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On stability of tangent bundle of toric varieties

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Abstract. Let X be a nonsingular complex projective toric variety. We address the question of semi-stability as well as stability for the tangent bundle TX. In particular, a complete answer is given when X is a Fano toric variety of dimension four with Picard number at most two, complementing the earlier work of Nakagawa (*Tohoku. Math. J.* **45** (1993) 297–310; **46** (1994) 125–133). We also give an infinite set of examples of Fano toric varieties for which TX is unstable; the dimensions of this collection of varieties are unbounded. Our method is based on the equivariant approach initiated by Klyachko (*Izv. Akad. Nauk. SSSR Ser. Mat.* **53** (1989) 1001–1039, 1135) and developed further by Perling (*Math. Nachr*. **263/264** (2004) 181–197) and Kool (Moduli spaces of sheaves on toric varieties, Ph.D. thesis (2010) (University of Oxford); *Adv. Math*. **227** (2011) 1700–1755).

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1. Introduction

Let X be a smooth complex projective variety. If the canonical line bundle K_X is ample, then from a theorem of [2,22], it follows that the tangent bundle TX is semistable (in the sense of Mumford and Takemoto) with respect to the polarization K_X. The variety X is said to be Fano if the anti-canonical line bundle K_X^{-1} is ample. Fano varieties are very basic objects in birational classification of complex algebraic varieties (minimal model program), for example, a theorem of Birkar *et al.* [3] says that every uniruled variety

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is birational to a variety which has a fibration with a Fano general fiber. Stability of the tangent bundle of a nonsingular Fano variety with respect to polarization K_X^{-1} is a question of interest originated mostly from a differential geometric point of view. The existence of Einstein–Kähler metric on X implies the polystability of the tangent bundle with respect to K_X^{-1} [13,16]. In general, the converse of this result is not true. The simplest example is the surface Σ_2 obtained by blowing up the complex plane $\mathbb{P}^2$ at two points. In this paper, we are interested in studying semi-stability as well as stability of the tangent bundle TX when X is a toric variety and in particular, when X is a Fano toric variety.

Let X be a nonsingular complex projective toric variety of dimension n, equipped with an action of the n-dimensional complex torus T. A coherent torsion-free sheaf $\mathcal{E}$ on X is said to be T-equivariant (or T-linearized) if it admits a lift of the T-action on X, which is linear on the stalks of $\mathcal{E}$. Fix a polarization H of X, where H is a T-equivariant very ample line bundle (equivalently, T-invariant very ample divisor) of X.

A T-equivariant coherent torsion-free sheaf $\mathcal{E}$ on X is said to be equivariantly stable (respectively, equivariantly semistable) if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$) for every proper T-equivariant proper subsheaf $\mathcal{F} \subset \mathcal{E}$ (see Section 2). From the uniqueness of the Harder–Narasimhan filtration, it follows easily that the notions of semi-stability and equivariant semi-stability of an equivariant torsion-free sheaf on a nonsingular toric variety are equivalent. Further, in case of equivariant torsion-free sheaves, the notions of equivariant stability and stability coincide (see Theorem 2.1 or [15, Proposition 4.13]). Using this equivariant approach, we investigate the stability and semi-stability of the tangent bundle of a nonsingular toric variety.

Our main results are as follows:

(1) Determination of the stability (or otherwise) of the tangent bundle of Hirzebruch surfaces for an arbitrary polarization; see Theorem 6.2 and Corollary 6.3.
(2) A very simple proof of the well-known result that $T\mathbb{P}^n$ is stable with respect to the anti-canonical polarization (Theorem 7.1).
(3) We identify all nonsingular Fano toric 4-folds with Picard number at most two that have semi-stable tangent bundle (Theorem 9.3). In particular, we get an example of a Fano toric 4-fold (namely, B_5) which has a strictly semi-stable tangent bundle, but does not admit Einstein–Kähler metric (cf. [18]).
(4) Construction of an infinite family of Fano toric varieties with unstable tangent bundle, consisting of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(m))$ for all $n \geq 2$ and $m \leq n$ (Theorem 8.1). The case $n = 2$ was settled earlier by Steffens in [21].

