Proc. Indian Acad. Sci. (Math. Sci.) (2023) 133:1
https://doi.org/10.1007/s12044-022-00721-3

A higher order Levin–Faĭnleĭb theorem

OLIVIER RAMARÉ[1], ALISA SEDUNOVA[2,*] and RITIKA SHARMA[3]

[1]CNRS/Institut de Mathématiques de Marseille, U.M.R. 7373, Aix Marseille Université, Site Sud, Campus de Luminy, Case 907, 13288 Marseille Cedex 9, France
[2]Georg-August-Universität Göttingen, Bunsenstrasse 3-5, 37073 Göttingen, Germany
[3]Stockholm University, Frescativägen, 11419 Stockholm, Sweden
*Corresponding Author.
E-mail: olivier.ramare@univ-amu.fr; alisa.sedunova@gmail.com; rish0842@student.su.se

MS received 20 January 2022; revised 12 September 2022; accepted 29 September 2022

Abstract. When restricted to some non-negative multiplicative function, say f, bounded on primes and that vanishes on non square-free integers, our result provides us with an asymptotic for $\sum_{n\le X} f(n)/n$ with error term $\mathcal{O}((\log X)^{\kappa-h-1+\varepsilon})$ (for any positive $\varepsilon > 0$) as soon as we have $\sum_{p\le Q} f(p)(\log p)/p = \kappa \log Q + \eta + \mathcal{O}(1/(\log 2Q)^h)$ for a non-negative κ and some non-negative integer h. The method generalizes the 1967-approach of Levin and Faĭnleĭb and uses a differential equation.

Keywords. Average orders; multiplicative functions.

2010 Mathematics Subject Classification. Primary: 11N37.

1. Introduction

In 1908, Landau [8] was the first to obtain an asymptotic formula for the number of integers up to a given number that are sum of two coprime squares. He used analytical method, which involves considering the square root of some analytical function and avoiding its pole through Hankel contour. Later, this procedure was further developed by H. Delange and A. Selberg allowing them to obtain asymptotic for partial sums of arithmetic functions whose Dirichlet series can be written in terms of complex powers of the Riemann ζ-function. This is now often referred to as the Selberg–Delange method. In [9], Levin and Faĭnleĭb established the logarithmic density of the same set by an elementary argument under more general conditions. When combined with the earlier method of Wirsing [19], as was done in [13], this leads to the determination of the natural density as well.

In [18], Serre used Landau's method to examine several other cases and deployed it to encompass not only the main term but also an asymptotic development, leading to a better error term. Extending the Levin and Faĭnleĭb approach in a similar fashion would allow for a more general hypotheses as well. This is the aim of the present paper. To express our results, we take a non-negative multiplicative function f and, following Levin and Faĭnleĭb, we associate to it the function $\Lambda_f(n)$ which is 0 when n is not a prime power

Published online: 12 January 2023

and which is otherwise defined by the formal power expansion:

$$\sum_{k\geq 0}\frac{\Lambda_f(p^k)}{p^{ks}}=\Big(\sum_{k\geq 1}\frac{f(p^k)\log p}{p^{ks}}\Big)\Big/\sum_{k\geq 0}\frac{f(p^k)}{p^{ks}}. \tag{1}$$

We recall some of its properties in Section 2. To handle the uniformity in our result, we recall that we use $f=\mathcal{O}^*(g)$ to mean that $|f|\leq g$ and $f=\mathcal{O}_{A,h,\kappa}(g)$ to mean that $|f|\leq C(A,\kappa,h)g$, where the constant $C(A,h,\kappa)$ depends only on the stated parameters. Here is our main theorem.

Theorem 1. *Let f be a non-negative multiplicative function. Assume that, for some integer $h\geq 0$, one has*

$$\forall Q\geq 1,\quad \sum_{m\leq Q}\frac{\Lambda_f(m)}{m}=\kappa\log Q+\eta_0+\mathcal{O}^*(A/\log^h(2Q)) \tag{H_h}$$

for some constants $\kappa\geq 0$, A and η_0. We further assume that $|\eta_0|\leq A$. Then there exist constants C and $(a_k)_{1\leq k\leq h}$ such that, when $X\geq 3$, we have

$$\sum_{n\leq X}\frac{f(n)}{n}(\log n)^{h+1}=C(\log X)^{\kappa+h+1}\Big(1+\frac{a_1}{\log X}+\cdots+\frac{a_h}{(\log X)^h}\Big)$$
$$+\mathcal{O}_{A,\kappa,h}\big((\log X)^{\kappa}(\log\log(3X))^{\frac{(h+2)(h+1)}{2}}\big),$$

where

$$C=\frac{1}{\Gamma(\kappa+1)}\prod_{p\geq 2}\Big(\Big(1-\frac{1}{p}\Big)^{\kappa}\sum_{\nu\geq 0}\frac{f(p^\nu)}{p^\nu}\Big).$$

