



A normal variety of invariant connexions on Hermitian symmetric spaces

INDRANIL BISWAS^{1,*} and HARALD UPMEIER²

¹School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

²Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Lahnberge, Hans-Meerwein-Strasse, 35032 Marburg, Germany

*Corresponding Author

E-mail: indranil@math.tifr.res.in; upmeier@mathematik.uni-marburg.de

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Abstract. We introduce a class of G -invariant connexions on a homogeneous principal bundle Q over a Hermitian symmetric space $M = G/K$. The parameter space carries the structure of normal variety and has a canonical anti-holomorphic involution. The fixed points of the anti-holomorphic involution are precisely the integrable invariant complex structures on Q . This normal variety is closely related to quiver varieties and, more generally, to varieties of commuting matrix tuples modulo simultaneous conjugation.

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1. Introduction

In [3], we undertook a systematic study of G -invariant connexions on principal H -bundles, for a complex algebraic group H , over G -homogeneous spaces G/K , in particular, over Hermitian symmetric spaces. The main result there gave an intrinsic “curvature” characterization of invariant connexions which produce (invariant) integrable complex structures. In addition, it was shown that this characterization holds in both the compact and non-compact Hermitian symmetric cases.

In this paper, we introduce the concept of “pure” invariant connexion (related to the curvature condition above) and regard the space of all such pure invariant connexions as an interesting moduli space in the sense of algebraic geometry. We prove that this moduli space is an algebraic variety endowed with a canonical anti-holomorphic involution, and investigate the question whether this variety is normal. In dimension 1 (the Hermitian symmetric space is the unit disk) and, more generally, for rank 1 (the Hermitian symmetric space is a unit ball), this problem is solved very explicitly, by realizing the moduli space as a quiver variety. For symmetric spaces of higher rank, we establish a close connexion to varieties of commuting matrix tuples (modulo joint conjugation).

2. Principal bundles

As this paper is a continuation of our paper [3], we first recall some concepts and notation from [3]. For a Lie group H , we consider C^∞ principal H -bundles Q over a given manifold M , with projection map $\pi : Q \rightarrow M$. For another principal H' -bundle $\pi' : Q' \rightarrow M'$, a bundle morphism $F : Q \rightarrow Q'$ is given by a commuting diagram

$$\begin{array}{ccc} Q & \xrightarrow{F} & Q' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{g} & M' \end{array}$$

of smooth maps, such that there exists a Lie group homomorphism $f : H \rightarrow H'$ satisfying the condition

$$F(qh) = F(q)f(h)$$

for all $q \in Q, h \in H$. We say that F is a bundle map over g , associated to the homomorphism f . If F is a diffeomorphism, $M = M'$ and $g = id_M$, we obtain a (bundle) isomorphism.

Two special cases of bundle maps arise as follows.

DEFINITION 2.1

Let H be a complex Lie group, and let $H_{\mathbb{R}} \subset H$ be a maximal compact subgroup. A *Hermitian structure* on a principal H -bundle Q is a subbundle $Q_{\mathbb{R}} \subset Q$ with structure group $H_{\mathbb{R}}$. Thus the inclusion map $\iota : Q_{\mathbb{R}} \hookrightarrow Q$ is a bundle morphism

$$\begin{array}{ccc} Q_{\mathbb{R}} & \xrightarrow{\iota} & Q \\ \pi \searrow & & \swarrow \pi \\ & M & \end{array} \tag{2.1}$$

over id_M , associated to the inclusion $H_{\mathbb{R}} \hookrightarrow H$. A morphism $(Q, Q_{\mathbb{R}}) \rightarrow (Q', Q'_{\mathbb{R}})$ between Hermitian structures over M is a morphism $F : Q \rightarrow Q'$ such that $F(Q_{\mathbb{R}}) \subset Q'_{\mathbb{R}}$. Putting $F_{\mathbb{R}} := F|_{Q_{\mathbb{R}}}$, we have a commuting diagram

$$\begin{array}{ccccc} Q & \xrightarrow{F} & Q' & & \\ \pi \searrow & & \swarrow \pi' & & \\ & Q_{\mathbb{R}} & \xrightarrow{F_{\mathbb{R}}} & Q'_{\mathbb{R}} & \\ \pi \searrow & & \swarrow \pi' & & \\ & M & \xrightarrow{g} & M' & \end{array}$$

Let H be a connected reductive affine algebraic group defined over \mathbb{C} , and let $H_{\mathbb{R}} \subset H$ be a maximal compact subgroup. Then there exists an anti-automorphism $h \mapsto h^*$ of order 2 on H such that

$$H_{\mathbb{R}} = \{h \in H \mid h^* = h^{-1}\}.$$

So $H_{\mathbb{R}}$ consists of the “unitary” elements in H . In other words, the corresponding anti-holomorphic involution (automorphism of order 2)

$$h \mapsto h^{\sharp} := h^{*-1} = h^{-1*}$$

has the fixed point subgroup $H_{\mathbb{R}}$. Its differential $\sharp = d_e \sharp$ is an anti-linear involution on the Lie algebra \mathfrak{h} , preserving the Lie bracket operation.

DEFINITION 2.2

Let H be a complex reductive group, and let $H_{\mathbb{R}} \subset H$ be a maximal compact subgroup [4]. A principal H -bundle Q is called *involutive* if there exists a morphism of order two

$$\begin{array}{ccc} Q & \xrightarrow{\kappa} & Q \\ & \searrow \pi & \swarrow \pi \\ & & M \end{array} \tag{2.2}$$

over id_M , associated with the involution $h \mapsto h^{\sharp}$ of H . A morphism between involutive principal bundles (Q, κ) and (Q', κ') is described by a commuting diagram

$$\begin{array}{ccccc} Q & \xrightarrow{F} & & & Q' \\ & \searrow \kappa & & & \swarrow \kappa' \\ & & Q & \xrightarrow{F} & Q' \\ & \searrow \pi & \downarrow \pi & & \swarrow \pi' \\ & & M & \xrightarrow{g} & M' \end{array}$$

It turns out that these two concepts are equivalent, as shown by the following proposition.

PROPOSITION 2.3

For an involutive principal H -bundle (Q, κ) , the fixed point set

$$Q_{\mathbb{R}} = \{p \in Q \mid \kappa(p) = p\} \tag{2.3}$$

is a subbundle which defines a Hermitian structure on Q . Conversely, every Hermitian structure $(Q, Q_{\mathbb{R}})$ induces a bundle involution κ on Q with fixed point subbundle $Q_{\mathbb{R}}$.

Proof. Consider the free action $Q \times H \rightarrow Q$ with quotient $Q/H = M$. For $p \in Q_{\mathbb{R}}$, $\ell \in H_{\mathbb{R}}$, we have

$$\kappa(p\ell) = \kappa(p)\ell^{\sharp} = p\ell$$

since κ is a bundle map and $Q_{\mathbb{R}}$ and $H_{\mathbb{R}}$ are fixed pointwise under the involutions κ and \sharp respectively. Thus we have a (free) action $Q_{\mathbb{R}} \times H_{\mathbb{R}} \rightarrow Q_{\mathbb{R}}$. Clearly, $p \sim p'$ modulo $H_{\mathbb{R}}$ implies $p \sim p'$ modulo H , so there is a map $Q_{\mathbb{R}}/H_{\mathbb{R}} \rightarrow Q/H$. This map is injective, since $p' = ph$ for some $h \in H$ implies that

$$ph = p' = \kappa(p') = \kappa(ph) = \kappa(p)h^{\sharp} = ph^{\sharp}.$$

Therefore, we have $h = h^{\sharp} \in H_{\mathbb{R}}$.

In order to show that the above map $Q_{\mathbb{R}}/H_{\mathbb{R}} \rightarrow Q/H$ is surjective, let $q \in Q$ be arbitrary. Since κ preserves base points, there exists a unique $k \in H$ such that $\kappa q = qk$. It follows that

$$q = \kappa(\kappa q) = \kappa(qk) = (\kappa q)k^{\sharp} = qkk^{\sharp}.$$

Therefore, $kk^{\sharp} = 1$ and $k = (k^{\sharp})^{-1} \in H$ is “unitary”. Hence there exists $h \in H$ such that $k = h(h^{\sharp})^{-1}$. For this element h , we have $\kappa(qh) = (\kappa q)h^{\sharp} = qkh^{\sharp} = qh$. Hence $p := qh \in Q_{\mathbb{R}}$ and $p \sim q \pmod H$. This shows that $Q_{\mathbb{R}}/H_{\mathbb{R}} = Q/H$. Thus every involutive principal H -bundle Q gives rise to a Hermitian structure.

Conversely, let $(Q, Q_{\mathbb{R}})$ be a Hermitian structure and let $q \in Q$ be arbitrary. Since $Q/H = Q_{\mathbb{R}}/H_{\mathbb{R}}$, there exists $h \in H$ such that $p := qh \in Q_{\mathbb{R}}$. Define a map $\kappa : Q \rightarrow Q$ by

$$\kappa q := p(h^{\sharp})^{-1} = qh(h^{\sharp})^{-1}. \tag{2.4}$$

In order to show that (2.4) is independent of the choice of h , let $qh_1 \in Q_{\mathbb{R}}$ for some $h_1 \in H$. Since qh and qh_1 have the same base point in M , there exists $\ell \in H_{\mathbb{R}}$ such that $qh_1 = qh\ell$. Therefore, $h_1 = h\ell$, and hence

$$h_1(h_1^{\sharp})^{-1} = h\ell((h\ell)^{\sharp})^{-1} = h\ell(h^{\sharp}\ell)^{-1} = h(h^{\sharp})^{-1}.$$

It is now straightforward to show that $q \mapsto \kappa q$ is a bundle isomorphism of Q of order 2 whose fixed point locus is the subbundle $Q_{\mathbb{R}}$. □

3. Equivariant principal H -bundles

Let G be a real Lie group acting on M via transformations $L_g^M, g \in G$.

DEFINITION 3.1

A C^∞ principal H -bundle over M is called *equivariant* (under G) if Q carries a smooth (left) action $g \mapsto L_g^Q$ of G which commutes with the right action of H on Q , such that for each $g \in G$, the diagram

$$\begin{array}{ccc}
 Q & \xrightarrow{L_g^Q} & Q \\
 \pi \downarrow & & \downarrow \pi \\
 M & \xrightarrow{L_g^M} & M
 \end{array} \tag{3.1}$$

commutes. Thus L_g^Q is a bundle morphism over L_g^M , associated with id_H . A bundle morphism $F : Q \rightarrow Q'$ between equivariant H -bundles is called equivariant if it intertwines the actions of $G \times H$.

DEFINITION 3.2

For a complex Lie group H an involutive H -bundle (Q, κ) is called *equivariant*, if Q is equivariant and each L_g^Q is an isomorphism of the involutive bundle (Q, κ) , i.e.,

$$L_g^Q \circ \kappa = \kappa \circ L_g^Q$$

for every $g \in G$. An equivalent condition is that the involution κ is an equivariant morphism.