The general strategy here is as follows. We use the isotypical decomposition of an equivariant sheaf to describe it in terms of certain combinatorial data, following Perling [20] and Kool [14]. (Of course, both draw inspiration from the seminal work of Klyachko [12].) We prove a formula in Lemma 4.3 that calculates the rank of an equivariant torsion-free coherent sheaf on X from the combinatorial data. With a little bit of fine-tuning, this specializes to a very useful rank formula, see (5.6) for an arbitrary equivariant coherent subsheaf of the tangent bundle. We obtain a similar type of formula for the degree of such a subsheaf, see (5.9), using a formula of Kool for the first Chern class. Using these formulas, we can identify the combinatorial data that may be associated to a subsheaf of a given rank whose slope exceeds that of the tangent bundle; see Lemmas 6.1 and 9.1. We then examine if a subsheaf with the given rank and corresponding to such combinatorial data really exists by studying the transition maps associated to the combinatorial data.

2. From equivariant stability to stability

Given a coherent torsion-free sheaf $\mathcal{E}$ on a projective variety X of dimension n, the slope $\mu(\mathcal{E})$ with respect to a polarization H on X is defined as the ratio

$$\mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}},$$

where the degree of $\mathcal{E}$ is defined as the intersection product $\deg \mathcal{E} := c_1(\mathcal{E}) \cdot H^{n-1}$. A subsheaf $\mathcal{F}$ of $\mathcal{E}$ is said to be a proper subsheaf if $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$. A torsion-free sheaf $\mathcal{E}$ is said to be μ-stable (respectively, μ-semistable) if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$) for every proper subsheaf $\mathcal{F} \subset \mathcal{E}$. The notion of μ-stability (semistability) was first introduced by Mumford and Takemoto. In this article, (semi)stability we will always mean μ-(semi)stability, unless otherwise specified. Also, a sheaf $\mathcal{E}$ will be called unstable if $\mathcal{E}$ is not semistable.

Stable and semistable sheaves play an important role in the structure theory of coherent sheaves (cf. [10]). Every torsion-free coherent sheaf admits the Harder–Narasimhan filtration such that each successive quotient is semistable. A semistable sheaf, in turn, admits a Jordan–Holder filtration such that each successive quotient is stable of the same slope.

In this section, we give a proof of the crucial fact that for an equivariant torsion-free sheaf on a nonsingular toric variety, equivariant stability is equivalent to the usual stability. This result is also proved by Kool [15, Proposition 4.13] for reflexive sheaves. The proof given here is different.

Theorem 2.1. *Let $\mathcal{E}$ be an equivariant torsion-free sheaf on a projective toric variety X. Then $\mathcal{E}$ is equivariantly stable if and only if $\mathcal{E}$ is stable.*

Proof. If $\mathcal{E}$ is stable, then it is evidently equivariantly stable. We will prove that $\mathcal{E}$ is stable if it is equivariantly stable.

We first note that it is enough to prove this under the extra assumption that $\mathcal{E}$ is reflexive. Indeed, if $\mathcal{E}$ is torsion-free and equivariantly stable, then $\mathcal{E}^{\vee\vee}$ is reflexive and equivariantly stable. On the other hand, if $\mathcal{E}^{\vee\vee}$ is stable, then clearly $\mathcal{E}$ is also stable. Indeed, for any coherent subsheaf $\mathcal{F} \subset \mathcal{E}^{\vee\vee}$, we have $\deg \mathcal{F} = \deg(\mathcal{F} \cap \mathcal{E})$.

So assume that $\mathcal{E}$ is reflexive and equivariantly stable. We will first show that $\mathcal{E}$ is semistable.

To prove semistability by contradiction, assume that $\mathcal{E}$ is not semistable. Then there is a unique maximal destabilizing semistable subsheaf

$$\mathcal{F} \subset \mathcal{E}. \tag{2.1}$$

In other words, $\mathcal{F}$ is the smallest nonzero subsheaf of $\mathcal{E}$ in the Harder–Narasimhan filtration of $\mathcal{E}$.

