*A\geq AA^*$ or $A^*A\leq AA^*$ and $B=I$. Then one can verify that (by using concavity of $f(t)=t^{1/2}$) Theorem 13 gives better bound than that in Theorem 8.

Next, we give a complete characterization for the equality of $w^2(A\otimes B)=\frac{1}{4}\|A^*A\otimes B^*B+AA^*\otimes BB^*\|$.

PROPOSITION 15

Let $A\otimes B\in\mathbb{B}(\mathbb{H}\otimes\mathbb{K})$. Then the equality

$$w^2(A\otimes B)=\frac{1}{4}\|A^*A\otimes B^*B+AA^*\otimes BB^*\|$$

holds if and only if

$$\begin{aligned}\|e^{i\theta}A\otimes B+e^{-i\theta}A^*\otimes B^*\|^2&=\|e^{i\theta}A\otimes B-e^{-i\theta}A^*\otimes B^*\|^2\\&=\|A^*A\otimes B^*B+AA^*\otimes BB^*\|,\end{aligned}$$

for all $\theta\in\mathbb{R}$.

Proof. The sufficient part follows easily, we only prove the necessary part. Let $w^2(A\otimes B)=\frac{1}{4}\|A^*A\otimes B^*B+AA^*\otimes BB^*\|$. Then for any real number θ, we have

$$\begin{aligned}&\|A^*A\otimes B^*B+AA^*\otimes BB^*\|\\&=\frac{1}{2}\|(e^{i\theta}A\otimes B+e^{-i\theta}A^*\otimes B^*)^2-(e^{i\theta}A\otimes B-e^{-i\theta}A^*\otimes B^*)^2\|\\&\leq\frac{1}{2}(\|e^{i\theta}A\otimes B+e^{-i\theta}A^*\otimes B^*\|^2+\|e^{i\theta}A\otimes B-e^{-i\theta}A^*\otimes B^*\|^2)\\&\leq 4w^2\,(A\otimes B)\\&=\|A^*A\otimes B^*B+AA^*\otimes BB^*\|.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\|e^{i\theta}A\otimes B+e^{-i\theta}A^*\otimes B^*\|^2 &= \|e^{i\theta}A\otimes B-e^{-i\theta}A^*\otimes B^*\|^2\\ &= \|A^*A\otimes B^*B+AA^*\otimes BB^*\|,\end{aligned}$$

for all $\theta\in\mathbb{R}$. □

Now, in the following theorem, we obtain a lower bound for $w(A\otimes B)$ which is incomparable with the bound obtained in Theorem 13.

Theorem 16. *Let $A\otimes B\in\mathbb{B}(\mathbb{H}\otimes\mathbb{K})$. Then*

$$\begin{aligned}w^2(A\otimes B)\geq{}&\frac{\|A^*A\otimes B^*B+AA^*\otimes BB^*\|}{4}\\ &+\frac{|\,\|A\otimes B+iA^*\otimes B^*\|^2-\|A\otimes B-iA^*\otimes B^*\|^2\,|}{8}.\end{aligned}$$

Proof. It follows from the inequalities in (10) that

$$\begin{aligned}&w^2(A\otimes B)\\ &\geq\frac{1}{2}\max\{\|\Re(A\otimes B)+\Im(A\otimes B)\|^2,\|\Re(A\otimes B)-\Im(A\otimes B)\|^2\}\\ &=\frac{\|\Re(A\otimes B)+\Im(A\otimes B)\|^2+\|\Re(A\otimes B)-\Im(A\otimes B)\|^2}{4}\\ &\quad+\frac{|\|\Re(A\otimes B)+\Im(A\otimes B)\|^2-\|\Re(A\otimes B)-\Im(A\otimes B)\|^2|}{4}\\ &\geq\frac{\|(\Re(A\otimes B)+\Im(A\otimes B))^2+(\Re(A\otimes B)-\Im(A\otimes B))^2\|}{4}\\ &\quad+\frac{|\|\Re(A\otimes B)+\Im(A\otimes B)\|^2-\|\Re(A\otimes B)-\Im(A\otimes B)\|^2|}{4}\\ &=\frac{\|\Re^2(A\otimes B)+\Im^2(A\otimes B)\|}{2}\\ &\quad+\frac{|\|\Re(A\otimes B)+\Im(A\otimes B)\|^2-\|\Re(A\otimes B)-\Im(A\otimes B)\|^2|}{4}\\ &=\frac{\|A^*A\otimes B^*B+AA^*\otimes BB^*\|}{4}\\ &\quad+\frac{|\,\|A\otimes B+iA^*\otimes B^*\|^2-\|A\otimes B-iA^*\otimes B^*\|^2\,|}{8}.\end{aligned}$$

Thus, we get the desired inequality. □

Remark 17. If the equality $w^2(A\otimes B)=\frac{\|A^*A\otimes B^*B+AA^*\otimes BB^*\|}{4}$ holds, then $\|A\otimes B+iA^*\otimes B^*\|=\|A\otimes B-iA^*\otimes B^*\|$. However, the converse is not necessarily true.

Finally, we obtain the following inequality.

Theorem 18. *Let $A \otimes B \in \mathbb{B}(\mathbb{H} \otimes \mathbb{K})$. Then*

$$\|A \otimes B\|^2 - w^2(A \otimes B) \leq \inf_{\lambda \in \mathbb{C}} \{\|A \otimes B - \lambda I \otimes I\|^2 - c^2(A \otimes B - \lambda I \otimes I)\}.$$

Proof. Let $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$. Then for any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \|(A \otimes B)f\|^2 - |\langle (A \otimes B)f, f\rangle|^2 &= \|(A \otimes B - \lambda I \otimes I)\, f\|^2 \\ &\quad - |\langle (A \otimes B - \lambda I \otimes I)f, f\rangle|^2 \\ &\leq \|A \otimes B - \lambda I \otimes I\|^2 \\ &\quad -c^2(A \otimes B - \lambda I \otimes I). \end{aligned}$$

Therefore, taking supremum over all $f \in \mathbb{H} \otimes \mathbb{K}$ with $\|f\| = 1$, we get

$$\|A \otimes B\|^2 - w^2(A \otimes B) \leq \|A \otimes B - \lambda I \otimes I\|^2 - c^2(A \otimes B - \lambda I \otimes I). \quad (11)$$

As the inequality (11) holds for all $\lambda \in \mathbb{C}$, so taking infimum over all $\lambda \in \mathbb{C}$, we get the required inequality. □

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