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Relative grade and relative Gorenstein dimension with respect to a semidualizing module

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Abstract. Let R be a commutative Noetherian ring, and let C be a semidualizing R-module. For R-modules M and N, the notions $\mathrm{grade}_{\mathcal{P}_C}(M, N)$ and $\mathrm{grade}_{\mathcal{I}_C}(M, N)$ are introduced as the relative setting of the notion $\mathrm{grade}(M, N)$ with respect to C. Some results about $\mathrm{grade}_{\mathcal{P}_C}(M, N)$, $\mathrm{grade}_{\mathcal{I}_C}(M, N)$ and $\mathrm{grade}(M, N)$ are mentioned. For finitely generated R-modules M and N, we show that $\mathrm{grade}_{\mathcal{P}_C}(M, N) = \mathrm{grade}(M, N)$ ($\mathrm{grade}_{\mathcal{I}_C}(M, N) = \mathrm{grade}(M, N)$), provided we have some special conditions. Also, the notions of C-perfect and G_C-perfect R-modules are introduced as the relative setting of the notions of perfect and G-perfect R-modules with respect to C, and it is proven that several results for these new concepts are similar to the classical results. Finally, some results about relative grade of tensor and Hom functors with respect to C are given.

Keywords. Semidualizing module; grade of module; perfect module; G-perfect.

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring and all modules are unital. The notion of a "semidualizing module" is a central notion in relative homological algebra. Ever since this notion was independently introduced by Foxby [7], Vasconcelos [17] and Golod [8], its various aspects has been investigated by many authors from different stand points; see for example, [3,5,9,13,16].

Let M be a finitely generated R-module. In [12], the "grade of M" was defined by Rees as the least integer $t \geqslant 0$ such that $\mathrm{Ext}^t_R(M, R) \neq 0$, and "the grade of an ideal $\mathfrak{a}$" as the grade of R-module $R/\mathfrak{a}$. It is known that $\mathrm{grade}\, M \leqslant \mathrm{pd}_R\, M$ ($\mathrm{grade}\, M \leqslant \text{G-dim}_R\, M$), and M is said to be perfect (G-perfect) if the equality holds. Some properties of G-perfect modules are investigated in [19].

Let C be a semidualizing R-module, and let M and N be R-modules. In this paper, the notions of $\mathrm{grade}_{\mathcal{P}_C}(M, N)$ and $\mathrm{grade}_{\mathcal{I}_C}(M, N)$ are introduced as the relative setting of the notion $\mathrm{grade}(M, N)$ with respect to C. Some results about $\mathrm{grade}_{\mathcal{P}_C}(M, N)$, $\mathrm{grade}_{\mathcal{I}_C}(M, N)$ and $\mathrm{grade}(M, N)$ are mentioned. For finitely generated R-modules M and

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N, it is shown that $\mathrm{grade}_{\mathcal{P}_C}(M, N) = \mathrm{grade}(M, N)$ ($\mathrm{grade}_{\mathcal{I}_C}(M, N) = \mathrm{grade}(M, N)$), by providing some special conditions. Also, it is shown that for finitely generated R-module M, we have

$$\mathrm{grade}\, M = \mathrm{grade}(M, C) = \mathrm{grade}_{\mathcal{P}_C}(M, C) = \mathrm{grade}_{\mathcal{I}_C}(M, R).$$

In addition, the notions of C-perfect and G_C-perfect R-modules are introduced as the relative setting of the notions perfect and G-perfect R-modules with respect to C. We prove several results for the new concepts similar to the classical results. It is shown that for a G_C-perfect R-module M of grade n, the R-module $\mathrm{Ext}^n_R(M, C)$ is G_C-perfect of grade n. Also, it is shown that a finitely generated R-module M is Cohen–Macaulay if and only if it is G_C-perfect, provided that C is dualizing for the local ring R. Some inequalities on $\mathrm{grade}_{\mathcal{P}_C}(M, N)$ ($\mathrm{grade}_{\mathcal{I}_C}(M, N)$) and G_C-dimension are investigated. Finally, we get some results about relative grade of tensor and Hom functors with respect to C.

2. Preliminaries

Throughout this paper, $\mathcal{M}(R)$ is the category of R-modules. We use the term "subcategory of $\mathcal{M}(R)$" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R-modules M and N, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$."

This section contains definition and background information for use in the proof of our main results.

DEFINITION 2.1

An R-complex is a sequence

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

of R-modules and R-homomorphisms such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n.

The notion of semidualizing modules, defined next, goes back at least to Foxby [7], but was rediscovered by others.

DEFINITION 2.2

A finitely generated R-module C is called *semidualizing* if the natural homothety homomorphism $\chi^R_C : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^{\geqslant 1}_R(C, C) = 0$. An R-module D is called *dualizing* if it is semidualizing and has finite injective dimension.

For a semidualizing R-module C, we set

$$\mathcal{P}_C(R) = \{P \otimes_R C \mid P \text{ is a projective } R\text{-module}\},$$
$$\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\}.$$

The R-modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-projective* and *C-injective*, respectively. When $C = R$, we omit the subscript and recover the classes of projective and injective R-modules.

In this paper, all resolutions (co-resolutions) will be built from pre-covers (pre-envelops) which we now define.

DEFINITION 2.3

Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$ and let M be an R-module. A homomorphism $\varphi : X \to M$ with $X \in \mathcal{X}$ is an $\mathcal{X}$-*precover* of M if for every homomorphism $\psi : Y \to M$ with $Y \in \mathcal{X}$, there exists a homomorphism $f : Y \to X$ such that $\varphi f = \psi$. If every R-module admits an $\mathcal{X}$-precover, then we say that the class $\mathcal{X}$ is *pre-covering*.

An $\mathcal{X}$-*cover* of M is an $\mathcal{X}$-precover $\varphi : X \to M$ with the additional property that any endomorphism $f : X \to X$ with $\varphi = \varphi f$ must be an automorphism. If every R-module admits an $\mathcal{X}$-cover, then we say that the class $\mathcal{X}$ is *covering*.

Pre-envelopes and *envelopes* are defined dually, see [6, Chapter 5 and 6] for further details.

In [10], Holm and White proved that the class $\mathcal{P}_C(R)$ is pre-covering. So, for any R-module M, one can iteratively take pre-covers to construct an *augmented proper* $\mathcal{P}_C$-*projective resolution* of M, that is, a complex

$$X^+ = \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow M \longrightarrow 0$$

such that $\mathrm{Hom}_R(C \otimes_R Q, X^+)$ is exact for all projective R-modules Q. The truncated complex

$$X = \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow 0$$

is a *proper* $\mathcal{P}_C$-*projective resolution* of M.

Dually, in [10], it is shown that the class $\mathcal{I}_C(R)$ is enveloping. So, for an R-module N one can construct an *augmented proper* $\mathcal{I}_C$-*injective resolution*, that is, a complex

$$Y^+ = 0 \longrightarrow N \longrightarrow \mathrm{Hom}_R(C, I^0) \longrightarrow \mathrm{Hom}_R(C, I^1) \longrightarrow \cdots$$

such that $\mathrm{Hom}_R(Y^+, \mathrm{Hom}_R(C, I))$ is exact for all injective R-module I. The $\mathcal{P}_C$-*projective dimension* of M is

$$\mathcal{P}_C\text{-}\,\mathrm{pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is a proper } \mathcal{P}_C\text{-projective resolution of } M\}.$$

The $\mathcal{I}_C$-*injective dimension*, denoted by $\mathcal{I}_C\text{-}\,\mathrm{id}_R(-)$ is defined dually.