We have the same error term for the sum

$$\sum_{n\leq X}\frac{f(n)}{n}\Big(\log\frac{X}{n}\Big)^{h+1}.$$

We can also obtain $\sum_{n\leq X}\frac{f(n)}{n}(\log n)^k$ for any $k\in\{0,\ldots,h\}$ with an error term $\mathcal{O}((\log\log X)^{\frac{(h+2)(h+1)}{2}}/(\log X)^{h+1-k})$, by summation by parts, but some additional $\log\log X$ term may appear in the development when κ is an integer, which is why we state our result in this manner. The non-negative assumption is not essential in our method, nor is the fact that f is real-valued (but κ has to be a real number). We may instead assume that

$$\sum_{n\leq X}|f(n)|\ll(\log X)^{\kappa^*} \tag{2}$$

for some parameter κ^* and modify our error term $\mathcal{O}((\log X)^{\kappa}(\log\log X)^c)$ to $\mathcal{O}((\log X)^{\kappa^*}(\log\log X)^c)$. This is for instance the path chosen, when $h=0$ in Theorem 1.1 of the book by Iwaniec and Kowalski [6]. We did not try to optimize the power of $\log\log(3X)$ that appears. It is likely that no such term should be present in fact, but in practice, when our assumption holds for $h\geq 1$, it holds for any h. Using the result for $h+1$ removes this parasitic factor.

To measure the relative strength of our theorem, let us consider $h = 1$ and $f(n) = -\mu(n)$, where $\mu(n)$ is the Moebius function, hence $\Lambda_f(p^k) = (-1)^{k-1}\log p$, $k \geq 1$. Then the estimate $\sum_{p\leq X}(\log p)/p = \log X + c + \mathcal{O}(1/\log X)$ verifies (H_h) with $\kappa = 1$ and implies that $\sum_{n\leq X}\mu(n)/n \ll (\log\log X)^3/\log X$. The case $h > 1$ yields another proof of the results of Kienast in [7].

We started this project several years back, and got sidetracked by several events. In between Granville and Koukoulopoulos in [5] considered a similar question which they attacked via the Landau–Selberg–Delange method. Their work has been improved upon by de la Bretêche and Tenenbaum in [2]. The results obtained by this mean are more extensive than the ones we present here. However the main difference truly comes at the methodological level: our proof stays in the realm of real analysis and is rather 'elementary'. The readers may also consult [16, pp. 183–185], [15] and [10] on related issues.

The proof relies on a recursion on h. It is however easier to assume a more complete hypothesis.

Recursion Hypothesis (for h). For each $\ell \in [0, h]$, there exists a polynomial P_ℓ of degree ℓ such that

$$\sum_{n\leq X}\frac{f(n)}{n}(\log n)^\ell = \left(P_\ell(\log X) + \mathcal{O}\left((\log\log X)^{\frac{(h+1)(h+2)}{2}}\right)\right)(\log X)^\kappa. \tag{3}$$

We show during the proof that we may as well assume a similar hypothesis with $(\log(X/n))^\ell$ rather than $(\log n)^\ell$: this is a consequence of the functional relation we prove at the beginning of our proof, see (11). The Levin–Faĭnleĭb theorem gives a proof of this claim when $h = 0$ (and even better as the $\log\log(3X)$ is absent in this theorem). We provide in Section 8 a survey of the proof.

Notation. We set for typographical simplicity $g(n) = f(n)/n$. Next, for a non-negative integer j define

$$G_j(X) = \sum_{n\leq X} g(n)\log^j(X/n), \quad G_0(X) = G(X). \tag{4}$$

For $k \geq 0$, we define $H_k(\log X) = G_k(X)$.

2. On the function Λ_f

Let F denote the formal Dirichlet series of f, namely

$$F(s) = \sum_{n\geq 1}\frac{f(n)}{n^s}.$$

Note that Euler product formula gives

$$F(s) = \prod_{p\geq 2}\left(1 + \sum_{k\geq 1}\frac{f(p^k)}{p^{ks}}\right).$$

On taking the logarithmic derivative of $F(s)$, we find that

$$-\frac{F'(s)}{F(s)} = \sum_{p\geq 2}\left(\sum_{k\geq 1}\frac{f(p^k)}{p^{ks}}\log(p^k)\right)\left(1 + \sum_{k\geq 1}\frac{f(p^k)}{p^{ks}}\right)^{-1}$$
$$= \sum_{p\geq 2} Z_p(s)\log p.$$

Further, expanding the second product in $Z_p(s)$ and changing the order of summation, we find that

$$Z_p(s) = \sum_{k\geq 1} \frac{kf(p^k)}{p^{ks}} \sum_{r\geq 0}(-1)^r \sum_{\ell\geq 0} \sum_{k_1+k_2+\cdots+k_r=\ell} \frac{f(p^{k_1})\cdots f(p^{k_r})}{p^{\ell s}}$$
$$= \sum_{m\geq 1} \frac{1}{p^{ms}} \left(\sum_{k+k_1+\cdots+k_r=m} (-1)^r kf(p^k)f(p^{k_1})\cdots f(p^{k_r}) \right).$$