Similarly, a Hermitian structure $(Q, Q_{\mathbb{R}})$ is called *equivariant*, if Q is equivariant and each L_g^Q is an isomorphism of the Hermitian structure $(Q, Q_{\mathbb{R}})$, i.e., satisfies the condition $L_g^Q Q_{\mathbb{R}} = Q_{\mathbb{R}}$. Then $Q_{\mathbb{R}}$ becomes an equivariant bundle under the restricted action $L_g^{Q_{\mathbb{R}}} := L_g^Q|_{Q_{\mathbb{R}}}$, for which the inclusion map $\iota : Q_{\mathbb{R}} \hookrightarrow Q$ is an equivariant bundle morphism.

Equivariant morphisms between equivariant involutive H -bundles or equivariant Hermitian structures are defined in a natural way.

We shall apply these concepts to the homogeneous case, in particular, when $M = G/K$ is an irreducible symmetric space. For any $g \in G$, let

$$L_g^M : G/K \rightarrow G/K \tag{3.2}$$

be the left translation defined by $L_g^M(g'K) := gg'K$. In the homogeneous case, an equivariant principal H -bundle Q (and, similarly, an equivariant Hermitian structure $(Q, Q_{\mathbb{R}})$ or an involutive H -bundle (Q, κ)) will also be called *homogeneous*. By [3, Theorem 1.2], all equivariant homogeneous principal H -bundles, up to equivariant isomorphism, are classified by the set $\text{Hom}(K, H)/H$, by associating to a Lie group homomorphism $f : K \rightarrow H$, up to H -conjugacy, the homogeneous principal H -bundle

$$Q = G \times_{K,f} H = \{[g : h] = [gk : f(k)^{-1}h] \mid g \in G, h \in H, k \in K\}, \tag{3.3}$$

which is equivariant under the (well-defined) action

$$L_g^Q[g' : h] := [gg' : h]. \tag{3.4}$$

More precisely, for any choice of base point $\mathbf{q} \in Q_o$ in the fibre over $o = eK \in G/K$, the homomorphism $f : K \rightarrow H$ is uniquely determined by the condition

$$k\mathbf{q} = \mathbf{q}f(k) \tag{3.5}$$

for all $k \in K$; another choice $\mathbf{q}' = \mathbf{q}h^{-1} \in Q_o$ induces the conjugate homomorphism $I_h^H \circ f$, where

$$I_h^H h' = hh'h^{-1}$$

denotes the inner automorphism.

Similarly, all equivariant Hermitian structures $(Q_{\mathbb{R}}, Q)$, up to equivariant isomorphism, are parametrized by the discrete set $\text{Hom}(K, H_{\mathbb{R}})/H_{\mathbb{R}}$, mapping the $H_{\mathbb{R}}$ -conjugacy class of a homomorphism $f : K \rightarrow H_{\mathbb{R}}$ to the equivariant $H_{\mathbb{R}}$ -bundle

$$Q_{\mathbb{R}} = G \times_{K, f} H_{\mathbb{R}} \subset Q = G \times_{K, f} H$$

regarded as a subbundle of (3.3). Since for any homomorphism $f : K \rightarrow H$, the image is contained in a maximal compact subgroup $H_{\mathbb{R}}$, and for reductive Lie groups all maximal compact subgroups $H_{\mathbb{R}} \subset H$ are conjugate, the two classifying sets can be identified, i.e., any homogeneous H -bundle Q admits a Hermitian structure $Q_{\mathbb{R}} \subset Q$, which is unique after specifying the maximal compact subgroup $H_{\mathbb{R}}$.

Lemma 3.3. *Let H be a complex reductive group, and let $H_{\mathbb{R}} \subset H$ be a maximal compact subgroup. Then the associated equivariant bundle $Q = G \times_{K, f} H$ is involutive with respect to $H_{\mathbb{R}}$, under the involution*

$$\kappa[g : h] := [g : h^{\sharp}] \quad (3.6)$$

for all $g \in G, h \in H$. The associated Hermitian structure has the fixed point subbundle

$$Q_{\mathbb{R}} = G \times_{K, f} H_{\mathbb{R}}.$$

Proof. For all $k \in K$, we have $f(k) \in H_{\mathbb{R}}$, and hence $f(k)^{\sharp} = f(k)$. It follows that

$$\kappa[ghk : f(k)^{-1}h] = [ghk : (f(k)^{-1}h)^{\sharp}] = [ghk : f(k)^{-1}h^{\sharp}] = [g : h^{\sharp}].$$

Therefore the map in (3.6) is well-defined. It is clear that it commutes with the G -action on Q in (3.4). \square

While the principal bundle in (3.3) depends only on the homomorphism $f : K \rightarrow H$, the base point \mathbf{q} satisfying (3.5) is not unique. Define a subgroup

$$H^f := \{\eta \in H \mid f(k)\eta = \eta f(k) \quad \forall k \in K\} = \{\eta \in H \mid I_{\eta}^H \circ f = f\}. \quad (3.7)$$

Then for any $\eta \in H^f$, the point $\mathbf{q}' := \mathbf{q}\eta$ also satisfies (3.5), because

$$k\mathbf{q}' = k\mathbf{q}\eta = \mathbf{q}f(k)\eta = \mathbf{q}\eta f(k) = \mathbf{q}'f(k)$$

for all $k \in K$. For $f : K \rightarrow H_{\mathbb{R}}$, we put

$$H_{\mathbb{R}}^f := H_{\mathbb{R}} \cap H^f.$$

For $x \in M$, we define the evaluation map $R_x^M : G \rightarrow M$ by

$$R_x^M(g) := L_g^M(x) = gx.$$

Via the map $R_o^M : G \rightarrow M$, the group G can be regarded as a principal K -bundle over $M = G/K$, which is equivariant under the left translation action

$$L_g^G g' := gg'$$

for $g, g' \in G$. This follows from the commuting diagram

$$\begin{array}{ccc} G & \xrightarrow{L_g^G} & G \\ R_o^M \downarrow & & \downarrow R_o^M \\ M & \xrightarrow{L_g^M} & M \end{array}$$

Sometimes we write \mathbf{G} to emphasize the principal K -bundle structure. Now consider the homogeneous K -bundle

$$G \times_K K := \{[g : k] = [gk_1 : k_1^{-1}k] \mid g \in G, k, k_1 \in K\}$$

which is equivariant under the left action

$$g \cdot [g', k] = [gg', k]$$

for all $g, g' \in G$ and $k \in K$. Then the map

$$G \ni g \mapsto [g : e_H] = [gk : k^{-1}] \in G \times_K K$$

defines an equivariant isomorphism of principal K -bundles

$$\begin{array}{ccc} G & \xrightarrow{\quad} & G \times_K K \\ & \searrow R_o^M & \swarrow \pi \\ & & G/K \end{array}$$

Lemma 3.4. For a homomorphism $f : K \rightarrow H$, and $\eta \in H^f$, define

$$f_\eta(g) := [g : \eta] = g[e : \eta]$$

for all $g \in G$. Then f_η is an equivariant bundle map

$$\begin{array}{ccc} G & \xrightarrow{f_\eta} & G \times_{K,f} H \\ & \searrow R_o^M & \swarrow \pi \\ & & M \end{array}$$

over the identity map of $M = G/K$, associated with the homomorphism f .

Proof. This follows from the identity

$$\begin{aligned} f_\eta(gk) &= [gk : \eta] = [g : f(k)\eta] = [g : \eta f(k)] \\ &= [g : \eta]f(k) = f_\eta(g)f(k) \end{aligned}$$

for all $g \in G, k \in K$. □

In case $f : K \rightarrow H_{\mathbb{R}}$ and $\eta \in H_{\mathbb{R}}^f$, we obtain a similar G -equivariant bundle morphism

$$\begin{array}{ccc} G & \xrightarrow{f_\eta} & G \times_{K,f} H_{\mathbb{R}} \\ & \searrow R_o^M & \swarrow \pi \\ & & M \end{array}$$

Thus the various choices of base point in Q correspond to suitable bundle embeddings of G into Q .

PROPOSITION 3.5

For every $\eta \in H^f$, the map

$$L_\eta^Q [g : h] := [g : \eta h]$$

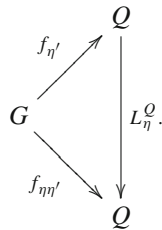
defines a bundle automorphism L_η^Q of $Q = G \times_{K,f} H$ satisfying $L_h^Q \mathbf{q} = \mathbf{q}'$.

Proof. For $k \in K$, we have $f(k)^{-1}\eta h = \eta f(k)^{-1}h$. This shows that L_h^Q is a well-defined bundle map. Moreover,

$$L_h^Q \mathbf{q} = L_h^Q [e : e_H] = [e : h] = [e : e_H]h = \mathbf{q}h = \mathbf{q}'$$

and the proposition is proved. □

There is a commuting diagram



4. Connexions

For a given principal H -bundle Q over M , we consider *connexions* Θ on Q [1, 10]. The space of all connexions on Q will be denoted by $\mathcal{C}(Q)$. For a connexion $\Theta \in \mathcal{C}(Q)$, let $T_q^\Theta Q \subset T_q Q$ be the horizontal tangent subspace at $q \in Q$.

We identify a connexion on Q with its *connexion 1-form*, on the total space of Q , which is a pseudo-tensorial \mathfrak{h} -valued 1-form

$$T_q Q \ni X \mapsto \Theta_q X \in \mathfrak{h}, \quad q \in Q$$

on Q , which is H -equivariant and coincides with the Maurer–Cartan form on the fibers of Q [11, Section II.1]. Let $\Omega_H^j(Q, \mathfrak{h})$ denote the vector space of all tensorial \mathfrak{h} -valued j -forms on Q that are H -equivariant. Given a fixed connexion $\Theta^0 \in \mathcal{C}(Q)$ on Q , all other connexions on Q are of the form

$$\Theta_q^\omega := \Theta_q^0 + \omega_q, \quad q \in Q, \tag{4.1}$$

where $\omega \in \Omega_H^1(Q, \mathfrak{h})$. In short, we have $\mathcal{C}(Q) = \Theta^0 + \Omega_H^1(Q, \mathfrak{h})$. This notation depends on the choice of Θ^0 . Let $\underline{\Theta} \in \Omega_H^2(Q, \mathfrak{h})$ denote the *curvature* of Θ . By [11, Section II.5], for each $\Omega \in \Omega_H^j(Q, \mathfrak{h})$, there exists a unique j -form $\tilde{\Omega}$ on M of type Ad_H , i.e., with values in the homogeneous vector bundle

$$Q \times_H \mathfrak{h} = \{[q : \beta] = [qh : Ad_{h^{-1}}\beta] \mid q \in Q, \beta \in \mathfrak{h}, h \in H\},$$

such that

$$\tilde{\Omega}_{\pi(q)}((d_q\pi)X_1, \dots, (d_q\pi)X_j) = [q : \Omega_q(X_1, \dots, X_j)]$$

for all $q \in Q$ and $X_i \in T_q Q$, where $1 \leq i \leq j$. We call $\tilde{\Omega}$ the *associated bundle-valued j -form*.