Note that X^+ and Y^+ need not be exact. In [16, Corollary 2.4], Takahashi and White proved that if M is in $\mathcal{B}_C(R)$ (resp. $\mathcal{A}_C(R)$), then every augmented proper $\mathcal{P}_C$-projective resolution (resp. $\mathcal{I}_C$-injective resolution) of M is exact. The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ defined next are collectively known as Foxby classes. The definitions are due to Foxby.

DEFINITION 2.4

Let C be a semidualizing R-module. The *Auslander class* with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that

(i) $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ for all $i \geqslant 1$, and
(ii) the natural map $\gamma_C^M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that

(i) $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ for all $i \geqslant 1$, and
(ii) the natural evaluation map $\xi_M^C : C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

In [8], Golod introduced the following notion, though elements of it can be traced to Foxby [7] and Vasconcelos [17].

DEFINITION 2.5

Let C be a semidualizing R-module. An R-module G is called *totally C-reflexive* if it satisfies the following conditions:

(i) G is finitely generated;
(ii) the biduality map $\delta_G^C : G \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(G, C), C)$, given by $\delta_G^C(n)(\psi) = \psi(n)$ for each $n \in G$ and $\psi \in \mathrm{Hom}_R(G, C)$, is an isomorphism; and
(iii) $\mathrm{Ext}_R^i(G, C) = 0 = \mathrm{Ext}_R^i(\mathrm{Hom}_R(G, C), C)$ for all $i \geqslant 1$.

In [8], Golod used the notion of semidualizing module to define G_C-dimension where C is a semidualizing module as follows.

DEFINITION 2.6

Let C be a semidualizing R-module and let M be a finitely generated R-module. An *augmented G_C-resolution* of M is an exact sequence

$$G^+ = \cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} G_{i-1} \xrightarrow{\partial_{i-1}^G} \cdots \xrightarrow{\partial_1^G} G_0 \xrightarrow{\partial_0^G} M \longrightarrow 0,$$

wherein each G_i is totally C-reflexive. The *G_C-resolution* of M associated to G^+ is the sequence obtained by truncating

$$G = \cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} G_{i-1} \xrightarrow{\partial_{i-1}^G} \cdots \xrightarrow{\partial_1^G} G_0 \longrightarrow 0.$$

Note that if C is a semidualizing R-module, then every finitely generated projective R-module is totally C-reflexive (see [14, Proposition 2.1.13]). It follows that every (augmented) resolution by finitely generated projective R-module is an (augmented) G_C-resolution.

DEFINITION 2.7

Let C be a semidualizing R-module and let M be a finitely generated R-module. If M admits a G_C-resolution G such that $G_i = 0$ for $i \gg 0$, then we say that M has finite G_C-dimension. More specifically, the *G_C-dimension* of M is the shortest such resolution:

$$G_C\text{-}\dim_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a } G_C\text{-resolution of } M\}.$$

By definition, we have $G_C\text{-}\dim_R(M) = -\infty$ if and only if $M = 0$. In the case $C = R$, we use the more common terminology "G-dimension" and the notation "$G\text{-}\dim_R(M)$".

Remark 2.8. Let C be a semidualizing R-module. A non-zero finitely generated R-module is totally C-reflexive if and only if it has G_C-dimension zero.

Remark 2.9. Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then $G_C\text{-}\dim_R(M) \leqslant \mathcal{P}_C\text{-}\mathrm{pd}_R(M)$ and $\mathrm{G}_C\text{-}\dim_R(M) \leqslant \mathrm{pd}_R(M)$. For every finitely generated projective R-module P, the modules P and $C \otimes_R P$ are totally C-reflexive by [14, Proposition 2.1.13].

The notion of G_C-dimension ($\mathcal{P}_C$-projective dimension) has many properties of G-dimension (projective dimension). In particular, the Auslander–Bridger formula holds with G_C-dimension ($\mathcal{P}_C$-projective dimension) instead of G-dimension (projective dimension).

Theorem 2.10. *Let R be a local ring, and let C be a semidualizing R-module. Assume that M is a finitely generated R-module. Then the following statements hold*:

(*i*) [14, *Proposition* 6.4.2]. If $\mathrm{G}_C\text{-}\dim_R(M) < \infty$, *then*

$$\mathrm{G}_C\text{-}\dim_R(M) = \operatorname{depth} R - \operatorname{depth}_R(M).$$

(*ii*) [18, *Corollary* 4.6]. If $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty$, *then*

$$\mathcal{P}_C\text{-}\operatorname{pd}_R(M) = \operatorname{depth} R - \operatorname{depth}_R(M).$$

PROPOSITION 2.11 [14, Proposition 6.1.7]

Let C be a semidualizing R-module, and let M be a finitely generated R-module such that $\mathrm{G}_C\text{-}\dim_R(M) < \infty$. *Then*

$$\mathrm{G}_C\text{-}\dim_R(M) = \sup\{i \geqslant 0 \mid \operatorname{Ext}^i_R(M, C) \neq 0\}.$$

Fact 2.12 [14, *Corollary* 2.1.17]. Let C be a semidualizing R-module, and let $M \neq 0$ be an R-module. Then $\operatorname{Hom}_R(C, M) \neq 0$ and $C \otimes_R M \neq 0$.

PROPOSITION 2.13

Let C be a semidualizing R-module, and let $M \neq 0$ be a finitely generated R-module. Then the following statements hold:

(*i*) $\operatorname{Supp}_R(\operatorname{Hom}_R(C, M)) = \operatorname{Supp}_R(M) = \operatorname{Supp}_R(C \otimes_R M)$.

(*ii*) $\dim_R(\operatorname{Hom}_R(C, M)) = \dim_R(M) = \dim_R(C \otimes_R M)$.

Proof.

(i) We just prove the first equality. The proof of the second one is similar. It is clear that $\operatorname{Supp}_R(\operatorname{Hom}_R(C, M)) \subseteq \operatorname{Supp}_R(M)$. Suppose that $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $M_\mathfrak{p} \neq 0$, and $C_\mathfrak{p}$ is a semidualizing $R_\mathfrak{p}$-module by [14, Proposition 2.2.3]. By Fact 2.12, $(\operatorname{Hom}_R(C, M))_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, M_\mathfrak{p}) \neq 0$. So, $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Hom}_R(C, M))$, and hence $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(\operatorname{Hom}_R(C, M))$.

(ii) It follows from item (i). □

The following functors are studied in [15,16].