Thus

$$-\frac{F'(s)}{F(s)} = \sum_{n\geq 1} \frac{\Lambda_f(n)}{n^s}, \tag{5}$$

where

$$\Lambda_f(p^m) = \sum_{k+k_1+\cdots+k_r=m} (-1)^r kf(p^k)f(p^{k_1})\cdots f(p^{k_r}) \log p \tag{6}$$

and $\Lambda_f(n) = 0$ when n is not a prime power. Note that $\Lambda_f(p^m)$ depends only on the local factor of $F(s)$ at prime p. In particular, $\Lambda_1(p^m) = \Lambda(p^m)$. Moreover, when $f(p^m) = 1_{p\in\mathcal{P}}$, we have $\Lambda_f(p^m) = \Lambda(p^m)\cdot f(p^m)$ (here $1_X = 1$ if X is true and 0 otherwise). For example, let us select $\mathcal{P} = \{p \equiv 1 \pmod 4\}$. As mentioned above, the definition of $\Lambda_f(p^m)$ depends only on the local factor at prime p, hence we readily see that $\Lambda_f(p^m) = \Lambda(p^m)$ for $p \equiv 1 \pmod 4$ and 0 otherwise. Note that when f is supported on square-free integers, we get $\Lambda_f(p^m) = (-1)^{m-1} f(p)^m \log p$.

Lemma 1. *Let h be a non-negative real number. Then for any $k \leq h$, there exists a constant η_k, such that, under the assumption* (H_h), *we have*

$$\sum_{n\leq Q} \frac{\Lambda_f(n)\log^k n}{n} = \frac{\kappa}{k+1}\log^{k+1} Q + \eta_k + E_{k,h}(Q), \tag{A}$$

where $E_{k,h}(Q) \ll 1/\log^{h-k}(2Q)$ for $k < h$ and $E_{h,h}(Q) \ll \log\log(3Q)$.

Proof. Denote the sum on the left-hand side of (A) by $S_k(Q)$. Then using partial summation, we have

$$S_k(Q) = S_0(Q)\log^k Q - k\int_1^Q S_0(t)\log^{k-1} t \,\frac{dt}{t}.$$

Further, when $k < h$, we may apply (H_h) to get

$$S_k(Q) = \frac{\kappa}{k+1}\log^{k+1} Q + \eta_0 \log^k Q - \eta_0 k \int_1^Q \frac{\log^{k-1} t dt}{t}$$
$$- k\int_1^\infty \Big(S_0(t) - \kappa\log t - \eta_0\Big)\frac{\log^{k-1} t dt}{t}$$
$$+ \mathcal{O}\bigg(\frac{1}{\log^{h-k} Q} + \int_Q^\infty \frac{d\log t}{\log^{h-k+1} t}\bigg),$$

whence

$$S_k(Q) = \frac{\kappa}{k+1}\log^{k+1} Q + \eta_k + \mathcal{O}(1/\log^{h-k}(2Q))$$

as announced. Analogous argument gives the result for $k = h$. □

3. Generalizations of Λ_f

We will use the next formula several times.

Lemma 2 (*Faà di Bruno formula*). *We have*

$$\frac{d^n f(g(x))}{dx^n} = \sum_{\substack{m_1,m_2,\ldots,m_n\geq 0,\\ m_1+2m_2+\cdots+nm_n=n}} \frac{n!}{m_1!m_2!\cdots m_n!} f^{(m_1+m_2+\cdots+m_n)}(g(x)) \prod_{j=1}^{n}\left(\frac{g^{(j)}(x)}{j!}\right)^{m_j}.$$

Here is a combinatorial identity, which is an immediate corollary of [14, Theorem 2.1], itself being a straightforward consequence of the Faà di Bruno formula.

Lemma 3. *Let F be a function and denote $Z_F = -F'/F$. We have*

$$F^{(h+1)} = F \sum_{\sum_{i\geq 1} ik_i = h+1} \frac{(h+1)!(-1)^{\sum_i k_i}}{k_1!k_2!\cdots(1!)^{k_1}(2!)^{k_2}\ldots} \prod_{k_i} Z_F^{(i-1)k_i}.$$

Notation $Z_F^{(i-1)k_i}$ denotes the $(i-1)$-th derivative multiplied k_i times by itself.

Proof. This is an immediate corollary of [14, Theorem 2.1] with $F = 1/G$ and hence $Z_F = -Z_G$. □

When $h = 1$, this gives $F'' = F(Z_F^2 - Z_F')$. We thus define

$$\sum_{n\geq 1}\frac{\Lambda_{f,h}(n)}{n^s} = (-1)^h \sum_{\sum_{i\geq 1} ik_i = h} \frac{h!(-1)^{\sum_i k_i}}{k_1!k_2!\cdots(1!)^{k_1}(2!)^{k_2}\ldots} \prod_{k_i} Z_F^{(i-1)k_i} \tag{7}$$

so that

$$f\log^h = f \star \Lambda_{f,h}.$$

When $f = 1$, these functions have their origin in the work of Selberg [17] around an elementary proof of the Prime Number Theorem. They have been generalized as above by Bombieri in [1], see also the papers [3] and [4]. Incidentally, Lemma 3 gives a non-recursive description of the functions $\Lambda_h = \Lambda_{1,h}$, something that is missing from the aforementioned works.