As before, $T \subset \operatorname{Aut}(X)$ is the torus acting on X. Let

$$\Phi : T \times \mathcal{E} \longrightarrow \mathcal{E}$$

be an action of the torus T on $\mathcal{E}$ lifting the action of T on X. For any element $t \in T$, the homomorphism

$$\mathcal{E} \longrightarrow \mathcal{E}, \quad v \longmapsto \Phi(t, v)$$

will be denoted by Φ_t. Note that Φ_t is an automorphism of $\mathcal{E}$ over the automorphism of X given by the action of t on it. For any $t \in T$, and any coherent subsheaf $\mathcal{V} \subset \mathcal{E}$, the coherent subsheaf $\Phi_t(\mathcal{V}) \subset \mathcal{E}$ will be denoted by $t \cdot \mathcal{V}$. The above automorphism Φ_t produces an isomorphism

$$(t^{-1})^*\mathcal{V} \xrightarrow{\sim} t \cdot \mathcal{V} \tag{2.2}$$

over the identity map of X.

Since $\mu(t^*\mathcal{V}) = \mu(\mathcal{V})$ for any coherent sheaf $\mathcal{V}$ on X, from (2.2) it follows that $\mu(t \cdot \mathcal{V}) = \mu(\mathcal{V})$ for every coherent subsheaf $\mathcal{V} \subset \mathcal{E}$. This and (2.2) together imply that the subsheaf $\mathcal{F}$ in (2.1) has the property that $t \cdot \mathcal{F}$ is also a maximal destabilizing subsheaf of $\mathcal{E}$. Therefore, from the uniqueness of the maximal destabilizing subsheaf it is deduced that

$$t \cdot \mathcal{F} = \mathcal{F} \subset \mathcal{E}.$$

Consequently, the action of T on $\mathcal{E}$ preserves the subsheaf $\mathcal{F}$. Therefore, $\mathcal{F}$ is equivariant. This contradicts the given condition that $\mathcal{E}$ is equivariantly semistable. Hence we conclude that $\mathcal{E}$ is semistable.

Let $\mathcal{H}$ be the socle of $\mathcal{E}$; in other words, $\mathcal{H}$ is the maximal polystable subsheaf of $\mathcal{E}$ with the same slope as $\mathcal{E}$ ([7, Proposition 3.1], [10, p. 23, Lemma 1.5.5]). From the uniqueness of $\mathcal{H}$, it follows that $t \cdot \mathcal{H} = \mathcal{H}$ for every $t \in T$. Therefore, $\mathcal{H}$ is a T-equivariant subsheaf of $\mathcal{E}$. Since $\mathcal{E}$ is equivariantly stable, and the slopes of $\mathcal{H}$ and $\mathcal{E}$ coincide, we must have $\mathcal{H} = \mathcal{E}$. This implies that $\mathcal{E}$ is polystable.

Since $\mathcal{E}$ is polystable, it suffices to show that $\mathcal{E}$ is indecomposable. Note that $\mathcal{E}$ is indecomposable if the dimension of a maximal torus in the algebraic group $\mathrm{Aut}(\mathcal{E})$ is one ([1, p. 201, Proposition 16]); the automorphism group $\mathrm{Aut}(\mathcal{E})$ is a Zariski open subset of the affine space $H^0(X, \mathrm{End}(\mathcal{E}))$.

The action of T on $\mathcal{E}$ produces an action of T on the group $\mathrm{Aut}(\mathcal{E})$:

$$(t \cdot A)(v) = t \cdot A(t^{-1} \cdot v), \quad \forall\ A \in \mathrm{Aut}(\mathcal{E}),\ v \in \mathcal{E},$$

for every $t \in T$. We will show that there is a maximal torus $\tilde{T} \subset \mathrm{Aut}(\mathcal{E})$ on which T acts trivially. For this, first consider the semi-direct product $\mathrm{Aut}(\mathcal{E}) \rtimes T$ for this action of T on $\mathrm{Aut}(\mathcal{E})$. Let $\tilde{T}' \subset \mathrm{Aut}(\mathcal{E}) \rtimes T$ be a maximal torus containing the subgroup T of $\mathrm{Aut}(\mathcal{E}) \rtimes T$. Then