DEFINITION 2.14

Let C be a semidualizing R-module, and let M and N be R-modules. Let L be a proper $\mathcal{P}_C$-resolution of M, and let J be a proper $\mathcal{I}_C$-coresolution of N. For each i, set

$$\operatorname{Ext}^i_{\mathcal{P}_C}(M, N) := \mathrm{H}_{-i}(\operatorname{Hom}_R(L, N))$$
$$\operatorname{Ext}^i_{\mathcal{I}_C}(M, N) := \mathrm{H}_{-i}(\operatorname{Hom}_R(M, J)).$$

PROPOSITION 2.15 [16, Theorem 3.2]

Let C be a semidualizing R-module, $n \geqslant 0$ be an integer, and let M be an R-module. Then $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) \leqslant n$ if and only if $\operatorname{Ext}^i_{\mathcal{P}_C}(M, -) = 0$ for all $i > n$.

3. Relative grade and relative perfect modules

Let C be a semidualizing R-module, and let M and N be R-modules. In this section, the notions $\operatorname{grade}_{\mathcal{P}_C}(M, N)$ and $\operatorname{grade}_{\mathcal{I}_C}(M, N)$ are introduced as the relative setting of the notion $\operatorname{grade}(M, N)$ with respect to C. Some results are mentioned about $\operatorname{grade}_{\mathcal{P}_C}(M, N)$, $\operatorname{grade}_{\mathcal{I}_C}(M, N)$ and $\operatorname{grade}(M, N)$.

DEFINITION 3.1

Let C be a semidualizing R-module, and let M and N be R-modules. We define

$$\operatorname{grade}(M, N) = \inf\{i \mid \operatorname{Ext}^i_R(M, N) \neq 0\},$$
$$\operatorname{grade}_{\mathcal{P}_C}(M, N) = \inf\{i \mid \operatorname{Ext}^i_{\mathcal{P}_C}(M, N) \neq 0\},$$
$$\operatorname{grade}_{\mathcal{I}_C}(M, N) = \inf\{i \mid \operatorname{Ext}^i_{\mathcal{I}_C}(M, N) \neq 0\}.$$

Lemma 3.2. *Let C be a semidualizing R-module, and let M and N be R-modules. Then* $\operatorname{grade}_{\mathcal{P}_C}(M, N) = \operatorname{grade}(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N))$ *and* $\operatorname{grade}_{\mathcal{I}_C}(M, N) = \operatorname{grade}(C \otimes_R M, C \otimes_R N)$.

Proof. For all $i \geqslant 0$, we have $\operatorname{Ext}^i_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N))$, and $\operatorname{Ext}^i_{\mathcal{I}_C}(M, N) \cong \operatorname{Ext}^i_R(C \otimes_R M, C \otimes_R N)$, by [16, Theorem 4.1]. Now the assertions follow from definition. □

Remark 3.3. Let M be a finitely generated R-module, and let I be an ideal of R such that $IM \neq M$. Then the following statements hold:

(i) All maximal M-sequence in I have the same length $\operatorname{grade}(R/I, M)$.

(ii) Let C be a semidualizing R-module. If $I \in \mathcal{A}_C(R)$, then all maximal $(\operatorname{Hom}_R(C, M))$-sequence in I have the same length $\operatorname{grade}_{\mathcal{P}_C}(C/IC, M)$. Since Lemma 3.2 implies that

$$\operatorname{grade}_{\mathcal{P}_C}(C/IC, M) = \operatorname{grade}(\operatorname{Hom}_R(C, (R/I) \otimes_R C), \operatorname{Hom}_R(C, M)).$$

Also, $\operatorname{Hom}_R(C, (R/I) \otimes_R C) \cong R/I$ since $R/I \in \mathcal{A}_C(R)$, by [14, Proposition 3.1.7].

(iii) Let C be a semidualizing R-module. If $R/I \in \mathcal{B}_C(R)$, then all maximal $(C \otimes_R M)$-sequence in I have the same length $\operatorname{grade}_{\mathcal{I}_C}(\operatorname{Hom}_R(C, R/I), M)$. Since Lemma 3.2 implies that

$$\operatorname{grade}_{\mathcal{I}_C}(\operatorname{Hom}_R(C, R/I), M) = \operatorname{grade}(C \otimes_R \operatorname{Hom}_R(C, R/I), C \otimes_R M).$$

Also, $C \otimes_R \operatorname{Hom}_R(C, R/I) \cong R/I$ since $R/I \in \mathcal{B}_C(R)$.

Theorem 3.4. *Let C be a semidualizing R-module, and let M and N be finitely generated R-modules. Then the following statements hold*:

(*i*) $\operatorname{grade}(M, N) = \inf\{\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}$.
(*ii*) $\operatorname{grade}_{\mathcal{P}_C}(M, N) = \inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C, N)_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}$.
(*iii*) $\operatorname{grade}_{\mathcal{P}_C}(M, N) = \operatorname{grade}(M, \operatorname{Hom}_R(C, N))$.
(*iv*) $\operatorname{grade}_{\mathcal{I}_C}(M, N) = \inf\{\operatorname{depth}_{R_\mathfrak{p}}((C \otimes_R N)_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}$.
(*v*) $\operatorname{grade}_{\mathcal{I}_C}(M, N) = \operatorname{grade}(M, C \otimes_R N)$.

Proof.

(i) It is proved in [19, Theorem 1.2].
(ii) In the following, the first equality follows from Lemma 3.2, the second one follows from (i), and the third one follows from Proposition 2.13.

$$\begin{aligned}\operatorname{grade}_{\mathcal{P}_C}(M, N) &= \operatorname{grade}(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N))\\ &= \inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C, N)_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Hom}_R(C, M))\}\\ &= \inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C, N)_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}.\end{aligned}$$

(iii) It follows from (i) and (ii).
(iv) It is proved in the same argument of (ii).
(v) It follows from (i) and (iv). □

Remark 3.5. Let C be a semidualizing R-module. For a finitely generated R-module M, $\operatorname{grade}(M, R)$ is denoted by grade M and it is called the *grade of* M. This notion was defined by Rees in [12]. Note that $\operatorname{grade} M = \operatorname{grade}(M, C) = \operatorname{grade}_{\mathcal{P}_C}(M, C) = \operatorname{grade}_{\mathcal{I}_C}(M, R)$, by Theorem 3.4.

PROPOSITION 3.6

Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then the following statements hold:

(*i*) $\operatorname{grade} M \leqslant \operatorname{pd}_R(M)$.
(*ii*) $\operatorname{grade} M \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$.
(*iii*) $\operatorname{grade} M = \operatorname{grade}(\operatorname{Hom}_R(C, M))$.
(*iv*) $\operatorname{grade} M = \operatorname{grade}(C \otimes_R M)$.
(*v*) $\operatorname{grade} M \leqslant \text{G-}\dim_R(M)$.
(*vi*) $\operatorname{grade} M \leqslant \text{G}_C\text{-}\dim_R(M)$.

Proof. The items (i) and (ii) follow from Proposition 2.15.