Lemma 4. *Let θ_1 and θ_2 be two functions on the integers that satisfy*, for $i \in \{1, 2\}$,

$$\sum_{n \le X} \theta_i(n) = C_i (\log X)^{d_i} + Q_i(\log X) + \mathcal{O}(1/(\log 2X)^{h-d_i}),$$

where $d_i \ge 1$, Q_i is a polynomial of degree at most $d_i - 1$ and h is some fixed parameter. Then

$$\sum_{mn \le X} \theta_1(m)\theta_2(n) = C_1 C_2 \frac{d_1! d_2!}{(d_1 + d_2)!} (\log X)^{d_1+d_2} + Q(\log X) + \mathcal{O}\left(\frac{1}{(\log 2X)^{h-d_1-d_2}}\right),$$

where Q is a polynomial of degree at most $d_1 + d_2 - 1$.

Proof. We use the Dirichlet Hyperbola formula. We split the variables at $\sqrt{X}$ to get the announced error term. In order to compute the main term, it is enough to consider

$$S = \sum_{n \le X} \theta_1(n) C_2 \left(\log \frac{X}{n}\right)^{d_2}.$$

An integration by parts gives us

$$S = C_2 \sum_{n \le X} \theta_1(n) d_2 \int_1^{X/n} (\log t)^{d_2 - 1} \frac{dt}{t} = C_2 d_2 \int_1^X \sum_{n \le X/t} \theta_1(n) (\log t)^{d_2-1} \frac{dt}{t},$$

so that the principal part of the main term is given by

$$M = C_1 C_2 d_2 \int_1^X \left(\log \frac{X}{t}\right)^{d_1} (\log t)^{d_2-1} \frac{dt}{t} = C_1 C_2 d_2 (\log X)^{d_1+d_2} \int_0^1 (1-u)^{d_1} u^{d_2-1} du = C_1 C_2 \frac{d_1! d_2!}{(d_1+d_2)!} (\log X)^{d_1+d_2}$$

by the classical evaluation of the Euler beta-function. □

On iterating the previous lemma, we get the next one.

Lemma 5. *Let $(\theta_i)_{i \le r}$ be r functions on the integers that satisfy, for $i \in \{1, \ldots, r\}$,*

$$\sum_{n \le X} \theta_i(n) = C_i (\log X)^{d_i} + Q_i(\log X) + \mathcal{O}(1/(\log 2X)^{h-d_i}),$$

where $d_i \geq 1$, Q_i is a polynomial of degree at most $d_i - 1$ and h is some fixed parameter. Then

$$\sum_{m_1 \cdots m_r \leq X} \prod_{i \leq r} \theta_i(m_i) = \prod_{i \leq r} C_i \frac{d_1! \cdots d_r!}{(d_1 + \cdots + d_r)!} (\log X)^{d_1 + \cdots + d_r} + Q(\log X) + \mathcal{O}\left(\frac{1}{(\log 2X)^{h - d_1 - \cdots - d_r}} \right),$$

where Q is a polynomial of degree at most $d_1 + d_2 + \cdots + d_r - 1$.

Lemma 6. *Under* (H_h), *we have*

$$\sum_{n \leq X} \frac{\Lambda_{f,k}(n)}{n} = \frac{\kappa(\kappa+1)\cdots(\kappa+k-1)}{k!} (\log X)^k + Q(\log X) + \mathcal{O}\left(\frac{\log\log(3X)}{(\log 2X)^{h+1-k}} \right),$$

where Q is polynomial of degree at most $k - 1$.

Proof. Lemma 5 tells us that the sum reads

$$\sum_{n \leq X} \frac{\Lambda_{f,k}(n)}{n} = \sum_{\sum_{i \geq 1} i k_i = k} \frac{k!}{k_1! k_2! \cdots (1!)^{k_1} (2!)^{k_2} \cdots} \prod_i \frac{\kappa^{k_i} i!^{k_i}}{i^{k_i}} \frac{(\log X)^k}{k!} + Q(\log X) + \mathcal{O}\left(\frac{\log\log(3X)}{(\log 2X)^{h+1-k}} \right). \quad (8)$$

The main term simplifies into

$$\sum_{\sum_{i \geq 1} i k_i = k} \frac{1}{k_1! k_2! \cdots} \prod_i \frac{\kappa^{k_i}}{i^{k_i}} (\log X)^k.$$

The i-th derivative of $g(x) = -\kappa \log(1 - x)$ is $(i - 1)!\kappa/(1 - x)^i$ so that κ/i is also $g^{(i)}(0)/i!$. The Faà di Bruno formula for the k-th derivative of $\exp(g(x)) = (1 - x)^{-\kappa}$ tells us that

$$\sum_{\sum_{i \geq 1} i k_i = k} \frac{k!}{k_1! k_2! \cdots} \prod_i \frac{\kappa^{k_i}}{(i(1-x)^i)^{k_i}} = \frac{\kappa(\kappa+1)\cdots(\kappa+k-1)}{(1-x)^{\kappa+k}}.$$