We shall often use the concept of *induced connexion*. Let

$$\begin{array}{ccc} Q & \xrightarrow{F} & Q' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{g} & M' \end{array}$$

be a principal bundle map over a diffeomorphism $g : M \rightarrow M'$, associated to a Lie group homomorphism $f : H \rightarrow H'$. Then any connexion Θ on Q induces a connexion, denoted by $F_*\Theta$ on Q' , such that the bundle map F preserves the respective horizontal subspaces. By [11, Proposition II.6.1], the connexion and curvature forms of Θ and $F_*\Theta$ are related by

$$F^*(F_*\Theta) = \underline{f} \circ \Theta, \tag{4.2}$$

$$F^*(\underline{F_*\Theta}) = \underline{f} \circ \underline{\Theta}, \tag{4.3}$$

where F^* denotes the pull-back of differential forms, and $\underline{f} = d_e f : \mathfrak{h} \rightarrow \mathfrak{h}'$ is the differential of f at the unit element $e \in H$.

For a Hermitian structure $(Q, Q_{\mathbb{R}})$, we also consider $\mathfrak{h}_{\mathbb{R}}$ -valued connexion 1-forms Ξ on P . Given a fixed connexion Ξ^0 on $Q_{\mathbb{R}}$, all other connexions on $Q_{\mathbb{R}}$ have the form

$$\Xi_p^\omega := \Xi_p^0 + \omega_p, \quad p \in Q_{\mathbb{R}}, \tag{4.4}$$

where $\omega \in \Omega_{H_{\mathbb{R}}}^1(Q_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$. Thus we have $\mathcal{C}(Q_{\mathbb{R}}) = \Xi^0 + \Omega_{H_{\mathbb{R}}}^1(Q_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$. Under the inclusion map in (2.1), any connexion Ξ on $Q_{\mathbb{R}}$ induces a unique connexion $\iota_*\Xi$ on Q preserving horizontal subspaces. Thus we have a natural map $\iota_* : \mathcal{C}(Q_{\mathbb{R}}) \rightarrow \mathcal{C}(Q)$ whose image

$$\mathcal{C}_{\mathbb{R}}(Q) := \iota_*(\mathcal{C}(Q_{\mathbb{R}}))$$

consists of what are known as *Hermitian connexions* on Q .

PROPOSITION 4.1

Let $\Theta^0 := \iota_*\Xi^0$. Then

$$\iota_*\Xi^\omega = \Theta^{\iota_*\omega}$$

for any $\omega \in \Omega_{H_{\mathbb{R}}}^1(Q_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$, where $\iota : H_{\mathbb{R}} \hookrightarrow H$ is the inclusion map, with differential $\underline{\iota} : \mathfrak{h}_{\mathbb{R}} \hookrightarrow \mathfrak{h}$. The associated bundle-valued 1-forms satisfy

$$\underline{\iota} \tilde{\circ} \omega = \tilde{\iota} \circ \tilde{\omega},$$

where the vector bundle map $\tilde{\iota} : Q_{\mathbb{R}} \times_{H_{\mathbb{R}}} \mathfrak{h}_{\mathbb{R}} \rightarrow Q \times_H \mathfrak{h}$ is defined by

$$\tilde{\iota}[p, \beta] = [\iota p, \underline{\iota}\beta]$$

for all $p \in Q_{\mathbb{R}} \subset Q$ and $\beta \in \mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$.

If H is a complex Lie group, then $Q \times_H \mathfrak{h}$ is a complex vector bundle over M . Let (Q, κ) be an involutive H -bundle. Given a connexion Θ on Q , we define the connexion

$$\Theta^\sharp := \kappa_* \Theta$$

on Q induced by the involution in (2.2). We have $\Theta^{\sharp\sharp} = \Theta$ and thus obtain a mapping $\Theta \mapsto \Theta^\sharp$ of order 2 on $\mathcal{C}(Q)$.

PROPOSITION 4.2

If $\Theta^0 = (\Theta^0)^\sharp$, then

$$(\Theta^\omega)^\sharp = \Theta^{\sharp\circ\omega}$$

for any $\omega \in \Omega_H^1(Q, \mathfrak{h})$, where \sharp is the involution on H , with differential $\underline{\sharp} = d_e \sharp$ on \mathfrak{h} . The associated bundle-valued 1-forms satisfy

$$\underline{\sharp} \circ \tilde{\omega} = \tilde{\kappa} \circ \tilde{\omega},$$

where the (anti-linear) vector bundle map $\tilde{\kappa}$ on $Q \times_H \mathfrak{h}$ is defined by

$$\tilde{\kappa}[q : \beta] := [\kappa q : \underline{\sharp}\beta] \quad (4.5)$$

for all $q \in Q$ and $\beta \in \mathfrak{h}$.

PROPOSITION 4.3

A connexion Θ on an involutive principal H -bundle is Hermitian, i.e., it is of the form $\Theta = \iota_* \Xi$ for some connexion Ξ on the fixed point bundle $Q_{\mathbb{R}} \subset Q$, if and only if $\Theta^\sharp = \Theta$. Thus

$$\mathcal{C}_{\mathbb{R}}(Q) = \iota_* \mathcal{C}(Q_{\mathbb{R}}) = \{\Theta \in \mathcal{C}(Q) \mid \Theta^\sharp = \Theta\}.$$

Proof. Since $\kappa \circ \iota = \iota$, it follows that

$$(\iota_* \Xi)^\sharp = \kappa_*(\iota_* \Xi) = (\kappa \circ \iota)_* \Xi = \iota_* \Xi.$$

Thus a Hermitian connexion Θ satisfies the condition $\Theta^\sharp = \Theta$.

Conversely, the condition $\kappa_* \Theta = \Theta$ implies that $\Theta = \Theta^\omega$ for some $\omega \in \Omega_H^1(Q, \mathfrak{h})$ satisfying $\omega = \underline{\sharp} \circ \omega$. Therefore, $\omega = \underline{\iota} \circ \omega_{\mathbb{R}}$ for some $\omega_{\mathbb{R}} \in \Omega_{H_{\mathbb{R}}}^1(Q_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$. Setting $\Xi := \Xi^{\omega_{\mathbb{R}}}$, we conclude that $\iota_* \Xi = \Theta$. \square

We now introduce the central concept of this paper.

DEFINITION 4.4

Suppose that M is a complex manifold and H is a complex Lie group. We say that $\Omega \in \Omega_H^2(Q, \mathfrak{h})$ is *pure* if for every $z \in M$, the associated bundle-valued 2-form $\tilde{\Omega}_z$ is of

type $(1, 1)$, i.e., its (unique) \mathbb{C} -bilinear extension $\tilde{\Omega}_z : T_z^{\mathbb{C}}M \wedge T_z^{\mathbb{C}}M \rightarrow (Q \times_H \mathfrak{h})_z$ satisfies the condition

$$\tilde{\Omega}_z(v_1, v_2) = 0 = \tilde{\Omega}_z(\bar{v}_1, \bar{v}_2) \tag{4.6}$$

for all holomorphic tangent vectors $v_1, v_2 \in T_z^{1,0}M$ and all anti-holomorphic tangent vectors $\bar{v}_1, \bar{v}_2 \in T_z^{0,1}M$. A connexion Θ on a principal H -bundle Q over M is called *pure*, if its curvature form $\underline{\Theta}$ is pure.

Let

$$\mathcal{L}(Q) \subset \mathcal{C}(Q) \tag{4.7}$$

denote the set of all pure connexions on Q .

PROPOSITION 4.5

Let (Q, κ) be an involutive principal H -bundle over a complex manifold M . If a connexion Θ on Q is pure, then the connexion Θ^\sharp is also pure.

Proof. We have

$$\tilde{\Theta}_z((d_q\pi)X, (d_q\pi)Y) = [q : \underline{\Theta}_q(X, Y)]$$

for all $X, Y \in T_qQ$, where $z = \pi(q)$. For the connexion $\Theta^\sharp = \kappa_*\Theta$, the curvature satisfies

$$\kappa^*\underline{\Theta}^\sharp = \kappa^*\underline{\kappa_*\Theta} = \underline{\sharp} \circ \underline{\Theta},$$

according to (4.3). This means that

$$\underline{\sharp}(\underline{\Theta}_q(X, Y)) = (\kappa^*\underline{\Theta}^\sharp)_q(X, Y) = \underline{\Theta}^\sharp_{\kappa q}((d_{\kappa q}\kappa)X, (d_{\kappa q}\kappa)Y).$$

Using the bundle map in (4.5), we have

$$\begin{aligned} \tilde{\kappa}(\tilde{\Theta}_z((d_q\pi)X, (d_q\pi)Y)) &= \tilde{\kappa}[q : \underline{\Theta}_q(X, Y)] = [\kappa q : \underline{\sharp}(\underline{\Theta}_q(X, Y))] \\ &= [\kappa q : \underline{\Theta}^\sharp_{\kappa q}((d_{\kappa q}\kappa)X, (d_{\kappa q}\kappa)Y)] = \tilde{\Theta}^\sharp_z((d_{\kappa q}\pi)(d_{\kappa q}\kappa)X, (d_{\kappa q}\pi)(d_{\kappa q}\kappa)Y) \\ &= \tilde{\Theta}^\sharp_z((d_q\pi)X, (d_q\pi)Y) \end{aligned}$$

because $\pi \circ \kappa = \pi$. Thus the equality

$$\tilde{\kappa}(\tilde{\Theta}_z(v_1, v_2)) = \tilde{\Theta}^\sharp_z(v_1, v_2)$$

holds for all $v_i \in T_zM$. For the \mathbb{C} -bilinear extension, we obtain the equality

$$\tilde{\Theta}^\sharp_z(v_1, v_2) = \tilde{\kappa}(\tilde{\Theta}_z(\bar{v}_1, \bar{v}_2))$$

for all $v_i \in T_z^{\mathbb{C}}M$, where $v \mapsto \bar{v}$ is the canonical conjugation with fixed point set T_zM . Thus $\tilde{\Theta}^\sharp_z$ satisfies the condition in (4.6) as well. □

It follows that there is a period 2 map $\Theta \mapsto \Theta^\sharp$ on the space $\mathcal{L}(Q)$ (see (4.7)) of pure connexions, whose fixed point set $\mathcal{L}_{\mathbb{R}}(Q)$ consists of all pure Hermitian connexions on Q .