In item (iii), the first and the third equalities follow from Remark 3.5 and the second equality follows from Lemma 3.2:

$$\begin{aligned}\operatorname{grade} M &= \operatorname{grade}_{\mathcal{P}_C}(M, C)\\ &= \operatorname{grade}(\operatorname{Hom}_R(C, M), R)\\ &= \operatorname{grade}(\operatorname{Hom}_R(C, M)).\end{aligned}$$

In item (iv), the first and the third equalities follow from Remark 3.5 and the second equality follows from Lemma 3.2:

$$\begin{aligned}\operatorname{grade} M &= \operatorname{grade}_{\mathcal{I}_C}(M, R)\\ &= \operatorname{grade}(C \otimes_R M, C)\\ &= \operatorname{grade}(C \otimes_R M).\end{aligned}$$

The items (v) and (vi) follow from Proposition 2.11. □

Recall that the finitely generated R-module M is called *perfect* if grade $M = \mathrm{pd}_R(M)$, and is called *G-perfect* if grade $M = \text{G-dim}_R(M)$. In [19], Yassemi *et al.* proved several results for G-perfect concept similar to the classical results for perfect concept. In the following, we introduce relative version of the notions of perfect and G-perfect modules.

DEFINITION 3.7

Let C be a semidualizing R-module, and let M be a finitely generated R-module. Then

(i) M is called *C-perfect* if grade $M = \mathcal{P}_C\text{-}\mathrm{pd}_R(M)$.
(ii) M is called *G_C-perfect* if grade $M = \mathrm{G}_C\text{-dim}_R(M)$.

The functors $C \otimes_R - : \mathcal{A}_C(R) \to \mathcal{B}_C(R)$ and $\mathrm{Hom}_R(C, -) : \mathcal{B}_C(R) \to \mathcal{A}_C(R)$ establish an equivalence of categories between the Auslander class and the Bass class. This is usually called the Foxby equivalence between the two classes. Now, Proposition 3.6 leads to the following propositions.

PROPOSITION 3.8

Let C be a semidualizing R-module, and let M be a non-zero finitely generated R-module. Then the following statements hold:

(*i*) *M is perfect if and only if $C \otimes_R M$ is C-perfect.*
(*ii*) *M is C-perfect if and only if $\mathrm{Hom}_R(C, M)$ is perfect.*

Proof. Note that grade M = grade$(\mathrm{Hom}_R(C, M))$ = grade$(C \otimes_R M)$, by Proposition 3.6. Also, $\mathcal{P}_C\text{-}\mathrm{pd}_R(C \otimes_R M) = \mathrm{pd}_R(M)$ and $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) = \mathrm{pd}_R(\mathrm{Hom}_R(C, M))$, by [16, Theorem 2.11]. So, we get the assertions. □

PROPOSITION 3.9

Let C be a semidualizing R-module, and let M be a non-zero finitely generated R-module. Then the following statements hold:

(*i*) *If $M \in \mathcal{A}_C(R)$, then M is G-perfect if and only if $C \otimes_R M$ is G_C-perfect.*
(*ii*) *If $M \in \mathcal{B}_C(R)$, then M is G_C-perfect if and only if $\mathrm{Hom}_R(C, M)$ is G-perfect.*

Proof.

(i) Note that grade M = grade$(C \otimes_R M)$ by Proposition 3.6. Also,

$$\mathrm{Ext}^i_R(C \otimes_R M, C) \cong \mathrm{Ext}^i_R(C \otimes_R M, C \otimes_R R) \cong \mathrm{Ext}^i_R(M, R)$$

for all $i \geqslant 0$, by [14, Lemma 3.1.13]. So, we get the assertion.

(ii) Note that grade M = grade$(\mathrm{Hom}_R(C, M))$ by Proposition 3.6. Also,

$$\begin{aligned}\mathrm{Ext}^i_R(M, C) &\cong \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C, C)) \\ &\cong \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, M), R)\end{aligned}$$

for all $i \geqslant 0$, by [14, Lemma 3.1.13]. So, we get the assertions. □

PROPOSITION 3.10

Let C be a semidualizing R-module, and let M be a non-zero finitely generated G_C-perfect R-module. Then for any prime ideal $\mathfrak{p} \in \mathrm{Supp}_R(M)$ the following statements are equivalent:

(*i*) $\mathfrak{p} \in \mathrm{Ass}_R\, M$.
(*ii*) $\mathrm{depth}_{R_\mathfrak{p}}(C_\mathfrak{p}) = \mathrm{grade}\, M$.

Furthermore, $\mathrm{grade}\, R/\mathfrak{p} = \mathrm{grade}\, M$ *for all prime ideals* $\mathfrak{p} \in \mathrm{Ass}_R(M)$.

Proof. Note that $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ by [14, Proposition 2.1.16], and for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $C_\mathfrak{p}$ is a semidualizing $R_\mathfrak{p}$-module by [14, Proposition 2.2.3]. Hence for every $\mathfrak{p} \in \mathrm{Supp}_R(M)$, we have

$$\begin{aligned}\mathrm{grade}\, M &\leqslant \mathrm{grade}\, M_\mathfrak{p}\\ &\leqslant \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p})\\ &\leqslant \mathrm{G}_C\text{-}\dim_R(M).\end{aligned}$$

Since M is G_C-perfect, the above inequalities become equalities. On the other hand, for every $\mathfrak{p} \in \mathrm{Supp}_R(M)$, we have $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{depth}\, M_\mathfrak{p} = \mathrm{depth}\, R_\mathfrak{p}$ by Theorem 2.10. Hence $\mathrm{depth}\, M_\mathfrak{p} = 0$ if and only if $\mathrm{grade}\, M = \mathrm{depth}\, R_\mathfrak{p}$. Also, $\mathrm{depth}\, R_\mathfrak{p} = \mathrm{depth}_{R_\mathfrak{p}}(C_\mathfrak{p})$ by [14, Theorem 2.2.6]. Furthermore, if $\mathfrak{p} \in \mathrm{Ass}_R(M)$, then $\mathfrak{p} \supseteq \mathrm{Ann}_R\, M$, and hence $\mathrm{Supp}_R(R/\mathfrak{p}) \subseteq \mathrm{Supp}_R(M)$. By Remark 3.4, we get that $\mathrm{grade}\, R/\mathfrak{p} \geqslant \mathrm{grade}\, M$. For the converse, from (ii), we have $\mathrm{grade}\, M = \mathrm{depth}_{R_\mathfrak{p}}(C_\mathfrak{p}) = \mathrm{depth}\, R_\mathfrak{p} \geqslant \mathrm{grade}\, R/\mathfrak{p}$, since M is G_C-perfect. □

Remark 3.11. Let C be a semidualizing R-module. Note that Proposition 3.10 also holds in the case that M is a non-zero finitely generated C-perfect R-module.

In the following, for a G_C-perfect module M of grade n, we show that the R-module $\mathrm{Ext}^n_R(M, C)$ is also G_C-perfect of grade n, where C is a semidualizing R-module.