We evaluate this equality at $x = 0$. □

4. Auxiliary results

Lemma 7. *For $k \geq 1$, we have*

$$G_k(X) = k \int_1^X G_{k-1}(t) \frac{dt}{t}.$$

Proof. Notice that by a simple change of variable $t = \log(u/n)$, we have

$$\frac{1}{k}\left(\log\frac{X}{n}\right)^k = \int_0^{\log(X/n)} t^{k-1}dt = \int_n^X \left(\log\frac{u}{n}\right)^{k-1}\frac{du}{u}.$$

Using the above together with the definition of G_k, we directly compute

$$\begin{aligned}\int_1^X G_{k-1}(t)\frac{dt}{t} &= \sum_{n\le X} g(n)\int_n^X \left(\log\frac{t}{n}\right)^{k-1}\frac{dt}{t}\\ &= \frac{1}{k}\sum_{n\le X} g(n)\left(\log\frac{X}{n}\right)^k = \frac{G_k(X)}{k}\end{aligned}$$

as claimed in the lemma. □

Here is a direct consequence of the previous lemma, on recalling that $H_k(\log X) = G_k(X)$.

Lemma 8. *When* $\ell \in \{0, \ldots, k\}$, *we have*

$$H_k^{(\ell)}(u) = \frac{k!}{(k-\ell)!}G_{k-\ell}(e^u).$$

Lemma 9. *When* $k \ge 0$, *we have*

$$\sum_{n\le e^u} g(n)(\log n)^k = \frac{u^{k+1}}{k!}(H_k(u)/u)^{(k)}.$$

Proof. This lemma is true for $k = 0$. For $k = 1$, we find that

$$u^2(H_1(u)/u)^{(1)} = uH_1'(u) - H_1(u) = \sum_{n\le e^u} g(n)\big(u - (u - \log n)\big)$$

as required. For general k, write

$$\begin{aligned}\sum_{n\le e^u} g(n)(\log n)^k &= \sum_{n\le e^u} g(n)\left(u - \log\frac{e^u}{n}\right)^k\\ &= \sum_{0\le j\le k}\binom{k}{j}u^j(-1)^{k-j}G_{k-j}(e^u)\\ &= \sum_{0\le j\le k}\binom{k}{j}u^j(-1)^{k-j}\frac{(k-j)!}{k!}H_k^{(j)}(u).\end{aligned}$$

We next notice that

$$\frac{d^\ell}{du^\ell}\frac{1}{u} = \frac{(-1)^\ell \ell!}{u^{\ell+1}}$$

so that

$$\sum_{n\le e^u} g(n)(\log n)^k = \frac{u^{k+1}}{k!}\sum_{0\le j\le k}\binom{k}{j}(-1)^{k-j}\frac{(k-j)!}{u^{k-j+1}}H_k^{(j)}(u)$$
$$= \frac{u^{k+1}}{k!}(H_k(u)/u)^{(k)}$$

as announced. □

5. Approximate solutions of an Euler differential equation

Popa and Pugna in [11], and building on [12] studied perturbation of an Euler differential equation, say

$$u^r y^{(r)}(u) + \sum_{0\le i\le r-1} b_i u^i y^{(i)}(u) \tag{9}$$

for a function y that is in $C^r(I)$ for some interval $I \subset [0,\infty)$. On looking more closely at their work which goes by iteration, one sees that the last derivative does not need to be continuous provided one may integrate, and so may be simply *absolutely continuous on every subinterval of* I. We denote this class by $C^{r-}(I)$.

We next need a second modification of their work. For any $c \in I$, any complex number α and any suitable function φ, they consider

$$\Phi^*_{\alpha,c}(\varphi)(x) = x^{\Re\alpha}\left|\int_c^x u^{-\Re\alpha}\varphi(u)\frac{du}{u}\right|.$$

Notice that Popa and Pugna [11] forgot this change of variable that is necessary between their Theorems 2.1 and 2.3. This explains our notation Φ^* rather than the Φ that they got. We have added the index c to their notation and we may in fact take $c=\infty$ (and reverse the order of integration as usual). We select r parameters $c_1,\dots,c_r$, some of them maybe be infinite.

Following [11], we consider the roots $\lambda_1,\dots,\lambda_r$ of the equation

$$b_0 + \sum_{1\le s\le r}\lambda(\lambda-1)\cdots(\lambda-s+1)b_s = 0, \quad b_r = 1. \tag{10}$$

We also select a function S in $C^r(I)$. With these notations, here is the version of [11, Theorem 2.3] that we shall use.