5. Invariant connexions

For a G -equivariant principal bundle Q , the group G acts on $\mathcal{C}(Q)$ as follows: For any $g \in G$, the bundle map in (3.1), associated with the identity map id_H , yields the induced connexion

$$g \cdot \Theta := (L_g^Q)_* \Theta$$

satisfying $(L_g^Q)^*(g \cdot \Theta) = \Theta$. Thus we have

$$g \cdot \Theta = (L_{g^{-1}}^Q)^* \Theta. \tag{5.1}$$

Let $\mathcal{C}(Q)^G \subset \mathcal{C}(Q)$ denote the set of all connexions on Q which are invariant under the action of G . For an equivariant Hermitian structure $(Q, Q_{\mathbb{R}})$, the group G acts on $\mathcal{C}(Q_{\mathbb{R}})$ in a manner analogous to (5.1), and we let $\mathcal{C}(Q_{\mathbb{R}})^G \subset \mathcal{C}(Q_{\mathbb{R}})$ be the fixed point locus for this action of G on $\mathcal{C}(Q_{\mathbb{R}})$. The extension map $\Xi \mapsto \iota_* \Xi$ commutes with the left action of G on connexions. It follows that there is a commuting diagram

$$\begin{array}{ccc} \mathcal{C}^G(Q_{\mathbb{R}}) & \subset & \mathcal{C}(Q_{\mathbb{R}}) \\ \downarrow \iota_* & & \downarrow \iota_* \\ \mathcal{C}^G(Q) & \subset & \mathcal{C}(Q) \end{array}$$

PROPOSITION 5.1

If a connection Θ^0 is G -invariant, then

$$g \cdot \Theta^\omega = \Theta^0 + g \cdot \omega = \Theta^{g \cdot \omega}$$

for all $\omega \in \Omega_H^1(Q, \mathfrak{h})$. If $(Q, Q_{\mathbb{R}})$ is a Hermitian structure, and Ξ^0 is invariant, then

$$g \cdot \Xi^\omega = \Xi^0 + g \cdot \omega = \Xi^{g \cdot \omega}$$

for all $\omega \in \Omega_{H_{\mathbb{R}}}^1(Q_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$. In both cases, the associated bundle-valued 1-forms satisfy

$$\widetilde{g \cdot \omega}_{g_x} = \tilde{g} \cdot (\tilde{\omega}_x \circ (d_{g_x} L_g^M)),$$

using the action

$$\tilde{g} \cdot [q, \beta] := [gq, \beta] \tag{5.2}$$

of $g \in G$ on $Q \times_H \mathfrak{h}$ (respectively, $Q \times_{H_{\mathbb{R}}} \mathfrak{h}_{\mathbb{R}}$). In particular, Θ^ω (respectively, Ξ^ω) is invariant if and only if

$$\tilde{\omega}_{g_x} \circ (d_x L_g^M) = g \cdot \tilde{\omega}_x$$

for all $g \in G$ and $x \in M$.

Proof. Since $\Theta^\omega = \Theta^0 + \omega$ and Θ^0 is invariant, from (5.1), it follows that $g \cdot \omega = (L_{g^{-1}}^Q)^*\omega$. Thus we have

$$(g \cdot \omega)_{gq} = ((L_{g^{-1}}^Q)^*\omega)_{gq} = \omega_q \circ (d_{gq}L_{g^{-1}}^Q).$$

For every $X \in T_{gq}Q$, it follows that

$$\begin{aligned} g \cdot \omega_{gx}(d_{gq}\pi)X &= [gq : (g \cdot \omega)_{gq}X] = [gq : \omega_q(d_{gq}L_{g^{-1}}^Q)X] \\ &= g \cdot [q : \omega_q(d_{gq}L_{g^{-1}}^Q)X] = g \cdot \tilde{\omega}_x(d_q\pi)(d_{gq}L_{g^{-1}}^Q)X \\ &= g \cdot \tilde{\omega}_x(d_xL_{g^{-1}}^M)(d_{gq}\pi)X, \end{aligned}$$

using the identity $\pi \circ L_{g^{-1}}^Q = L_{g^{-1}}^M \circ \pi$. Thus

$$\widetilde{g \cdot \omega}_{gx}v = g \cdot \tilde{\omega}_x(d_{gx}L_{g^{-1}}^M)v$$

for all $v \in T_{gx}M$. □

PROPOSITION 5.2

In case G acts holomorphically on a complex manifold M , the G -action on $\mathcal{C}(Q)$ preserves the Hodge types of the curvature. In particular, if Θ is a pure connexion, then $g \cdot \Theta$ is also pure for every $g \in G$.

Proof. Since $g \cdot \Theta = (L_{g^{-1}}^Q)^*\Theta$, we have

$$\underline{g \cdot \Theta}_{gq}(X, Y) = \underline{\Theta}_q((d_{gq}L_{g^{-1}}^Q)X, (d_{gq}L_{g^{-1}}^Q)Y)$$

for all $X, Y \in T_{gq}Q$, and hence

$$\begin{aligned} \widetilde{\underline{g \cdot \Theta}}_{gz}((d_{gq}\pi)X, (d_{gq}\pi)Y) &= [gq : \underline{g \cdot \Theta}_{gq}(X, Y)] \\ &= [gq : \underline{\Theta}_q((d_{gq}L_{g^{-1}}^Q)X, (d_{gq}L_{g^{-1}}^Q)Y)] \\ &= g \cdot [q : \underline{\Theta}_q((d_{gq}L_{g^{-1}}^Q)X, (d_{gq}L_{g^{-1}}^Q)Y)] \\ &= g \cdot \tilde{\Theta}_z((d_q\pi)(d_{gq}L_{g^{-1}}^Q)X, (d_q\pi)(d_{gq}L_{g^{-1}}^Q)Y) \\ &= g \cdot \tilde{\Theta}_z((d_{gz}L_{g^{-1}}^M)(d_{gq}\pi)X, (d_{gz}L_{g^{-1}}^M)(d_{gq}\pi)Y). \end{aligned}$$

It follows that

$$\widetilde{\underline{g \cdot \Theta}}_{gz}(v_1, v_2) = \tilde{g} \cdot \tilde{\Theta}_z((d_{gz}L_{g^{-1}}^M)v_1, (d_{gz}L_{g^{-1}}^M)v_2)$$

for all $v_i \in T_{gz}M$. Since the transformations $L_{g^{-1}}^M$ are holomorphic, and the action in (5.2) is \mathbb{C} -linear on the fibres, it follows that $\widetilde{\underline{g \cdot \Theta}}$ has the same Hodge type as that of $\tilde{\Theta}$. □

From Proposition 5.2, it follows that the action of G preserves the subset $\mathcal{L}(Q)$ in (4.7). Let

$$\mathcal{L}(Q)^G = \mathcal{L}(Q) \cap \mathcal{C}(Q)^G$$

denote the set of all invariant pure connexions on Q . The involution $\Theta \mapsto \Theta^\sharp$ on the space $\mathcal{L}(Q)^G$ of all pure invariant connexions on Q has the fixed point set $\mathcal{L}_{\mathbb{R}}(Q)^G$ consisting of all pure invariant Hermitian connexions.

Now consider the special case where $M = G/K$ is a *symmetric space*. Let \mathfrak{g} (respectively, \mathfrak{k}) be the Lie algebra of G (respectively, K). Both \mathfrak{g} and \mathfrak{k} are K -modules by the adjoint action. The Killing form on \mathfrak{g} is non-degenerate. Let

$$\mathfrak{p} = \mathfrak{k}^\perp \subset \mathfrak{g}$$

be the orthogonal complement of \mathfrak{k} for the Killing form. Then the natural homomorphism

$$\mathfrak{k} \oplus \mathfrak{p} \longrightarrow \mathfrak{g} \tag{5.3}$$

is an isomorphism. The translates of \mathfrak{p} by the left-translation action of G on itself define a distribution $D \subset TG$ which is in fact preserved by the right-translation action of K on G because the decomposition in (5.3) is K -equivariant. From this, it follows immediately that D defines a connexion, denoted by Γ^0 , on the principal K -bundle $\mathbf{G} := G \times_K K$. Since D is preserved by the left-translation action of G on itself, we conclude that Γ^0 is invariant. Its connexion 1-form coincides with the canonical projection

$$\Gamma_e^0 : T_e G = \mathfrak{g} \longrightarrow \mathfrak{k}$$

at the identity element, and

$$\Gamma_g^0 := \Gamma_e^0 \circ (d_g L_{g^{-1}}^G)$$

for every $g \in G$, using the left translations of G on itself.

Consider the Lie bracket operation composed with the projection to \mathfrak{k} ,

$$\mathfrak{p} \otimes \mathfrak{p} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{k}. \tag{5.4}$$

Left translations by elements of G of this composition of homomorphisms defines a \mathcal{C}^∞ two-form on G/K with values in the adjoint vector bundle $\text{ad}(G)$. This $\text{ad}(G)$ -valued 2-form is in fact the curvature of Γ^0 .

PROPOSITION 5.3

For any homomorphism $f : K \longrightarrow H$ and $\eta \in H^f$, the tautological connexion

$$\Theta^0 := (f_\eta)_* \Gamma^0$$

induced by the bundle map in (2.1) is invariant. If $f : K \longrightarrow H_{\mathbb{R}}$ and $\eta \in H_{\mathbb{R}}$, then the induced connexion

$$\Xi^0 := (f_\eta)_* \Gamma^0$$

on $Q_{\mathbb{R}} = G \times_{K,f} H_{\mathbb{R}}$ is invariant.

Proof. Since f_η intertwines the G -actions L_g^G and L_g^Q on \mathbf{G} and Q respectively, we have

$$\begin{aligned} f_\eta^*(L_g^Q)^*\Theta^0 &= (L_g^Q \circ f_\eta)^*\Theta^0 = (f_\eta \circ L_g^G)^*\Theta^0 \\ &= (L_g^G)^*f_\eta^*\Theta^0 = (L_g^G)^*\Gamma^0 = \Gamma^0 = f_\eta^*\Theta^0 \end{aligned}$$

for the respective connexion 1-forms. It follows that $(L_g^Q)^*\Theta^0 = \Theta^0$ for all $g \in G$. More directly, we can argue that

$$\begin{aligned} g \cdot \Theta^0 &= (L_g^Q)_*\Theta^0 = (L_g^Q)_*(f_\eta)_*\Gamma^0 = (L_g^Q \circ f_\eta)_*\Gamma^0 \\ &= (f_\eta \circ L_g^G)_*\Gamma^0 = (f_\eta)_*((L_g^G)_*\Gamma^0) = (f_\eta)_*\Gamma^0 = \Theta^0. \end{aligned}$$

This proves the proposition. □

In [3, Section 3], the connexion Θ^0 (for $\eta = e_H$) is described more explicitly.

A real-linear map $\mathbf{c} : T_oM \rightarrow \mathfrak{h}$ is called *f-covariant* if

$$\mathbf{c} \circ (d_oL_k^M) = Ad_{f(k)}^H \circ \mathbf{c} \tag{5.5}$$

for all $k \in K$. Let $\mathcal{L}(T_oM, \mathfrak{h})^f$ denote the vector space of all *f-covariant* linear mappings. If $f : K \rightarrow H_\mathbb{R}$, we define $\mathcal{L}(T_oM, \mathfrak{h}_\mathbb{R})^f$ in a similar way.