Theorem 3.12. *Let C be a semidualizing R-module, and let M be a G_C-perfect R-module. If* $\mathrm{grade}\, M = n$, *then the R-module* $\mathrm{Ext}^n_R(M, C)$ *is G_C-perfect of grade n.*

Proof. The proof is by induction on n. If $n = 0$, then $\mathrm{G}_C\text{-}\dim_R(M) = 0$ and hence M is a totally C-reflexive R-module. By [14, Proposition 5.1.4], $\mathrm{Hom}_R(M, C)$ is totally C-reflexive and hence $\mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(M, C)) = 0$. Thus $\mathrm{Hom}_R(M, C)$ is G_C-perfect of grade 0. Now let $n > 0$. Then there exists an R-regular element x such that $xM = 0$. By [14, Theorem 2.2.6], x is C-regular and C/xC is a semidualizing (R/xR)-module. Also, $\mathrm{Ext}^{i+1}_R(M, C) \cong \mathrm{Ext}^i_{R/xR}(M, C/xC)$ for every integer i, by [4, Lemma 3.1.16]. Thus $\mathrm{grade}_{R/xR}\, M = \mathrm{grade}_{R/xR}(M, C/xC) = n - 1$, and $\mathrm{G}_{C/xC}\text{-}\dim_{R/xR}(M) = n-1$. So M is $G_{C/xC}$-perfect as an (R/xR)-module and hence by induction hypothesis $\mathrm{Ext}^{n-1}_{R/xR}(M, C/xC)$ is $G_{C/xC}$-perfect of grade $n-1$. By [4, Lemma 3.1.16], $\mathrm{Ext}^{n-1}_{R/xR}(M, C/xC) \cong \mathrm{Ext}^n_R(M, C)$, and hence $\mathrm{grade}_{R/xR}(\mathrm{Ext}^n_R(M, C)) = \mathrm{G}_{C/xC}\text{-}\dim_{R/xR}(\mathrm{Ext}^n_R(M, C)) = n - 1$. This implies that $\mathrm{grade}_R(\mathrm{Ext}^n_R(M, C)) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^n_R(M, C)) = n$. □

4. Inequalities on relative grades

In this section, we bring some inequalities on relative grades and relative G-dimension with respect to a semidualizing R-module.

Theorem 4.1. *Let C be a semidualizing R-module, and let M and N be finitely generated R-modules. Then the following statements hold*:

(*i*) $\operatorname{depth}_R(\operatorname{Hom}_R(C,N)) - \dim_R(M) \leqslant \operatorname{grade}_{\mathcal{P}_C}(M,N)$.
(*ii*) *If* $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(\operatorname{Hom}_R(C,N))$, *then*

$$\operatorname{grade}_{\mathcal{P}_C}(M,N) \leqslant \dim_R(N) - \dim_R(M).$$

(*iii*) $\operatorname{depth}_R(C \otimes_R N) - \dim_R(M) \leqslant \operatorname{grade}_{\mathcal{I}_C}(M,N)$.
(*iv*) *If* $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(C \otimes_R N)$, *then*

$$\operatorname{grade}_{\mathcal{I}_C}(M,N) \leqslant \dim_R(N) - \dim_R(M).$$

Proof. Note that $\operatorname{grade}_{\mathcal{P}_C}(M,N) = \operatorname{grade}(M, \operatorname{Hom}_R(C,N))$ and $\operatorname{grade}_{\mathcal{I}_C}(M,N) = \operatorname{grade}(M, C \otimes_R N)$, by Theorem 3.4. Also, $\dim_R(\operatorname{Hom}_R(C,N)) = \dim_R(N) = \dim_R(C \otimes_R N)$, by Proposition 2.13. Now the assertions follow from [19, Theorem 2.1]. □

For any R-module M, the *Cohen–Macaulay defect*, $\dim_R M - \operatorname{depth}_R(M)$ is denoted by $\operatorname{cmd} M$ and the *imperfection of* M, $\operatorname{depth} R - \operatorname{depth}_R(M) - \operatorname{grade} M$ is denoted by $\operatorname{imp} M$.

COROLLARY 4.2

Let R be a local ring, and let C be a semidualizing R-module. Assume that M and N are finitely generated R-modules such that $\mathrm{G}_C\text{-}\dim_R(M) < \infty$. *Then the following statements hold*:

(*i*) *If* $\mathrm{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,N)) < \infty$, *then*

$$\mathrm{G}_C\text{-}\dim_R(M) - \operatorname{grade}_{\mathcal{P}_C}(M,N) \leqslant \mathrm{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,N)) + \operatorname{cmd} M.$$

(*ii*) *If* $\mathrm{G}_C\text{-}\dim_R(C \otimes_R N) < \infty$, *then*

$$\mathrm{G}_C\text{-}\dim_R(M) - \operatorname{grade}_{\mathcal{I}_C}(M,N) \leqslant \mathrm{G}_C\text{-}\dim_R(C \otimes_R N) + \operatorname{cmd} M.$$

Proof. It follows from Theorems 2.10 and 4.1. □

PROPOSITION 4.3

Let R be a local ring, and let C be a semidualizing R-module. If M is a finitely generated R-module, then

$$\operatorname{imp} M \leqslant \operatorname{cmd} M \leqslant \operatorname{imp} M + \operatorname{cmd} R.$$

In particular, if R is a Cohen–Macaulay ring, then $\operatorname{cmd} M = \operatorname{imp} M$. *In addition, if M is G_C-perfect, then* $\operatorname{cmd} M \leqslant \operatorname{cmd} R$.

Proof. By Remark 3.5, $\mathrm{grade}_{\mathcal{P}_C}(M, C) = \mathrm{grade}\, M$. Now by putting $N = C$ in Theorem 4.1, items (i) and (ii), we get the assertions. □

COROLLARY 4.4

Let R be a local ring, and let C be a semidualizing R-module. Let M be a finitely generated R-module such that $\mathrm{G}_C\text{-}\dim_R M < \infty$. Then the following statements hold:

(*i*) *If M is Cohen–Macaulay, then M is G_C-perfect.*
(*ii*) *If R is Cohen–Macaulay and M is G_C-perfect, then M is Cohen–Macaulay.*

Proof.

(i) In the following, the first equality follows from Theorem 2.10, and the inequality follows from Theorem 4.1:

$$\begin{aligned} \mathrm{G}_C\text{-}\dim_R(M) &= \mathrm{depth}\, R - \mathrm{depth}_R(M) \\ &= \mathrm{depth}\, R - \dim_R M \\ &= \mathrm{depth}_R(\mathrm{Hom}_R(C, C)) - \dim_R(M) \\ &\leqslant \mathrm{grade}_{\mathcal{P}_C}(M, C). \end{aligned}$$

Also, $\mathrm{grade}_{\mathcal{P}_C}(M, C) = \mathrm{grade}\, M$ by Remark 3.5. So, we get the assertion.

(ii) By Proposition 4.3, we have $\mathrm{cmd}\, M = \mathrm{imp}\, M$. Hence

$$\begin{aligned} \dim_R(M) - \mathrm{depth}_R(M) &= \mathrm{depth}\, R - \mathrm{depth}_R(M) - \mathrm{grade}\, M \\ &= \mathrm{depth}\, R - \mathrm{depth}_R(M) - \mathrm{G}_C\text{-}\dim_R(M) \\ &= 0. \end{aligned}$$

So, M is Cohen–Macaulay. □

It is known that if R is a Gorenstein local ring, and M is a finitely generated R-module, then M is a Cohen–Macaulay R-module if and only if M is G-perfect, see [19, Remark 2.5]. In the following, we generalize this result.

COROLLARY 4.5

Let R be a local ring, and let C be a dualizing R-module. A finitely generated R-module M is Cohen–Macaulay if and only if it is G_C-perfect.