Lemma 10. *Let* $\varphi : I \to [0,\infty)$ *be such that* $\Phi^*_{\lambda_r,c_r}\circ\cdots\circ\Phi^*_{\lambda_1,c_1}(\varphi)$ *exists and is finite. Then for every* $y \in C^{r-}(I)$ *satisfying*

$$\forall u \in I,\quad \left|u^r y^{(r)}(u) + \sum_{0\le i\le r-1} b_i u^i y^{(i)}(u) - S(u)\right| \le \varphi(u),$$

there exists a solution y_0 *of*

$$u^r y^{(r)}(u) + \sum_{0\le i\le r-1} b_i u^i y^{(i)}(u) = 0$$

with the property

$$\forall u \in I, \quad |y(u) - y_0(u)| \leq \Phi^*_{\lambda_r, c_r} \circ \cdots \circ \Phi^*_{\lambda_1, c_1}(\varphi)(u).$$

6. A differential equation

On using Lemmas 3 and 6, we get

$$\sum_{n \leq X} g(n)(\log n)^{h+1} = \sum_{n \leq X} g(n)\Bigg(\frac{\kappa(\kappa+1)\cdots(\kappa+h)}{(h+1)!}\left(\log\frac{X}{n}\right)^{h+1} + Q(\log(X/n)) + \mathcal{O}(\log\log(3X))\Bigg). \tag{11}$$

Hence, by our recursion hypothesis in h, we get

$$\sum_{n \leq X} g(n)(\log n)^{h+1} = \frac{\kappa(\kappa+1)\cdots(\kappa+h)}{(h+1)!}G_{h+1}(X) + P(\log X)(\log X)^{\kappa} + \mathcal{O}((\log X)^{\kappa}(\log\log X)^{\frac{(h-1)h}{2}}) \tag{12}$$

for some polynomial P of degree at most h. Here we have used the recursion hypothesis with $(\log X/n)^k$. It is precisely Equation (12) that allows us to switch easily from one form of our hypothesis to the other. When $h = 1$, then $h - 1 = 0$, and we do not have the power of $\log\log X$.

We may express the left-hand side by Lemma 9, getting our first fundamental formula:

$$u^{h+1}\left(\frac{H_{h+1}(u)}{u}\right)^{(h+1)} = \frac{(\kappa+h)!}{(\kappa-1)!}\frac{H_{h+1}(u)}{u} + (h+1)!P(u)u^{\kappa-1} + \mathcal{O}(u^{\kappa-1}(\log u)^{\frac{h(h-1)}{2}}), \tag{13}$$

where we use the shortcut

$$\frac{(\kappa+h)!}{(\kappa+h-j)!} = (\kappa+h)\cdots(\kappa+h-j+1).$$

This is an *Euler* differential equation. As mentioned before, it may be reduced to a linear differential equation with constant coefficients with the change of variables $u = e^v$, but we shall skip this step and use an already formed result. It is technically clearer to first extract a 'simplifying term' and this is our first step.

Simplifying the equation. Since we may assume that the polynomial P has no constant coefficient, we set

$$(h+1)!P(u) = \sum_{1 \leq s \leq h} q_s u^s.$$

We define, for $0 \leq s \leq h-1$, the real number a_s by

$$\left(\frac{(\kappa+s)!}{(\kappa-1)!} - \frac{(\kappa+h)!}{(\kappa-1)!}\right)a_s = q_{s-1}.$$

We then check that $K(u) = \sum_{0\le s\le s-1} a_s u^{s+\kappa}$ satisfies

$$u^{h+1}K^{(h+1)}(u) = \frac{(\kappa+h)!}{(\kappa-1)!}K(u) + (h+1)!P(u)u^{\kappa-1}.$$

Note that we could have added any monomial $a_h u^{h+\kappa}$ to $K(u)$.

From the approximate differential equation to the exact one. We define $W(u) = H_{h+1}(u)u^{-1} - K(u)$. This function satisfies

$$u^{h+1}W^{(h+1)}(u) = \frac{(\kappa+h)!}{(\kappa-1)!}W(u) + \mathcal{O}(u^{\kappa-1}(\log u)^{\frac{h(h-1)}{2}}).$$

We can now use Lemma 10. At the beginning, we should consider the roots $\lambda_1 = \kappa + h, \ldots, \lambda_r$ of the equation

$$\lambda(\lambda-1)\cdots(\lambda-h) = \kappa(\kappa+1)\cdots(\kappa+h)$$

that are such that $\lambda_i > \kappa - 1$. Set $\varphi(u) = Cu^{\kappa-1}(\log 2u)^{\frac{h(h-1)}{2}}$ for a large enough constant C, so that

$$\left|u^{h+1}W^{(h+1)}(u) - \frac{(\kappa+h)!}{(\kappa-1)!}W(u)\right| \le \varphi(u).$$

We find that

$$\Phi^*_{\lambda_i,c_i}(\varphi)(u) = Cu^{\Re\lambda_i}\left|\int_{c_i}^{u} t^{(\kappa-1-\Re\lambda_i)u}\log(2t)^{\frac{h(h-1)}{2}}dt\right|.$$

When $\kappa - \Re\lambda_i > 0$, we select $c_i = 1$ and get that $\Phi^*_{\lambda_i,c_i}(\varphi)(u) \ll u^{\kappa-1}\log(2u)$. When $\kappa - \Re\lambda_i < 0$, we select $c_i = \infty$ and get the same result. There remains the case $\kappa = \Re\lambda_i$ where we select $c_i = 1$ and get a further power of $\log u$. By Lemma 10, there exist parameters $C_1, \ldots, C_r$ such that

$$\left|W(u) - \sum_{1\le s\le r} C_s u^{\lambda_s}\right| \le u^{\kappa-1}(\log 2u)^{\frac{h(h+1)}{2}}.$$

At this level, we still have not proved that the relevant roots λ_s that have a non-zero coefficient C_s are of the form $\kappa + h - \ell$.