PROPOSITION 5.4

There is a one-to-one correspondence

$$\mathcal{L}(T_oM, \mathfrak{h})^f \rightarrow \Omega_H^1(Q, \mathfrak{h})^G, \quad \mathbf{c} \mapsto \omega \tag{5.6}$$

which is uniquely determined by

$$\omega_{\mathbf{q}}X = \mathbf{c}((d_{\mathbf{q}}\pi)X) \tag{5.7}$$

for all $X \in T_{\mathbf{q}}Q$, where $\mathbf{q} \in Q_o$ satisfies (3.5). The associated bundle-valued 1-form is given by

$$\tilde{\omega}_x v := [g\mathbf{q}, \mathbf{c}(d_xL_{g^{-1}}^M)v] \tag{5.8}$$

for all $v \in T_xM$, where $x = g(o)$ for some $g \in G$. For a Hermitian structure $(Q, Q_\mathbb{R})$, there is a similar one-to-one correspondence

$$\mathcal{L}(T_oM, \mathfrak{h}_\mathbb{R})^f \rightarrow \Omega_{H_\mathbb{R}}^1(Q_\mathbb{R}, \mathfrak{h}_\mathbb{R})^G, \quad \mathbf{c} \mapsto \omega \tag{5.9}$$

such that (5.7) and (5.8) hold for any $\mathbf{q} \in (Q_\mathbb{R})_o$.

Proof. Since

$$\begin{aligned} [g\mathbf{q}, \mathbf{c}(d_xL_{(gk)^{-1}}^M)v] &= [g\mathbf{q}f(k), \mathbf{c}(d_oL_{k^{-1}}^M)(d_xL_{g^{-1}}^M)v] \\ &= [g\mathbf{q}, Ad_{f(k)}^H \mathbf{c}(d_oL_{k^{-1}}^M)(d_xL_{g^{-1}}^M)v] \end{aligned}$$

for all $k \in K$, the covariance condition in (5.5) ensures that $\tilde{\omega}_x$ is well-defined, i.e., it is independent of the choice of g . Conversely, every G -invariant 1-form of type Ad_H on M arises this way. \square

The group H^f defined in (3.7) acts on $\mathcal{L}(T_oM, \mathfrak{h})^f$ via $\mathbf{c} \mapsto I_\eta^H \circ \mathbf{c}$, for $\eta \in H^f$. Similarly, for $f : K \rightarrow H_{\mathbb{R}}$, the group $H_{\mathbb{R}}^f$ acts on $\mathcal{L}(T_oM, \mathfrak{h}_{\mathbb{R}})^f$.

The next result is a sharpening of [3, Proposition 3.3].

Theorem 5.5. *For a fixed homomorphism $f : K \rightarrow H$, the assignment*

$$\mathbf{c} \mapsto \Theta^{\mathbf{c}} := \Theta^\omega = \Theta^0 + \omega \quad (5.10)$$

described in (5.6) yields a canonical bijection

$$\mathcal{L}(T_oM, \mathfrak{h})^f / H^f \xrightarrow{\sim} \mathcal{C}(Q)^G$$

which classifies all invariant connexions on $Q = G \times_{K,f} H$. If $f : K \rightarrow H_{\mathbb{R}}$, the assignment

$$\mathbf{c} \mapsto \Xi^{\mathbf{c}} := \Xi^\omega = \Xi^0 + \omega$$

specified in (5.9) yields a canonical bijection

$$\mathcal{L}(T_oM, \mathfrak{h}_{\mathbb{R}})^f / H_{\mathbb{R}}^f \xrightarrow{\sim} \mathcal{C}(Q_{\mathbb{R}})^G$$

which classifies all invariant connexions on $Q_{\mathbb{R}} = G \times_{K,f} H_{\mathbb{R}}$.

Proof. Using Proposition 5.4 and [3, Proposition 3.3], it can be shown that every invariant connexion Θ on Q is of the form $\Theta^{\mathbf{c}}$ for some $\mathbf{c} \in \mathcal{L}(T_oM, \mathfrak{h})^f$. However, the assignment in (5.10) is not injective, because the tautological connexion Θ^0 depends on the choice of base point $\mathbf{q} \in Q_o$ satisfying (3.5). For any $\eta \in H^f$, we may choose $\mathbf{q}' = \mathbf{q}\eta$ as an equivalent base point, which leads to a different choice of Θ^0 and accordingly, to another realization of \mathbf{c} . It is easy to see that now \mathbf{c} is replaced by $\mathbf{c}' := I_\eta^H \circ \mathbf{c}$. \square

6. Hermitian symmetric spaces

Now suppose that $M = G/K$ is a Hermitian symmetric space. Then the complexified tangent space $T_o^{\mathbb{C}}M$ has a decomposition

$$T_o^{\mathbb{C}}M = Z \times \bar{Z},$$

where $Z = T_o^{1,0}M$ carries the structure of a Hermitian Jordan triple, and $\bar{Z} = T_o^{0,1}M$. For background on Jordan triples and the Jordan theoretic realization of bounded symmetric domains, we refer to [7, 13, 14]. The ‘real’ tangent space is recovered as the diagonal

$$T_oM = \{v + \bar{v} \mid v \in Z\}.$$

In this realization, we have

$$(d_oL_k^M)(v + \bar{v}) = kv + \bar{k}\bar{v}$$

for all $k \in K$; this way we obtain the entire identity component of the Jordan triple automorphism group of Z . Every real linear map $\mathbf{c} : T_oM \rightarrow \mathfrak{h}$ has a unique decomposition

$$\mathbf{c}(v + \bar{v}) = \mathbf{a}v + \mathbf{b}\bar{v},$$

where $\mathbf{a} : Z \rightarrow \mathfrak{h}$ and $\mathbf{b} : \bar{Z} \rightarrow \mathfrak{h}$ are linear (respectively, anti-linear) mappings. For $f : K \rightarrow H$, if \mathbf{c} is f -covariant, the identity

$$\begin{aligned} \mathbf{a}(kv) + \mathbf{b}(\bar{k}\bar{v}) &= \mathbf{c}((d_oL_k^M)(v + \bar{v})) = Ad_{f(k)}^H \mathbf{c}(v + \bar{v}) \\ &= Ad_{f(k)}^H \mathbf{a}(v) + Ad_{f(k)}^H \mathbf{b}(\bar{v}) \end{aligned}$$

shows that \mathbf{a} and \mathbf{b} are f -covariant in the sense that

$$\mathbf{a}(kv) = Ad_{f(k)}^H \mathbf{a}(v), \quad \mathbf{b}(\bar{k}\bar{v}) = Ad_{f(k)}^H \mathbf{b}(\bar{v})$$

for all $k \in K$. Here we use that L_k^M is holomorphic and $Ad_{f(k)}^H$ is \mathbb{C} -linear. We write $\mathbf{c} = (\mathbf{a}, \mathbf{b})$ and $\Theta^{\mathbf{c}} = \Theta^{\mathbf{a}, \mathbf{b}}$. Let $\mathcal{L}(Z, \mathfrak{h})^f$ (respectively, $\mathcal{L}(\bar{Z}, \mathfrak{h})^f$) denote the \mathbb{C} -vector space of all \mathbb{C} -linear (respectively, anti-linear) mappings which are f -covariant. The group H^f defined in (3.7) acts on both $\mathcal{L}(Z, \mathfrak{h})^f$ and $\mathcal{L}(\bar{Z}, \mathfrak{h})^f$. If we have $f : K \rightarrow H_{\mathbb{R}}$, then the involution $\Theta \mapsto \Theta^{\sharp}$ is realized as

$$(\Theta^{\mathbf{a}, \mathbf{b}})^{\sharp} = \Theta^{\sharp \circ \mathbf{b}, \sharp \circ \mathbf{a}}.$$

Moreover, the f -covariant maps $\mathbf{c} : T_oM \rightarrow \mathfrak{h}_{\mathbb{R}}$ have the form

$$\mathbf{c}(v + \bar{v}) = \mathbf{a}(v) + \sharp \mathbf{a}(v),$$

where $a \in \mathcal{L}(Z, \mathfrak{h})^f$.

Applying Theorem 5.5, we obtain the following.

Theorem 6.1. *Let $M = G/K$ be Hermitian symmetric. For any homomorphism $f : K \rightarrow H$, there is a canonical bijection*

$$(\mathcal{L}(Z, \mathfrak{h})^f \times \mathcal{L}(\bar{Z}, \mathfrak{h})^f) / H^f \xrightarrow{\sim} \mathcal{C}(Q)^G, \quad (\mathbf{a}, \mathbf{b}) \mapsto \Theta^{\mathbf{a}, \mathbf{b}},$$

which classifies all invariant connexions on the homogeneous H -bundle $Q = G \times_{K, f} H$. If $f : K \rightarrow H_{\mathbb{R}}$, there is a canonical bijection

$$\mathcal{L}(Z, \mathfrak{h})^f / H^f \xrightarrow{\sim} \mathcal{C}_{\mathbb{R}}(Q)^G, \quad \mathbf{a} \mapsto \Theta^{\mathbf{a}, \sharp \mathbf{a}},$$

which classifies all invariant Hermitian connexions on the Hermitian structure $(Q, Q_{\mathbb{R}})$.

COROLLARY 6.2

For any Hermitian symmetric space $M = G/K$, the space $\mathcal{C}(Q)^G$ is a normal variety, such that the involution $\Theta \mapsto \Theta^{\sharp}$ is anti-holomorphic.

Proof. By Theorem 6.1,

$$\mathcal{C}(Q)^G = (\mathcal{L}(Z, \mathfrak{h})^f \times \mathcal{L}(\bar{Z}, \mathfrak{h})^f) / H^f$$

is a quotient variety of a linear space under the action of an algebraic subgroup H^f of H . By [2, Theorem 16.1.1 and Lemma 16.1.2], this is a normal variety. □

Now we turn to the classification of *pure* connexions. The group G acts by holomorphic automorphisms L_g^M on the Hermitian symmetric space M . The complexification of \mathfrak{p} has a decomposition

$$\mathfrak{p}^{\mathbb{C}} := \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-.$$

This decomposition produces the Hodge type decomposition of the complexified tangent bundle

$$T(G/K) \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}(G/K) \oplus T^{0,1}(G/K).$$

The complexification of the composition in (5.4) vanishes on both $\mathfrak{p}_+ \otimes \mathfrak{p}_+$ and $\mathfrak{p}_- \otimes \mathfrak{p}_-$. Consequently, both the (2, 0) and (0, 2) type components of the curvature of Γ^0 vanish. Therefore, the curvature of Γ^0 is of Hodge type (1, 1), showing that the connexion Γ^0 , extended to $G \times_K K^{\mathbb{C}}$ via the embedding $K \subset K^{\mathbb{C}}$, is pure. This implies that the induced tautological connexion Θ^0 on Q is also pure. In more detail, we see the following.