Proof. Note that R is a Cohen–Macaulay ring, since R has a dualizing module C. Also, $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ by [14, Proposition 6.4.6]. Now the assertion follows from Corollary 4.4. □

Theorem 4.6. *Let C be a semidualizing R-module, and let M, N and L be finitely generated R-modules such that $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(L)$. Then the following statements hold:*

(*i*) *If* $\mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C, N)) < \infty$ *and* $\mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C, L)) < \infty$, *then*

$$\mathrm{grade}\, L + \mathrm{grade}_{\mathcal{P}_C}(M, L) \leqslant \mathrm{grade}_{\mathcal{P}_C}(M, N) + \mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C, N)).$$

(*ii*) *If* $\mathrm{G}_C\text{-}\dim_R(C \otimes_R N) < \infty$ *and* $\mathrm{G}_C\text{-}\dim_R(C \otimes_R L) < \infty$, *then*

$$\mathrm{grade}\, L + \mathrm{grade}_{\mathcal{I}_C}(M, L) \leqslant \mathrm{grade}_{\mathcal{I}_C}(M, N) + \mathrm{G}_C\text{-}\dim_R(C \otimes_R N).$$

Proof. We just prove item (i). The proof of item (ii) is similar.

(i) By Theorem 3.4, we have

$$\operatorname{grade}_{\mathcal{P}_C}(M,N)=\inf\{\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C,N))_\mathfrak{p}\,|\,\mathfrak{p}\in\operatorname{Supp}_R(M)\}.$$

Choose $\mathfrak{q}\in\operatorname{Supp}_R(M)$ such that $\operatorname{grade}_{\mathcal{P}_C}(M,N)=\operatorname{depth}_{R_\mathfrak{q}}(\operatorname{Hom}_R(C,N))_\mathfrak{q}$. So,

$$\begin{aligned}\operatorname{grade}_{\mathcal{P}_C}(M,N)&=\operatorname{depth}R_\mathfrak{q}-\operatorname{G}_{C_\mathfrak{q}}\text{-}\dim_{R_\mathfrak{q}}(\operatorname{Hom}_R(C,N))_\mathfrak{q}\\&=\operatorname{depth}_{R_\mathfrak{q}}(\operatorname{Hom}_R(C,L))_\mathfrak{q}+\operatorname{G}_{C_\mathfrak{q}}\text{-}\dim_{R_\mathfrak{q}}(\operatorname{Hom}_R(C,L))_\mathfrak{q}\\&\quad-\operatorname{G}_{C_\mathfrak{q}}\text{-}\dim_{R_\mathfrak{q}}(\operatorname{Hom}_R(C,N))_\mathfrak{q}\\&\geqslant\operatorname{grade}_{\mathcal{P}_C}(M,L)+\operatorname{grade}(\operatorname{Hom}_R(C,L))\\&\quad-\operatorname{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,N))\\&=\operatorname{grade}_{\mathcal{P}_C}(M,L)+\operatorname{grade}L-\operatorname{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,N)).\end{aligned}$$

□

COROLLARY 4.7

Let C be a semidualizing R-module, and let M, N and L be finitely generated R-modules. Then the following statements hold:

(*i*) *If* $\operatorname{Supp}_R(M)\subseteq\operatorname{Supp}_R(L)$ *and* $\operatorname{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,L))<\infty$, *then*

$$\operatorname{grade}L+\operatorname{grade}_{\mathcal{P}_C}(M,L)\leqslant\operatorname{grade}M.$$

(*ii*) *If* $\operatorname{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,N))<\infty$, *then*

$$\operatorname{grade}M\leqslant\operatorname{grade}_{\mathcal{P}_C}(M,N)+\operatorname{G}_C\text{-}\dim_R(\operatorname{Hom}_R(C,N)).$$

(*iii*) *If* $\operatorname{Supp}_R(M)\subseteq\operatorname{Supp}_R(L)$ *and* $\operatorname{G}_C\text{-}\dim_R(C\otimes_R L)<\infty$, *then*

$$\operatorname{grade}L+\operatorname{grade}_{\mathcal{I}_C}(M,L)\leqslant\operatorname{grade}M.$$

(*iv*) *If* $\operatorname{G}_C\text{-}\dim_R(C\otimes_R N)<\infty$, *then*

$$\operatorname{grade}M\leqslant\operatorname{grade}_{\mathcal{I}_C}(M,N)+\operatorname{G}_C\text{-}\dim_R(C\otimes_R N).$$

Proof.

(i) Put $N=C$ in Theorem 4.6, item (i).
(ii) Put $L=C$ in Theorem 4.6, item (i).
(iii) Put $N=R$ in Theorem 4.6, item (ii).
(iv) Put $L=R$ in Theorem 4.6, item (ii). □

In the following, for finitely generated R-modules M and N, and semidualizing R-module C, we show that $\operatorname{grade}_{\mathcal{P}_C}(M,N)=\operatorname{grade}(M,N)$ ($\operatorname{grade}_{\mathcal{I}_C}(M,N)=\operatorname{grade}(M,N)$), provided we have some special conditions.

COROLLARY 4.8

Let C be a semidualizing R-module, and let M and N be finitely generated R-modules such that $\operatorname{Supp}_R(M)\subseteq\operatorname{Supp}_R(N)$. *Then the following statements hold*:

(*i*) *If* $\mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C,N)) < \infty$, *then*

$$\begin{aligned}&\mathrm{grade}_{\mathcal{P}_C}(M,N) + \mathrm{grade}\, N \leqslant \mathrm{grade}\, M\\ &\quad \leqslant \mathrm{grade}_{\mathcal{P}_C}(M,N) + \mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C,N)).\end{aligned}$$

In particular, if $\mathrm{Hom}_R(C,N)$ *is* G_C*-perfect, then*

$$\mathrm{grade}_{\mathcal{P}_C}(M,N) = \mathrm{grade}\, M - \mathrm{grade}\, N.$$

In addition, if N *is* G*-perfect, then* $\mathrm{grade}_{\mathcal{P}_C}(M,N) = \mathrm{grade}(M,N)$.

(*ii*) *If* $\mathrm{G}_C\text{-}\dim_R(C \otimes_R N) < \infty$, *then*

$$\begin{aligned}&\mathrm{grade}_{\mathcal{I}_C}(M,N) + \mathrm{grade}\, N \leqslant \mathrm{grade}\, M\\ &\quad \leqslant \mathrm{grade}_{\mathcal{I}_C}(M,N) + \mathrm{G}_C\text{-}\dim_R(C \otimes_R N).\end{aligned}$$

In particular, if $C \otimes_R N$ *is* G_C*-perfect, then*

$$\mathrm{grade}_{\mathcal{I}_C}(M,N) = \mathrm{grade}\, M - \mathrm{grade}\, N.$$

In addition, if N *is* G*-perfect, then* $\mathrm{grade}_{\mathcal{I}_C}(M,N) = \mathrm{grade}(M,N)$.

Proof. We just prove item (i). The proof of item (ii) is similar.