$$\tilde{T} := \tilde{T}' \cap \mathrm{Aut}(\mathcal{E}) \subset \mathrm{Aut}(\mathcal{E})$$

is a maximal torus of $\mathrm{Aut}(\mathcal{E})$ on which T acts trivially. Now $\tilde{T}$ produces an eigenspace decomposition of $\mathcal{E}$ for the characters of $\tilde{T}$

$$\mathcal{E} = \bigoplus_{\chi \in \tilde{T}^*} \mathcal{E}_\chi\,;$$

any $t \in \tilde{T}$ acts on $\mathcal{E}_\chi$ as multiplication by $\chi(t)$. The direct summands in this decomposition are preserved by the action of T on $\mathcal{E}$, because T acts trivially on $\tilde{T}$ (see [6, p. 55, Proposition 1.2] for a general result). But $\mathcal{E}$ is equivariantly stable, so it does not admit any nontrivial T-equivariant decomposition. This implies that $\dim \tilde{T} = 1$, because that action of $\tilde{T}$ on $\mathcal{E}$ is faithful. As noted before, this implies that $\mathcal{E}$ is indecomposable. □

3. Equivariant coherent sheaves on X

We briefly review the classification of equivariant coherent sheaves on a nonsingular toric variety, following Perling [20]. The notation established in this section will be used extensively in the rest of the paper.

Let X be a nonsingular complex projective toric variety of dimension n, equipped with the action of an n-dimensional torus T. Let M and N denote the group of characters of T and the group of one-parameter subgroups of T respectively. Then both M and N are free $\mathbb{Z}$-modules of rank n that are naturally dual to one another. Let Δ denote the fan of X. It is a collection of rational cones in the real vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$ closed under the operations of taking faces, and performing intersections. Denote the set of d-dimensional cones of the fan Δ of X by $\Delta(d)$. For any one-dimensional cone (ray) $\alpha \in \Delta(1)$, its primitive co-character generator is also denoted by α; this is for notational convenience. We refer the reader to [9,19] for details on toric varieties.

Let $\mathcal{E}$ be a T-equivariant coherent sheaf over X of rank r. Let X_σ be any affine toric subvariety of X corresponding to a cone σ. Denote by S_σ the semigroup

$$\{m \in M \mid \langle m, \alpha \rangle \geq 0 \ \forall \alpha \in \sigma\} \subseteq M.$$

Let $k[S_\sigma]$ be the finitely generated semigroup algebra which is the coordinate ring of X_σ. Let E^σ denote the $k[S_\sigma]$-module $\Gamma(X_\sigma, \mathcal{E})$ consisting of sections of $\mathcal{E}$ over X_σ. Consider the isotypical decomposition

$$E^\sigma = \bigoplus_{m \in M} E^\sigma_m. \tag{3.1}$$

For any $m' \in M$ with $\chi(m') \in k[S_\sigma]$, there is a natural multiplication map

$$\chi(m') : E^\sigma_m \longrightarrow E^\sigma_{m+m'}. \tag{3.2}$$

This homomorphism is injective if $\mathcal{E}$ is torsion-free.

Let

$$S_{\sigma^\perp} = \{m \in M \mid \langle m, \alpha \rangle = 0 \ \forall \alpha \in \sigma\}. \tag{3.3}$$

Define

$$M_\sigma = M / S_{\sigma^\perp}. \tag{3.4}$$

Denote by $[m]$ the equivalence class of m in M_σ. Note that M_σ may be identified with the character group of an appropriate subtorus T_σ of T, namely the maximal subtorus of T that has a fixed point in X_σ. Let A_σ be the subvariety of X_σ defined by the ideal generated by the set $\{\chi(m) - 1 \mid m \in S_{\sigma^\perp}\}$. Then A_σ has a dense T_σ-orbit. So elements of M_σ generate the field of rational functions on A_σ.