PROPOSITION 6.3

Let $M = G/K$ be a Hermitian symmetric space. For any homomorphism $f : K \rightarrow H$ and any $\eta \in H^f$, the tautological connexion $\Theta^0 = (f_\eta)_* \Gamma^0$ on Q is pure.

If $f : K \rightarrow H_{\mathbb{R}}$ and $\eta \in H_{\mathbb{R}}^f$, then the tautological connexion $\Xi^0 = (f_\eta)_* \Gamma^0$ on $Q_{\mathbb{R}}$ has a pure Hermitian extension $\iota_* \Xi^0$.

Proof. By [11, Proposition II.6.2], the connexion 1-form and curvature 2-form are related by

$$\begin{aligned} f_\eta^* \Theta^0 &= f_\eta^*((f_\eta)_* \Gamma^0) = \underline{f} \circ \Gamma^0, \\ f_\eta^* \underline{\Theta}^0 &= \underline{f_\eta^*(f_\eta)_* \Gamma^0} = \underline{f} \circ \underline{\Gamma}^0. \end{aligned}$$

Thus we have a commuting diagram

$$\begin{array}{ccc} \wedge^2 T_g G & \xrightarrow{d_g f_\eta} & \wedge^2 T_q Q \\ \Gamma_g^0 \downarrow & & \downarrow \Theta_q^0 \\ \mathfrak{k} & \xrightarrow{\underline{f}} & \mathfrak{h} \end{array} \tag{6.1}$$

where $g \in G$ and $q = f_\eta(g)$. Consider the induced vector bundle

$$G \times_K \mathfrak{k} = \{[g : \alpha] = [gk : Ad_{k^{-1}}^K \alpha] \mid g \in G, \alpha \in \mathfrak{k}, k \in K\}.$$

Then f_η induces a vector bundle homomorphism $\tilde{f}_\eta : G \times_K \mathfrak{k} \rightarrow Q \times_H \mathfrak{h}$ over M , defined by

$$\tilde{f}_\eta[g, \alpha] := [f_\eta(g), \underline{f}\alpha].$$

Now let $v_1, v_2 \in T_z M$, where $z = g(o)$. Choose $w_i \in T_g G$ with $(d_g R_o^M)w_i = v_i$. Define $\tilde{v}_i := (d_g f_\eta)w_i$. Then the commuting diagram in (6.1) implies that

$$(d_q \pi)\tilde{v}_i = (d_q \pi)(d_g f_\eta)w_i = (d_g R_o^M)w_i = v_i.$$

It now follows that

$$\begin{aligned} \tilde{\Theta}_z^0(v_1, v_2) &= [q : \Theta_q^0(\tilde{v}_1, \tilde{v}_2)] = [q : \Theta_q^0((d_g f_\eta)w_1, (d_g f_\eta)w_2)] \\ &= [q : (f_\eta^* \Theta_g^0)(w_1, w_2)] = [q : \underline{f}(\underline{\Gamma}_g^0(w_1, w_2))] = \tilde{f}_\eta(\tilde{\Gamma}_z^0(v_1, v_2)). \end{aligned}$$

Consider the complexified principal bundle $G \times_K \mathfrak{k}^\mathbb{C}$ and the map $\tilde{f}_\eta : G \times_K \mathfrak{k}^\mathbb{C} \rightarrow Q \times_H \mathfrak{h}$ which is \mathbb{C} -linear on the fibres. Passing to the complexification of $\underline{\Gamma}^0$, it follows that the relation

$$\tilde{\Theta}_z^0(v_1, v_2) = \tilde{f}_\eta(\tilde{\Gamma}_z^0(v_1, v_2))$$

also holds for complexified tangent vectors $v_i \in T_z^\mathbb{C}M$. There exist $u_i \in T_oM$ such that $v_i = T_o(L_g^M)u_i$. Since Γ^0 is invariant, we have

$$\tilde{\Gamma}_z^0(v_1, v_2) = \tilde{\Gamma}_{g(o)}^0((d_o L_g^M)u_1, (d_o L_g^M)u_2) = \tilde{\Gamma}_o^0(u_1, u_2) = 0$$

because the constant vector fields

$$u_i \frac{\partial}{\partial z} \in \mathfrak{g}^\mathbb{C}$$

commute. Similarly, we have

$$\tilde{\Theta}_z^0(\tilde{v}_1, \tilde{v}_2) = \tilde{f}_\eta(\tilde{\Gamma}_z^0(\tilde{v}_1, \tilde{v}_2))$$

for anti-holomorphic tangent vectors, and

$$\tilde{\Gamma}_z^0(\tilde{v}_1, \tilde{v}_2) = \tilde{\Gamma}_{g(o)}^0((d_o L_g^M)\bar{u}_1, (d_o L_g^M)\bar{u}_2) = \tilde{\Gamma}_o^0(\bar{u}_1, \bar{u}_2) = 0$$

since the quadratic vector fields

$$\bar{u}_i = \{z; u_i; z\} \frac{\partial}{\partial z} \in \mathfrak{g}^\mathbb{C}$$

commute as a consequence of the Jordan triple identity. □

The following theorem is our first main result.

Theorem 6.4. *Let $M = G/K$ be a Hermitian symmetric space. Consider the homogeneous complex principal H -bundle $Q = G \times_{K,f} H$ induced by the homomorphism $f : K \rightarrow H$. Then an invariant connexion $\Theta^c = \Theta^{a,b}$, associated with an f -covariant (real) linear map*

$$c = a + b : Z_\mathbb{R} \rightarrow \mathfrak{h}$$

is pure if and only if

$$a \wedge a = 0 = b \wedge b. \tag{6.2}$$

Proof. Write $\Theta^c = \Theta^0 + \omega$. Then [11, Theorem II.5.2 and Proposition II.5.5] imply that

$$\begin{aligned}\underline{\Theta}_q^c(X, Y) &= (d\Theta^c)_q(X, Y) + \frac{1}{2}[\Theta_q^c X, \Theta_q^c Y] \\ &= (d\Theta^0 + d\omega)_q(X, Y) + \frac{1}{2}[(\Theta_q^0 + \omega_q)X, (\Theta_q^0 + \omega_q)Y] \\ &= (d\Theta^0)_q(X, Y) + \frac{1}{2}[\Theta_q^0 X, \Theta_q^0 Y] + (d\omega)_q(X, Y) \\ &\quad + \frac{1}{2}[\Theta_q^0 X, \omega_q Y] + \frac{1}{2}[\omega_q X, \Theta_q^0 Y] + \frac{1}{2}[\omega_q X, \omega_q Y] \\ &= \underline{\Theta}_q^0(X, Y) + (D^0\omega)_q(X, Y) + (\omega \wedge \omega)_q(X, Y)\end{aligned}$$

for all $X, Y \in T_q Q$, where $D^0\omega$ is the covariant derivative of the tensorial 1-form ω with respect to Θ^0 . By Proposition 6.3, the curvature $\underline{\Theta}^0$ is of type (1, 1). Thus the main point of the proof is to show that

$$D^0\omega = 0. \quad (6.3)$$

The proof of (6.3) presented here is a simplification of a similar argument in [3, Theorem 5.2]. For any $\gamma \in \mathfrak{g}$, consider the induced vector field

$$\gamma_x^M = \left. \frac{\partial}{\partial t} \right|_{t=0} (\exp(t\gamma)x) = \left. \frac{\partial}{\partial t} \right|_{t=0} (R_x^M \exp(t\gamma)) = (d_e R_x^M)\gamma$$

on M induced by the left G -action. Let γ^Q denote the Θ^0 -horizontal lift of γ^M .

The following two lemmas would be needed to prove (6.3).

Lemma 6.5. For any $g \in G$ and $q = g\mathbf{q} = \tilde{f}(g)$, the following holds:

$$(\omega\gamma^Q)(\tilde{f}(g)) = \mathbf{c}((d_e R_o^M)Ad_{g^{-1}}^G\gamma).$$

Proof. Put $z = g(o) = \pi(q)$. The identity $L_{g^{-1}}^M \circ R_z^M = R_o^M \circ I_{g^{-1}}^G$ implies that

$$(d_z L_{g^{-1}}^M)\gamma_z = (d_z L_{g^{-1}}^M)(d_e R_z^M)\gamma = (d_e R_o^M)Ad_{g^{-1}}^G\gamma$$

and hence

$$\begin{aligned}[q : (\omega\gamma^Q)(q)] &= [q : \omega_q\gamma_q^Q] = \tilde{\omega}_z(d_q\pi)\gamma_q^Q = \tilde{\omega}_z\gamma_x = [q : \mathbf{c}(d_z L_{g^{-1}}^M)\gamma_z] \\ &= [q : \mathbf{c}(d_e R_o^M)Ad_{g^{-1}}^G\gamma].\end{aligned}$$

□

Lemma 6.6. For any $\gamma, \eta \in \mathfrak{g}$, the following holds:

$$(\gamma^Q(\omega\eta^Q))(\mathbf{q}) = -\mathbf{c}(d_e R_o^M)[\gamma, \eta].$$

Proof. Let $\gamma \in \mathfrak{p}$. Then

$$(d_{\mathfrak{q}}\pi)(d_e\tilde{f})\gamma = (d_e(\pi \circ \tilde{f}))\gamma = (d_eR_o^M)\gamma = \gamma_o^M = (d_{\mathfrak{q}}\pi)\gamma_{\mathfrak{q}}^Q$$

and $(d_e\tilde{f})\gamma \in (d_e\tilde{f})\mathfrak{p} = (d_e\tilde{f})T_e^0G = T_{\mathfrak{q}}^0Q$. It follows that

$$\gamma_{\mathfrak{q}}^Q = (d_e\tilde{f})\gamma.$$

Let $g_t := \exp(t\gamma) \in G$. Since $d_eR_o^M$ is linear, Lemma 6.5 implies that

$$\begin{aligned} (\gamma^Q(\omega\eta^Q))(\mathfrak{q}) &= (d_{\mathfrak{q}}(\omega\eta^Q))\gamma_{\mathfrak{q}}^Q = (d_{\mathfrak{q}}(\omega\eta^Q))(d_e\tilde{f})\gamma \\ &= \frac{\partial}{\partial t}\Big|_{t=0}(\omega\eta^Q)\tilde{f}(g_t) = \frac{\partial}{\partial t}\Big|_{t=0}\mathbf{c}((d_eR_o^M)Ad_{g_t}^G\eta) \\ &= \mathbf{c}((d_eR_o^M)\frac{\partial}{\partial t}\Big|_{t=0}Ad_{g_t}^G\eta) = -\mathbf{c}(d_eR_o^M)[\gamma, \eta]. \end{aligned}$$

This completes the proof. □

We can now conclude the proof of (6.3). For $u \in T_oM$, consider the vector field

$$\dot{u} = u + \epsilon\{z; u; z\} \in \dot{\mathfrak{p}}$$

on M , where $\epsilon = -1$ in the non-compact case and $\epsilon = 1$ in the compact case. Then

$$(D^0\omega)(\dot{u}^Q, \dot{v}^Q) = d\omega(\dot{u}^Q, \dot{v}^Q) = \dot{u}^Q(\omega\dot{v}^Q) - \dot{v}^Q(\omega\dot{u}^Q) - \omega[\dot{u}^Q, \dot{v}^Q]$$

and hence

$$(D^0\omega)(\dot{u}^Q, \dot{v}^Q)(\mathfrak{q}) = \dot{u}^Q(\omega\dot{v}^Q)(\mathfrak{q}) - \dot{v}^Q(\omega\dot{u}^Q)(\mathfrak{q}) - \omega_{\mathfrak{q}}[\dot{u}^Q, \dot{v}^Q]_{\mathfrak{q}}.$$

By Lemma 6.6, we have

$$\dot{u}^Q(\omega\dot{v}^Q)(\mathfrak{q}) = -\mathbf{c}(d_eR_o^M)[\dot{u}, \dot{v}] = 0$$

since $[\dot{u}, \dot{v}] \in \mathfrak{k}$, and $R_o^Mk = ko = o$ implies that $(d_eR_o^M)\mathfrak{k} = 0$. Similarly, we have $\dot{v}^Q(\omega\dot{u}^Q)(\mathfrak{q}) = 0$.