(i) By Corollary 4.7, we get that

$$\begin{aligned}&\mathrm{grade}_{\mathcal{P}_C}(M,N) + \mathrm{grade}\, N \leqslant \mathrm{grade}\, M\\ &\quad \leqslant \mathrm{grade}_{\mathcal{P}_C}(M,N) + \mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C,N)).\end{aligned}$$

If $\mathrm{Hom}_R(C,N)$ is G_C-perfect, then

$$\mathrm{G}_C\text{-}\dim_R(\mathrm{Hom}_R(C,N)) = \mathrm{grade}(\mathrm{Hom}_R(C,N)) = \mathrm{grade}\, N,$$

by Proposition 3.6. So, $\mathrm{grade}_{\mathcal{P}_C}(M,N) = \mathrm{grade}\, M - \mathrm{grade}\, N$. In addition, if N is G-perfect, then $\mathrm{grade}(M,N) = \mathrm{grade}\, M - \mathrm{grade}\, N$, by [19, Corollary 2.9]. Hence $\mathrm{grade}_{\mathcal{P}_C}(M,N) = \mathrm{grade}(M,N)$. □

PROPOSITION 4.9

Let C *be a semidualizing* R*-module, and let* M, N *and* L *be finitely generated* R*-modules such that* $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(L)$. *Then the following statements hold*:

(*i*) *If* $\mathrm{G}\text{-}\dim_R(\mathrm{Hom}_R(C,N)) < \infty$ *and* $\mathrm{G}\text{-}\dim_R(\mathrm{Hom}_R(C,L)) < \infty$, *then*

$$\begin{aligned}&\mathrm{grade}\, L + \mathrm{grade}_{\mathcal{P}_C}(M,L)\\ &\quad \leqslant \mathrm{grade}_{\mathcal{P}_C}(M,N) + \mathrm{G}\text{-}\dim_R(\mathrm{Hom}_R(C,N)).\end{aligned}$$

(*ii*) *If* $\mathrm{G}\text{-}\dim_R(C \otimes_R N) < \infty$ *and* $\mathrm{G}\text{-}\dim_R(C \otimes_R L) < \infty$, *then*

$$\begin{aligned}&\mathrm{grade}\, L + \mathrm{grade}_{\mathcal{I}_C}(M,L)\\ &\quad \leqslant \mathrm{grade}_{\mathcal{I}_C}(M,N) + \mathrm{G}\text{-}\dim_R(C \otimes_R N).\end{aligned}$$

Proof. We just prove item (i). The proof of item (ii) is similar.

(i) By Lemma 3.2, $\mathrm{grade}_{\mathcal{P}_C}(M, L) = \mathrm{grade}(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C, L))$, and $\mathrm{grade}_{\mathcal{P}_C}(M, N) = \mathrm{grade}(\mathrm{Hom}_R(C, M), \mathrm{Hom}_R(C, N))$. Also, $\mathrm{grade}(\mathrm{Hom}_R(C, L)) = \mathrm{grade}\, L$, by Proposition 3.6. Now the assertion follows from [19, Theorem 2.6], since $\mathrm{Supp}_R(\mathrm{Hom}_R(C, M)) \subseteq \mathrm{Supp}_R(\mathrm{Hom}_R(C, L))$ by Proposition 2.13.

□

Using the same method of the proof of Corollaries 4.7 and 4.8, we obtain the following corollaries.

COROLLARY 4.10

Let C be a semidualizing R-module, and let M, N and L be finitely generated R-modules. Then the following statements hold:

(*i*) *If* $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(L)$ *and* $\mathrm{G\text{-}dim}_R(\mathrm{Hom}_R(C, L)) < \infty$, *then*

$$\mathrm{grade}\, L + \mathrm{grade}_{\mathcal{P}_C}(M, L) \leqslant \mathrm{grade}\, M.$$

(*ii*) *If* $\mathrm{G\text{-}dim}_R(\mathrm{Hom}_R(C, N)) < \infty$, *then*

$$\mathrm{grade}\, M \leqslant \mathrm{grade}_{\mathcal{P}_C}(M, N) + \mathrm{G\text{-}dim}_R(\mathrm{Hom}_R(C, N)).$$

(*iii*) *If* $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(L)$ *and* $\mathrm{G\text{-}dim}_R(C \otimes_R L) < \infty$, *then*

$$\mathrm{grade}\, L + \mathrm{grade}_{\mathcal{I}_C}(M, L) \leqslant \mathrm{grade}\, M.$$

(*iv*) *If* $\mathrm{G\text{-}dim}_R(C \otimes_R N) < \infty$, *then*

$$\mathrm{grade}\, M \leqslant \mathrm{grade}_{\mathcal{I}_C}(M, N) + \mathrm{G\text{-}dim}_R(C \otimes_R N).$$

COROLLARY 4.11

Let C be a semidualizing R-module, and let M and N be finitely generated R-modules such that $\mathrm{Supp}_R(M) \subseteq \mathrm{Supp}_R(N)$. *Then the following statements hold*:

(*i*) *If* $\mathrm{G\text{-}dim}_R(\mathrm{Hom}_R(C, N)) < \infty$, *then*

$$\begin{aligned}\mathrm{grade}_{\mathcal{P}_C}(M, N) + \mathrm{grade}\, N &\leqslant \mathrm{grade}\, M \\ &\leqslant \mathrm{grade}_{\mathcal{P}_C}(M, N) + \mathrm{G\text{-}dim}_R(\mathrm{Hom}_R(C, N)).\end{aligned}$$

In particular, if $\mathrm{Hom}_R(C, N)$ *is G-perfect, then*

$$\mathrm{grade}_{\mathcal{P}_C}(M, N) = \mathrm{grade}\, M - \mathrm{grade}\, N.$$

In addition, if N is G-perfect, then $\mathrm{grade}_{\mathcal{P}_C}(M, N) = \mathrm{grade}(M, N)$.

(*ii*) *If* $\mathrm{G\text{-}dim}_R(C \otimes_R N) < \infty$, *then*

$$\begin{aligned}\mathrm{grade}_{\mathcal{I}_C}(M, N) + \mathrm{grade}\, N &\leqslant \mathrm{grade}\, M \\ &\leqslant \mathrm{grade}_{\mathcal{I}_C}(M, N) + \mathrm{G\text{-}dim}_R(C \otimes_R N).\end{aligned}$$

In particular, if $C \otimes_R N$ *is G-perfect, then*

$$\mathrm{grade}_{\mathcal{I}_C}(M, N) = \mathrm{grade}\, M - \mathrm{grade}\, N.$$

In addition, if N is G-perfect, then $\mathrm{grade}_{\mathcal{I}_C}(M, N) = \mathrm{grade}(M, N)$.

5. Relative grade of tensor and Hom functors

In this section, we get some results about relative grade of tensor and Hom functors with respect to a semidualizing R-module. First, we recall a result from Auslander which is known as depth formula.

Depth Formula [2, *Theorem* 1.2]. Let R be a local ring, and let M and N be finitely generated R-modules such that $\mathrm{pd}_R(N) < \infty$, and $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$. Then $\mathrm{depth}_R(M \otimes_R N) = \mathrm{depth}_R(M) - \mathrm{pd}_R(N)$.