From W to H_{h+1}. The determination of W via (14) goes to $H_{h+1}^{(h+1)}$ by (14) and the definition $W(u) = H_{h+1}(u)u^{-1} - K(u)$. We thus obtain that

$$\sum_{n\le X} g(n)(\log n)^{h+1} = \sum_i C_i(\log X)^{\theta_i} + \mathcal{O}((\log X)^{\kappa}(\log\log(3X))^{\frac{h(h+1)}{2}}),$$

where the sequence (θ_i) is the union of the one of λ_s and of $\kappa+h, \kappa+h-1, \ldots, \kappa$, coming from $K(u)$. By our functional equation (12), we have a similar development when we replace $(\log n)^{h+1}$ by $(\log X/n)^{h+1}$.

7. Ruling out the parasiting solutions

When $h = 1$, the two roots are $\kappa + 1$ and $-\kappa$. Lemma 10 then implies that we can find a and b such that

$$|W(u) - au^{\kappa+1} - bu^{-\kappa}| \ll u^{\kappa-1}\log(2u).$$

This reduces to $|W(u) - au^{\kappa+h}| \ll u^{\kappa-1}\log(2u)$ when $\kappa \geq 1/2$. But what happens when $\kappa < 1/2$?

A stability remark. Assume we have a non-negative multiplicative function f that satisfies the assumptions of our Theorem 1. Assume further we have distinct exponents $\kappa_0 = \kappa, \kappa_1, \ldots, \kappa_r \geq \kappa$ such that

$$\sum_{n\leq X}\frac{f(n)}{n}\left(\log\frac{X}{n}\right)^{h+1} = \sum_{0\leq s\leq r} C_s(\log X)^{h+1+\kappa_s} + \mathcal{O}((\log X)^{\kappa}(\log\log(3X))^{C})$$

for some non-zero constants $C_0, \ldots, C_r$ and $C \geq 0$. Select a positive integer K and consider the function τ_K that counts the number of K-tuples of divisors, so that τ_2 is the usual divisor function. Next, we consider the multiplicative function $f \star \tau_K$ that equally satisfies the assumptions of Theorem 1, with $\kappa + K$ instead of κ. By the Dirichlet Hyperbola formula, we find that

$$\sum_{n\leq X}\frac{(f\star\tau_K)(n)}{n}\left(\log\frac{X}{n}\right)^{h+1} = \sum_{0\leq s\leq r}\ \sum_{\substack{\ell\geq 0,\\ h+1+K+\kappa_s-\ell>\kappa-1}} C'_{s,\ell}(\log X)^{h+1+K+\kappa_s-\ell} + \mathcal{O}((\log X)^{K+\kappa}(\log\log(3X))^{C})$$

for some constants $C'_0, \ldots, C'_r$. This tells us that the set of exponents for $f \star \tau_K$ is $\kappa_0 + K, \ldots, \kappa_r + K$. Let κ_s denote the largest, if it exists, of the κ_i's that is not of the form $\kappa + h$ minus some integer. Then the coefficient $C'_{s,0}$ comes from the main term of

$$C_s\sum_{n\leq X}\frac{\tau_K(n)}{n}\left(\log\frac{X}{n}\right)^{h+1+K+\kappa_s}$$

and is thus a non-zero multiple of C_s.

General case. In general, the discussion of the previous subsection applies. We only need to consider the roots of λ of

$$R_h(\lambda,\kappa) = \lambda(\lambda-1)\cdots(\lambda-h) - \kappa(\kappa+1)\cdots(\kappa+h)$$

that are such that $R_h(\lambda+K, \kappa+K) = 0$ when K is a positive integer. This leads to a polynomial in K of degree $h+1$ that vanishes at these points (λ, κ). The coefficient of K^h is

$$(h+1)\lambda - (1+2+\cdots+h-1) - (h+1)(\kappa+h) + (1+2+\cdots+h-1)$$

and since it vanishes, we must have $\lambda = \kappa + h$. In short, only integer translates of κ may appear, and this concludes the proof of Theorem 1.

8. Technical remarks

The Levin–Faĭnleĭb's beginning, namely the link between $\sum_{n\leq x} g(n)\log n$ and $\sum_{n\leq x} g(n)$ where $g(n) = f(n)/n$, has had many application, so it is worth providing a sketch of the present method. When $h = 1$ and f is restricted to square-free integers, our method relies on the identity (as noticed immediately after Lemma 3)

$$\sum_{n\le X}\frac{f(n)}{n}(\log n)^2 = \sum_{m\le X}\frac{f(m)}{m}\Bigg(\sum_{p_1p_2\le X/m}\frac{f(p_1)f(p_2)(\log p_1)(\log p_2)}{p_1p_2} + \sum_{p\le X/m}\frac{f(p)(\log p)^2}{p}\Bigg) + \text{error}.$$