By definition, we have $(d_{\mathfrak{q}}\pi)\dot{u}_{\mathfrak{q}}^Q = \dot{u}_{\pi(\mathfrak{q})}$. Now applying [9, Proposition I.3.3], it follows that $(d_{\mathfrak{q}}\pi)[\dot{u}^Q, \dot{v}^Q]_{\mathfrak{q}} = [\dot{u}, \dot{v}]_o = 0$. Hence $\omega_{\mathfrak{q}}[\dot{u}^Q, \dot{v}^Q]_{\mathfrak{q}} = 0$, and therefore,

$$(D^0\omega)_{\mathfrak{q}}(\dot{u}_{\mathfrak{q}}^Q, \dot{v}_{\mathfrak{q}}^Q) = 0.$$

This proves (6.3) because $D^0\omega$ is G -invariant. It follows that pure invariant connexions are characterized by the condition that $\mathbf{c} \wedge \mathbf{c}$ is of type $(1, 1)$. This is equivalent to (6.2). □

A shorter proof of (6.3) (and hence of Theorem 6.4) uses the fact that $D^0\omega$ is a G -invariant section of $\text{ad}(Q) \otimes (\wedge^2 T_{\mathbb{C}}^*(G/K))$. Therefore, to prove (6.3), it suffices to show that

$$(D^0\omega)_{eK} = 0. \tag{6.4}$$

Let $\sqrt{-1}E = \sqrt{-1}z \frac{\partial}{\partial z}$ generate the center of \mathfrak{k} . Put

$$\mathfrak{h}_{\pm} := \{\beta \in \mathfrak{h} \mid [f(\sqrt{-1}E), \beta] = \pm\sqrt{-1}\beta\}.$$

Then $(D^0\omega)_{e_K}$ is an element of $(\mathfrak{h}_+ \oplus \mathfrak{h}_-) \otimes \wedge^2 \mathfrak{p}^{\mathbb{C}}$. Now the center \mathbf{T} of K acts on $\wedge^2 \mathfrak{p}^{\mathbb{C}}$ with weights 2, 0, -2 , while \mathbf{T} acts on $\mathfrak{h}_+ \oplus \mathfrak{h}_-$ with weights 1, -1 . Therefore, there is no nonzero \mathbf{T} -invariant element in $(\mathfrak{h}_+ \oplus \mathfrak{h}_-) \otimes \wedge^2 \mathfrak{p}^{\mathbb{C}}$. This proves (6.4). Hence (6.3) holds because $D^0\omega$ is G -invariant.

Put

$$\mathcal{L}(Z, \mathfrak{h})^f := \{\mathbf{a} \in \mathcal{L}(Z, \mathfrak{h})^f \mid \mathbf{a} \wedge \mathbf{a} = 0\}$$

and define $\mathcal{L}(\bar{Z}, \mathfrak{h})^f$ in a similar way. The group H^f defined in (3.7) acts on both $\mathcal{L}(Z, \mathfrak{h})^f$ and $\mathcal{L}(\bar{Z}, \mathfrak{h})^f$.

Combining Theorem 6.1 and Theorem 6.4 yields the following.

Theorem 6.7. *For any homomorphism $f : K \rightarrow H$, there is a canonical bijection*

$$(\mathcal{L}(Z, \mathfrak{h})^f \times \mathcal{L}(\bar{Z}, \mathfrak{h})^f) / H^f \xrightarrow{\sim} \mathcal{C}(Q)^G, \quad (\mathbf{a}, \mathbf{b}) \mapsto \Theta^{\mathbf{a}, \mathbf{b}},$$

which classifies all invariant pure connexions on the homogeneous principal H -bundle $Q = G \times_{K, f} H$.

If $f : K \rightarrow H_{\mathbb{R}}$, there is a canonical bijection

$$\mathcal{L}(Z, \mathfrak{h})^f / H^f \xrightarrow{\sim} \mathcal{C}_{\mathbb{R}}(Q)^G, \quad \mathbf{a} \mapsto \Theta^{\mathbf{a}, \sharp \mathbf{a}},$$

which classifies all invariant pure Hermitian connexions on the Hermitian structure $(Q, Q_{\mathbb{R}})$.

In terms of the preceding theorem, the involution $\Theta \mapsto \Theta^{\sharp}$ acts on the space $\mathcal{C}(Q)^G$, with fixed point set $\mathcal{C}_{\mathbb{R}}(Q)^G$ consisting of all pure invariant Hermitian connexions.

7. Fine structure of moduli spaces

We will now analyze the structure of the moduli space $\mathcal{C}(Q)^G$ of all pure invariant connexions on a homogeneous H -bundle $Q = G \times_{K, f} H$ in more detail. The goal is to show that $\mathcal{C}(Q)^G$ is a normal complex variety on which the canonical involution $\Theta \mapsto \Theta^{\sharp}$ is anti-holomorphic. According to Theorem 6.7, we start with a homomorphism $f : K \rightarrow H$ and two f -covariant linear maps $\mathbf{a} : Z \rightarrow \mathfrak{h}$, $\mathbf{b} : \bar{Z} \rightarrow \mathfrak{h}$, where Z is a Hermitian Jordan triple identified with the holomorphic tangent space $T_o^{1,0}M$ of the Hermitian symmetric space $M = G/K$ at the origin. We may assume that Z is irreducible of rank r . Let e_1, \dots, e_r be a frame of minimal orthogonal tripotents $e_i \in Z$. By the spectral theorem for Jordan triples [13], every $z \in Z$ has a representation

$$z = \sum_i t_i k e_i,$$

where $k \in K$ and the (uniquely determined) ‘singular values’ satisfy $0 \leq t_1 \leq t_2 \leq \dots \leq t_r$. For each i , we put

$$K_i := \{k \in K \mid k e_i = e_i\}.$$

These subgroups of K are all conjugate.

Lemma 7.1. *Let $\mathbf{a} \in \mathcal{L}(Z, \mathfrak{h})^f$ and $\mathbf{b} \in \mathcal{L}(\bar{Z}, \mathfrak{h})^f$. Then $\mathbf{a} \wedge \mathbf{a} = 0$ (respectively, $\mathbf{b} \wedge \mathbf{b} = 0$) if and only if $[\mathbf{a}(e_i), \mathbf{a}(e_j)] = 0$ (respectively, $[\mathbf{b}(e_i), \mathbf{b}(e_j)] = 0$) for all $1 \leq i, j \leq r$.*

Proof. Working out the first case, the condition that $\mathbf{a} \wedge \mathbf{a} = 0$ is equivalent to the condition that

$$[\mathbf{a}(z), \mathbf{a}(w)] = 0 \tag{7.1}$$

for all $z, w \in Z$. Now assume that $[\mathbf{a}(e_i), \mathbf{a}(e_j)] = 0$ for all $1 \leq i, j \leq r$. Assume that $k \in K$ satisfies the condition $ke_j = e_j$ for some fixed j . Then we have $Ad_{f(k)}^H \mathbf{a}(e_j) = \mathbf{a}(e_j)$. In view of (2.2), we obtain the equality

$$[\mathbf{a}(z), \mathbf{a}(e_j)] = \sum_i \zeta_i Ad_{f(k)}^H [\mathbf{a}(e_i), \mathbf{a}(e_j)] = 0. \tag{7.2}$$

For any fixed j , the orbit $\{\sum_i \zeta_i ke_i \mid \zeta_i \in \mathbb{C}, k \in K, ke_j = e_j\}$ has a non-empty interior. Hence from (7.2), it follows that $[\mathbf{a}(z), \mathbf{a}(e_j)] = 0$ for all $z \in Z$. Using again the spectral theorem for Jordan triples, we conclude that (7.1) holds. \square

PROPOSITION 7.2

For rank $r = 1$, corresponding to the unit ball in $G/K = \mathbb{C}^d$ (non-compact version) and the projective space $G/K = \mathbb{C}P^d$ (compact version), every invariant connexion Θ on Q is pure. Hence, by Corollary 6.2, $\mathcal{L}(Q)^G = \mathcal{C}(Q)^G$ is a normal variety.

Proof. For $Z = \mathbb{C}$ (corresponding to the unit disk and the Riemann sphere, respectively), the condition $\mathbf{a} \wedge \mathbf{a} = 0 = \mathbf{b} \wedge \mathbf{b}$ is trivially satisfied since $Z \wedge Z = 0$. In the higher-dimensional rank 1 case, we have only one tripotent $e = e_1$, and Lemma 7.1 shows that the condition $\mathbf{a} \wedge \mathbf{a} = 0 = \mathbf{b} \wedge \mathbf{b}$ is again satisfied. \square

In the 1-dimensional case, we can describe the moduli space very explicitly. Consider the case where $Z = \mathbb{C}$, $K = U(1) = \mathbf{T}$. Let $H = GL_N(\mathbb{C})$, $H_{\mathbb{R}} = U(N)$. The vector space $V = \mathbb{C}^N$ has a (finite) direct sum decomposition

$$V = \sum_{m \in \mathbf{Z}} V_m, \tag{7.3}$$

where

$$V_m = \{v \in V \mid f(\vartheta)v = \vartheta^m v \ \forall \vartheta \in \mathbf{T}\}$$

is the weight space with respect to the action of \mathbf{T} . Up to a permutation, we order the weight spaces according to strictly increasing weights $m_0 < m_1 < \dots < m_\ell$. Any matrix $A \in \mathfrak{h} = \mathfrak{gl}_N(\mathbb{C})$ can be written as a block matrix $A = (A_i^j)$ with $A_i^j : V_{m_j} \rightarrow V_{m_i}$. Since

$$f(\vartheta) = \begin{pmatrix} \vartheta^{m_0} I_{m_0} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \vartheta^{m_\ell} I_{m_\ell} \end{pmatrix},$$

it follows that $f(\vartheta)Af(\bar{\vartheta})$ is represented by the matrix $(\vartheta^{m_i-m_j} A_i^j)$. Now write $\mathbf{a}(z) = zA$, where $A := \mathbf{a}(1)$. Then the f -covariance condition yields

$$\vartheta A = \mathbf{a}(\vartheta) = f(\vartheta)Af(\bar{\vartheta}).$$

Thus

$$\vartheta A_i^j = \vartheta^{m_i-m_j} A_i^j$$

for all $\vartheta \in \mathbf{T}$. This shows that the condition $A_i^j \neq 0$ implies that $m_i - m_j = 1$.