If $m' \in S_{\sigma^\perp}$, then $\chi(m')$ in (3.2) is an isomorphism. Denote the isomorphism class of $E^\sigma_{m'}$, for $m' \in m + S_{\sigma^\perp}$, by $E^\sigma_{[m]}$. The space $E^\sigma_{[m]}$ may be identified with the space of sections of the T_σ-equivariant bundle $\mathcal{E}|_{A_\sigma}$ of weight $[m] \in M_\sigma$. We have, in fact, an isotypical decomposition

$$\Gamma(A_\sigma, \mathcal{E}) = \bigoplus_{[m]\in M_\sigma} E^\sigma_{[m]}. \tag{3.5}$$

Moreover, for $[m] \in M_\sigma$ and any $m' \in M$ such that $\chi(m') \in k[S_\sigma]$, the map $\chi(m')$ in (3.2) induces a map

$$\chi^\sigma([m']) : E^\sigma_{[m]} \longrightarrow E^\sigma_{[m+m']}. \tag{3.6}$$

Here, we may naturally identify $\chi^\sigma([m'])$ with a character of T_σ.

DEFINITION 3.1

Define an equivalence relation $\leq_\sigma$ on M by setting $m \leq_\sigma m'$ if and only if $m' - m \in S_\sigma$. This yields a directed pre-order on M which is a partial order when σ is of top dimension.

If $m \leq_\sigma m'$, but $m' \leq_\sigma m$ does not hold, we say that $m <_\sigma m'$.

DEFINITION 3.2

Let $\{E^\sigma_m \mid m \in M\}$ be a family of k-vector spaces. For each relation $m \leq_\sigma m'$, let there be given a k-linear map

$$\chi^\sigma(m, m') : E^\sigma_m \longrightarrow E^\sigma_{m'}$$

such that $\chi^\sigma(m, m) = 1$ for all $m \in M$, and also

$$\chi^\sigma(m, m'') = \chi^\sigma(m', m'') \circ \chi^\sigma(m, m')$$

for all triples $m \leq_\sigma m' \leq_\sigma m''$. We refer to the $\chi^\sigma(m, m'')$'s as multiplication maps. Denote such data by $\hat{E}^\sigma$ and call it a σ-family. A morphism $\phi^\sigma : \hat{E}^\sigma \longrightarrow \hat{E}'^\sigma$ of σ-families is given by a collection of linear maps $\{\phi^\sigma_m : E^\sigma_m \longrightarrow E'^\sigma_m \mid m \in M\}$ respecting the multiplication maps.

DEFINITION 3.3

A σ-family $\hat{E}^\sigma$ is called finite if

(1) all the k-vector spaces E^σ_m are finite dimensional,
(2) for each chain $\cdots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \cdots$ of elements of M, there exists an $i_0 \in \mathbb{Z}$ such that $E^\sigma_{m_i} = 0$ for all $i < i_0$, and
(3) there are only finitely many vector spaces E^σ_m such that the map

$$\bigoplus_{m' <_\sigma m} E^\sigma_{m'} \longrightarrow E^\sigma_m$$

defined by the summation of $\chi^\sigma(m', m)$'s is not surjective.

Theorem 3.4 [20]. *The category of T-equivariant coherent sheaves on X_σ is equivalent to the category of finite σ-families $\{\hat{E}^\sigma\}$.*

Let $\tau \leq \sigma$ be a subcone. Let $i_{\tau,\sigma} : X_\tau \longrightarrow X_\sigma$ be the corresponding inclusion map. Define

$$i^*_{\tau\sigma}(E^\sigma) = E^\sigma \otimes_{k[S_\sigma]} k[S_\tau].$$

Note that $i^*_{\tau\sigma}(E^\sigma)$ has a natural M-grading.

DEFINITION 3.5

Let Δ be a fan. A collection $\{\hat{E}^\sigma \mid \sigma \in \Delta\}$ of finite σ-families is called a finite Δ-family, denoted by $\hat{E}^\Delta$, if for every pair $\tau < \sigma$, there is an isomorphism $\eta_{\tau\sigma} : i^*_{\tau\sigma}\hat{E}^\sigma \longrightarrow \hat{E}^\tau$, such that for every triple $\rho < \tau < \sigma$, the following holds:

$$\eta_{\rho\sigma} = \eta_{\rho\tau} \circ i^*_{\rho\tau}\, \eta_{\tau\sigma}.$$