Theorem 5.1. *Let C be a semidualizing R-module,and let M and N be finitely generated R-modules such that $\mathrm{pd}_R(N) < \infty$, and $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$. If $\mathrm{Ext}_R^{i>0}(C, M \otimes_R N) = 0$ and $\mathrm{Ext}_R^{i>0}(C, M) = 0$ (e.g., $M \otimes_R N \in \mathcal{B}_C(R)$, and $M \in \mathcal{B}_C(R)$), then for any finitely generated R-module L, we have*

$$\mathrm{grade}_{\mathcal{P}_C}(L, M) \leqslant \mathrm{grade}_{\mathcal{P}_C}(L, M \otimes_R N) + \mathrm{pd}_R(N).$$

In addition, if $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(N)$, then

$$\mathrm{grade}_{\mathcal{P}_C}(L, M \otimes_R N) + \mathrm{grade}\, N \leqslant \mathrm{grade}_{\mathcal{P}_C}(L, M).$$

In particular, if N is a perfect R-module, then the equalities hold.

Proof. By Theorem 3.4, choose $\mathfrak{q} \in \mathrm{Supp}_R(L)$ such that $\mathrm{grade}_{\mathcal{P}_C}(L, M \otimes_R N) = \mathrm{depth}_{R_\mathfrak{q}}(\mathrm{Hom}_R(C, M \otimes_R N)_\mathfrak{q})$. In the following, the first and the third equalities follow from [1, Lemma 4.1] and the second equality follows from depth formula:

$$\begin{aligned}\mathrm{depth}_{R_\mathfrak{q}}(\mathrm{Hom}_R(C, M \otimes_R N)_\mathfrak{q}) &= \mathrm{depth}_{R_\mathfrak{q}}((M \otimes_R N)_\mathfrak{q}) \\ &= \mathrm{depth}_{R_\mathfrak{q}}(M_\mathfrak{q}) - \mathrm{pd}_{R_\mathfrak{q}}(N_\mathfrak{q}) \\ &= \mathrm{depth}_{R_\mathfrak{q}}((\mathrm{Hom}_R(C, M))_\mathfrak{q}) - \mathrm{pd}_{R_\mathfrak{q}}(N_\mathfrak{q}) \\ &\geqslant \mathrm{grade}_{\mathcal{P}_C}(L, M) - \mathrm{pd}_R(N).\end{aligned}$$

Now suppose that $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(N)$. By Theorem 3.4, choose $\mathfrak{p} \in \mathrm{Supp}_R(L)$ such that $\mathrm{grade}_{\mathcal{P}_C}(L, M) = \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})$. In the following, the first and the third equalities follow from [1, Lemma 4.1] and the second equality follows from depth formula:

$$\begin{aligned}\mathrm{depth}_{R_\mathfrak{p}}(\mathrm{Hom}_R(C, M)_\mathfrak{p}) &= \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}((M \otimes_R N)_\mathfrak{p}) + \mathrm{pd}_{R_\mathfrak{p}}(N_\mathfrak{p}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}(\mathrm{Hom}_R(C, M \otimes_R N)_\mathfrak{p}) + \mathrm{pd}_{R_\mathfrak{p}}(N_\mathfrak{p}) \\ &\geqslant \mathrm{grade}_{\mathcal{P}_C}(L, M \otimes_R N) + \mathrm{grade}(N_\mathfrak{p}) \\ &\geqslant \mathrm{grade}_{\mathcal{P}_C}(L, M \otimes_R N) + \mathrm{grade}\, N.\end{aligned}$$

□

Theorem 5.2. *Let C be a semidualizing R-module. Let M and N be finitely generated R-modules such that $\mathrm{G}_C\text{-}\mathrm{dim}_R(N) < \infty$, and $\mathrm{Ext}_R^i(C \otimes_R M, N) = 0$ for all $i > 0$. Then for any finitely generated R-module L, we have*

$$\mathrm{grade}_{\mathcal{P}_C}(L, \mathrm{Hom}_R(M, N)) + \mathrm{G}_C\text{-}\mathrm{dim}\, N \geqslant \mathrm{grade}\, L.$$

In addition, if $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(\mathrm{Hom}_R(M, N))$, *then*

$$\mathrm{grade}\, L \geqslant \mathrm{grade}_{\mathcal{P}_C}(L, \mathrm{Hom}_R(M, N)) + \mathrm{grade}\, N.$$

In particular, if N *is a* G_C*-perfect* R*-module, then the equalities hold.*

Proof. By Theorem 3.4, choose $\mathfrak{q} \in \mathrm{Supp}_R(L)$ such that

$$\mathrm{grade}_{\mathcal{P}_C}(L, \mathrm{Hom}_R(M, N)) = \mathrm{depth}_{R_\mathfrak{q}}(\mathrm{Hom}_R(C, \mathrm{Hom}_R(M, N))_\mathfrak{q}).$$

In the following, the first equality follows from adjointness, the second one follows from [1, Lemma 4.1] and the third one follows from Theorem 2.10:

$$\begin{aligned}\mathrm{depth}_{R_\mathfrak{q}}(\mathrm{Hom}_R(C, \mathrm{Hom}_R(M, N))_\mathfrak{q}) &= \mathrm{depth}_{R_\mathfrak{q}}(\mathrm{Hom}_R(C \otimes_R M, N))_\mathfrak{q}) \\ &= \mathrm{depth}_{R_\mathfrak{q}}(N_\mathfrak{q}) \\ &= \mathrm{depth}\, R_\mathfrak{q} - \mathrm{G}_{C_\mathfrak{q}}\text{-}\mathrm{dim}_{R_\mathfrak{q}}(N_\mathfrak{q}) \\ &\geqslant \mathrm{grade}\, L - \mathrm{G}_C\text{-}\mathrm{dim}_R(N).\end{aligned}$$

Now suppose that $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(\mathrm{Hom}_R(M, N))$. By Theorem 3.4, choose $\mathfrak{p} \in \mathrm{Supp}_R(L)$ such that $\mathrm{grade}\, L = \mathrm{depth}\, R_\mathfrak{p}$. In the following, the first equality follows from Theorem 2.10, the second one follows from [1, Lemma 4.1], and the third one follows from adjointness:

$$\begin{aligned}\mathrm{depth}\, R_\mathfrak{p} &= \mathrm{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) + \mathrm{G}_{C_\mathfrak{p}}\text{-}\mathrm{dim}_{R_\mathfrak{p}}(N_\mathfrak{p}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}(\mathrm{Hom}_R(C \otimes_R M, N))_\mathfrak{p}) + \mathrm{G}_{C_\mathfrak{p}}\text{-}\mathrm{dim}_{R_\mathfrak{p}}(N_\mathfrak{p}) \\ &= \mathrm{depth}_{R_\mathfrak{p}}(\mathrm{Hom}_R(C, \mathrm{Hom}_R(M, N))_\mathfrak{p}) + \mathrm{G}_{C_\mathfrak{p}}\text{-}\mathrm{dim}_{R_\mathfrak{p}}(N_\mathfrak{p}) \\ &\geqslant \mathrm{grade}_{\mathcal{P}_C}(L, \mathrm{Hom}_R(M, N)) + \mathrm{grade}\, N.\end{aligned}$$

□

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