Similarly, consider $\mathbf{b}(z) = \bar{z}B$, where $B := \mathbf{b}(1)$. Then the f -covariance condition yields

$$\bar{\vartheta} B = \mathbf{b}(\vartheta) = f(\vartheta)Bf(\bar{\vartheta}).$$

Thus we have

$$\bar{\vartheta} B_i^j = \vartheta^{m_i-m_j} B_i^j$$

for all $\vartheta \in \mathbf{T}$. This shows that the condition $B_i^j \neq 0$ implies that $m_i - m_j = 1$.

It follows that for all ‘gaps’ $m_i < m_{i+1}$ of size > 1 , we have $A_{i+1}^i = 0 = B_i^{i+1}$. Thus there is a decomposition

$$\mathbb{C}^N = W^1 \oplus \dots \oplus W^k \tag{7.4}$$

such that A, B are block-diagonal matrices with respect to the decomposition in (7.4) and each direct summand W in (7.4) consists of a connected chain of weight spaces

$$W = \sum_{i=0}^{\ell} V_{m+i}$$

for a fixed integer m .

It would be sufficient to consider each summand W separately. The restrictions of A, B to W give rise to block-matrices

$$\begin{pmatrix} E_0 & B^1 & 0 & 0 & 0 \\ A_1 & E_1 & B^2 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & A_{\ell-1} & E_{\ell-1} & B^\ell \\ 0 & 0 & 0 & A_\ell & E_\ell \end{pmatrix}.$$

Thus the moduli space consists of maps

$$V_{m+i-1} \xrightarrow{A_i} V_{m+i}, \quad V_{m+i} \xrightarrow{B_i} V_{m+i-1},$$

for $1 \leq i \leq \ell$, modulo joint conjugation

$$\tilde{A}_i = g_i A_i g_{i-1}^{-1}, \quad \tilde{B}^i = g_{i-1} B^i g_i^{-1}, \quad 1 \leq i \leq \ell,$$

where $(g_i) \in \prod_{i=0}^{\ell} GL(V_{m+i})$ is arbitrary. We note that this is also the *quiver variety* of the oriented graph

$$V_{m+0} \xrightarrow{A_1} V_{m+1} \xrightarrow{A_2} \dots \xrightarrow{A_{\ell-1}} V_{m+\ell-1} \xrightarrow{A_\ell} V_{m+\ell} \tag{7.5}$$

whose double is

$$V_{m+0} \begin{matrix} \xrightarrow{A_1} \\ \xleftarrow{B^1} \end{matrix} V_{m+1} \begin{matrix} \xrightarrow{A_2} \\ \xleftarrow{B^2} \end{matrix} \dots \begin{matrix} \xrightarrow{A_{\ell-1}} \\ \xleftarrow{B^{\ell-1}} \end{matrix} V_{m+\ell-1} \begin{matrix} \xrightarrow{A_\ell} \\ \xleftarrow{B^\ell} \end{matrix} V_{m+\ell}. \tag{7.6}$$

For any quiver Q with index set I , the double quiver \bar{Q} gives rise to a moment map

$$\mu(x) := \left(\sum_{a_+=i} x_a x_{\bar{a}} - \sum_{a_-=i} x_{\bar{a}} x_a \right)_{i \in I} \in \prod_{i \in I} \mathfrak{gl}(V_i).$$

Here a runs over all (oriented) edges in \bar{Q} , with head a_+ and tail a_- , and \bar{a} denotes the opposite edge. In our case $I = \{0, 1, \dots, \ell\}$, and for $x = (A_i, B^i)_{i=0}^\ell$ we have

$$\begin{aligned} \mu(x)_0 &= B^1 A_1 - B^1 A_1 = 0, \\ \mu(x)_\ell &= A_\ell B^\ell - A_\ell B^\ell = 0, \\ \mu(x)_i &= A_i B^i + B^{i+1} A_{i+1} - A_i B^i - B^{i+1} A_{i+1} = 0 \end{aligned}$$

for $0 < i < \ell$. Thus the condition $\mu(x) = 0$ is trivially satisfied, and by [6, Theorem 1.1], the quotient

$$\text{Rep}(\bar{Q}) / \prod_{i=0}^\ell \text{GL}(V_{m+i})$$

is a normal variety. By [5, Theorem 1] the ring of polynomial invariants of the Mumford’s geometric invariant theoretic quotient $\mathbb{C}^N // G$ is generated by traces of oriented cycles of the quiver in (3.7) of length $\leq N^2$.

In the *higher dimensional case*, a first step is to find matrices $A_i = \mathbf{a}(e_i)$, $B_i = \mathbf{b}(e_i)$ as in Lemma 7.1. Consider first the unit ball. Here the non-zero tripotents are the unit vectors $e \in Z := \mathbb{C}^d$, regarded as a Jordan triple of rank 1. For $K = \text{U}(d)$, we put

$$K_e := \{\ell \in K \mid \ell e = e\} \approx \text{U}(d-1). \tag{7.7}$$

PROPOSITION 7.3

Let $A, B \in \mathfrak{h}$ satisfy

$$\text{Ad}_{f(\ell)}^H A = A, \text{Ad}_{f(\ell)}^H B = B \tag{7.8}$$

for all $\ell \in K_e$, and

$$\text{Ad}_{f(\zeta)}^H A = \zeta A, \text{Ad}_{f(\zeta)}^H B = \bar{\zeta} B \tag{7.9}$$

for all $\zeta \in \mathbf{T}$ (center of K). Then there exist unique f -covariant linear maps $\mathbf{a} : Z \rightarrow \mathfrak{h}$, $\mathbf{b} : \bar{Z} \rightarrow \mathfrak{h}$ satisfying

$$\mathbf{a}(e) = A, \mathbf{b}(e) = B.$$

Proof. The (Shilov) boundary $S = \mathbf{S}^{2d-1}$ of the unit ball D is the homogeneous space $S = K/K_e$. Define a real-analytic mapping

$$a : S \rightarrow \mathfrak{h}$$

by $a(ke) := \text{Ad}_{f(k)}^H A$ for all $k \in K$. This is well-defined since for $k, k' \in K$ satisfying the condition $ke = k'e$; we have $\ell := k^{-1}k' \in K_e$ and hence $\text{Ad}_{f(k)}^H A = \text{Ad}_{f(k')}^H A$ as a

consequence of (7.8). By construction, we have $a(e) = A$ and $a(ks) = Ad_{f(k)}^H a(s)$ for all $k \in K$, $s \in S$.

Now consider the harmonic extension $\hat{a} : D \rightarrow \mathfrak{h}$ of a , given by the Poisson integral. Then $\hat{a}|_S = a$ and \hat{a} is also f -covariant, since the Poisson integral operator commutes with the action of K . Moreover, the condition in (7.9) implies that

$$\hat{a}(\zeta z) = \zeta \hat{a}(z)$$

for all $\zeta \in \mathbf{T}$ and $z \in D$. Therefore \hat{a} is (the restriction of) a \mathbb{C} -linear map, again denoted by a , on Z . The anti-linear case is treated similarly. \square

In the higher rank case, things are more complicated. Let Z be an irreducible Jordan triple of rank r , and let e_1, \dots, e_r be a frame of minimal orthogonal tripotents. Given matrices $A_1, \dots, A_r \in \mathfrak{h}$, we put

$$e_I := \sum_{i \in I} e_i, \quad A_I := \sum_{i \in I} A_i \in \mathfrak{h}$$

for all subsets $I \subset \{1, \dots, r\}$. We seek conditions on A_i such that there exists a (unique) f -covariant \mathbb{C} -linear map

$$\mathbf{a} : Z \rightarrow \mathfrak{h}$$

satisfying the condition $\mathbf{a}(e_i) = A_i$ for all $1 \leq i \leq r$. If I, J are two subsets of equal cardinality $|I| = |J|$, then e_I and e_J are tripotents of equal rank, and there exists $k \in K$ such that $ke_I = e_J$. A necessary condition for the A_i is that

$$Ad_{f(k)}^H A_I = A_J \tag{7.10}$$

whenever $|I| = |J|$ and $k \in K$ satisfies the condition $ke_I = e_J$. This applies in particular for $I = J$. Applying (7.10) to $I = \{1, \dots, r\}$, we have

$$Ad_{f(k)}^H (A_1 + \dots + A_r) = A_1 + \dots + A_r$$

for all $k \in K$ fixing the maximal tripotent $e := e_1 + \dots + e_r$. Now $S = K/K_e$ becomes the Shilov boundary of the (spectral) unit ball $D \subset Z$ (a proper subset of the full boundary if $r > 1$) and, as in the rank 1 case, we may define a f -covariant real-analytic mapping $a : S \rightarrow \mathfrak{h}$ by putting

$$a(ke) := Ad_{f(k)}^H (A_1 + \dots + A_r)$$

for all $k \in K$. Consider its harmonic extension $\hat{a} : D \rightarrow \mathfrak{h}$ given by the higher-rank analogue of the Poisson integral [12]. A Cartan subpace of \mathfrak{k} is given by the commuting linear vector fields

$$\sqrt{-1} \{e_i; e_i; z\} \frac{\partial}{\partial z},$$

where $1 \leq i \leq r$. Integrating these vector fields we obtain a holomorphic torus action

$$(\vartheta, z) \mapsto \vartheta \cdot z$$

of $\vartheta \in \mathbf{T}^r$ on D such that

$$(\vartheta_1, \dots, \vartheta_r)e_i = \vartheta_i e_i$$

for all i . A second necessary condition, generalizing (7.9), is the following:

$$Ad_{f(\vartheta)}^H A_i = \vartheta_i A_i$$

for all $\vartheta \in \mathbf{T}^d \subset K$. Imposition this condition leads to a unique \mathbb{C} -linear extension

$$a : Z \longrightarrow \mathfrak{h}$$

which is still f -covariant. The anti-linear case of matrices B_i is treated similarly.

In this way, the moduli space of pure invariant connexions gives rise to varieties of commuting matrix tuples (modulo joint conjugation), on which there is a rich literature [8]. A detailed study is reserved for a future publication.

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