A similar equation could be reached by noticing that, by the Selberg formula, $\log^2 = 1\star(\Lambda\log + \Lambda\star\Lambda)$, we have

$$\sum_{n\le X}\frac{f(n)}{n}(\log n)^2 = \sum_{m\le X}\frac{f(m)}{m}\Bigg(\sum_{\substack{p_1p_2\le X/m,\\(p_1p_2,m)=1}}\frac{f(p_1p_2)(\log p_1)(\log p_2)}{p_1p_2} + \sum_{\substack{p\le X/m,\\(p,m)=1}}\frac{f(p)(\log p)^2}{p}\Bigg).$$

Our usage of Λ_f thus avoids the coprimality conditions that soon become a true combinatorial hurdle. Then by Lemma 5 (or Lemma 6), we approximate the sum of the two sums over primes above by $\kappa(\kappa+1)(\log Y)^2 + c(\log Y) + \mathcal{O}(\log\log 3Y)$ and we notice that

$$(\log n)^2 = \Big(\log X - \log\frac{X}{n}\Big)^2 = (\log X)^2 - 2(\log X)\log\frac{X}{n} + \Big(\log\frac{X}{n}\Big)^2.$$

This gives us

$$(\log X)^2G_0(X) - 2(\log X)G_1(X) + G_2(X) = \kappa(\kappa+1)G_2(X) + \mathcal{O}(G_0(X)\log\log 3X).$$

We then convert this in an approximate differential equation in H_2 of Euler's type, i.e. it can be reduced to an approximate linear differential equation, for which one can prove deformation results.

Acknowledgements

This paper started in 2018 when the first and third authors were invited by the Indian Statistical Institute, Delhi under Cefipra program 5401-A. It was continued when these authors were visiting Stockholm in early 2019 and then in July of the same year when the first and second authors were invited by the Max Planck Institute in Bonn. It was finalized in 2021 when the first author was invited by the Haussdorf Institut für Mathematik in Bonn and the second author was invited by the Max Planck Institute in Bonn. The authors would like to thank all these institutions for providing suitable conditions without which this piece of work would surely have died in their drawers.

References

[1] Bombieri E, The asymptotic sieve, *Rend. Accad. Naz. XL V. Ser.* **1–2** (1976) 243–269

[2] de la Bretèche R and Tenenbaum G, Remarks on the Selberg–Delange method, *Acta Arith.* **200(4)** (2021) 349–369

[3] Friedlander J and Iwaniec H, On Bombieri's asymptotic sieve, *Ann. Sc. Norm. Sup. (Pisa)* **5** (1978) 719–756

[4] Friedlander J and Iwaniec H, Bombieri's sieve, in: Analytic number theory, edited by Bruce C Berndt *et al.*, Vol. 1, Proceedings of a Conference in Honor of Heini Halberstam, May 16–20, 1995, Urbana, IL, USA, Boston, MA, volume 138 of *Birkhäuser. Prog. Math.* (1996) pp. 411–430

[5] Granville A and Koukoulopoulos D, Beyond the LSD method for the partial sums of multiplicative functions, *Ramanujan J.* **49(2)** (2019) 287–319

[6] Iwaniec H and Kowalski E, Analytic Number Theory, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI (2004) xii+615 pp

[7] Kienast A, Über die Äquivalenz zweier Ergebnisse der analytischen Zahlentheorie. *Math. Ann.* **95** (1926) 427–445 10.1007/BF01206619.

[8] Landau E, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate. *Arch. der Math. u. Phys. (3)* **13** (1908) 305–312

[9] Levin B V and Faĭnleĭb A S, Application of certain integral equations to questions of the theory of numbers, *Uspehi Mat. Nauk.* **22(3(135))** (1967) 119–197

[10] Moree P and te Riele Herman J J, The hexagonal versus the square lattice, *Math. Comp.* **73(245)** (2004) 451–473

[11] Popa D and Pugna G, Hyers–Ulam stability of Euler's differential equation, *Results Math.* **69(3–4)** (2016) 317–325

[12] Popa D and Raşa I, On the Hyers–Ulam stability of the linear differential equation, *J. Math. Anal. Appl.* **381(2)** (2011) 530–537

[13] Ramaré O, Arithmetical Aspects of the Large Sieve Inequality, volume 1 of Harish-Chandra Research Institute Lecture Notes (2009) (New Delhi: Hindustan Book Agency) with the collaboration of D. S. Ramana

[14] Ramaré O, From explicit estimates for the primes to explicit estimates for the Moebius function, *Acta Arith.* **157(4)** (2013) 365–379

[15] Ram Murty M and Saradha N, An asymptotic formula by a method of Selberg, *C. R. Math. Rep. Acad. Sci. Canada* **15(6)** (1993) 273–277

[16] Selberg A, Collected Papers, Vol. II (1991) (Berlin: Springer-Verlag) with a Foreword by K. Chandrasekharan

[17] Selberg A, An elementary proof of the prime-number theorem, *Ann. Math.* **50(2)** (1949) 305–313

[18] Serre J-P, Divisibilité de certaines fonctions arithmétiques, *Enseignement Math. (2)* **22(3–4)** (1976) 227–260

[19] Wirsing E, Das asymptotische Verhalten von Summen über multiplikative Funktionen, *Math. Ann.* **143** (1961) 75–102

Communicating Editor: Sanoli Gun