A morphism of finite Δ-families is a collection of morphisms

$$\{\phi^\sigma : \hat{E}^\sigma \longrightarrow \hat{E}'^\sigma \mid \sigma \in \Delta\}$$

of finite σ-families such that for all $\tau < \sigma$, the following diagram commutes:

$$\begin{array}{ccc} i^*_{\tau\sigma}(\hat{E}^\sigma) & \xrightarrow{i^*_{\tau\sigma}(\phi^\sigma)} & i^*_{\tau\sigma}(\hat{E}'^\sigma) \\ \eta_{\tau\sigma}\downarrow & & \eta'_{\tau\sigma}\downarrow \\ \hat{E}^\tau & \xrightarrow{\phi^\sigma} & \hat{E}'^\tau \end{array}.$$

Since $S_{\sigma^\perp} \subseteq S_{\tau^\perp}$, there is a surjective group homomorphism

$$M/S_{\sigma^\perp} \longrightarrow M/S_{\tau^\perp}.$$

Then $\eta_{\tau\sigma}$ induces an isomorphism $\eta_{\tau\sigma} : (i^*_{\tau\sigma}(E^\sigma_{[m]})) \longrightarrow E^\tau_{[m]}$.

Theorem 3.6 [20]. *The category of finite Δ-families is equivalent to the category of coherent T-equivariant sheaves over X.*

For a T-equivariant subsheaf $\mathcal{F}$ of $\mathcal{E}$, one has $F^\sigma_m \subseteq E^\sigma_m$ for every $\sigma \in \Delta$ and $m \in M$.

4. Rank of an equivariant torsion-free coherent sheaf

In this section, we derive a formula for the rank of an equivariant torsion-free coherent sheaf on X.

DEFINITION 4.1

For an equivariant torsion-free coherent sheaf $\mathcal{F}$ and an n-dimensional cone σ, define

$$Gen(\hat{F}^\sigma) = \{m' \in M \mid \dim F^\sigma_m < \dim F^\sigma_{m'} \ \forall\, m <_\sigma m'\}.$$

Since $\mathcal{F}$ is a coherent sheaf, it follows that $Gen(\hat{F}^\sigma)$ is finite for every σ. Note that the finite collection of graded vector spaces

$$\{F^\sigma_m \mid m \in Gen(\hat{F}^\sigma),\ \sigma \in \Delta(n)\},$$

and the isomorphisms $\eta_{\tau\sigma}$ of the previous section, together determine the Δ-family $\hat{F}^\Delta$.

A coherent sheaf is locally free on some open subset and its rank equals the rank of its restriction to such an open subset. By equivariance, the coherent sheaf $\mathcal{F}$ must be locally free on the dense torus orbit $X_{\{0\}}$, where $\{0\}$ denotes the trivial cone. By localizing to the dense torus orbit, we find that $\operatorname{rank} \mathcal{F} = \dim F^{\{0\}}_m$ for all $m \in M$. Now it is straight-forward to check that

$$\operatorname{rank} \mathcal{F} = \dim F^\sigma_m, \text{ where } m' <_\sigma m \text{ for all } m' \in Gen(\hat{F}^\sigma), \tag{4.1}$$

for any $\sigma \in \Delta(n)$.

Let α be any one dimensional cone. Note that the spaces F^α_m and $F^\alpha_{m'}$ are isomorphic if $m - m' \in S_{\alpha^\perp}$, or in other words, if $\langle \alpha, m\rangle = \langle \alpha, m'\rangle$.

DEFINITION 4.2

Let $\mathcal{F}$ be a T-equivariant torsion-free coherent sheaf on X. For a subcone $\alpha \in \Delta(1)$ and $\lambda \in \mathbb{Z}$, define

$$d(\mathcal{F}, \alpha, \lambda) = \dim F^\alpha_m, \quad \text{where } \lambda = \langle \alpha, m\rangle.$$

Define

$$e(\mathcal{F}, \alpha, \lambda) = d(\mathcal{F}, \alpha, \lambda) - d(\mathcal{F}, \alpha, \lambda - 1).$$

We remark that $d(\mathcal{F}, \alpha, \lambda) = \dim F^\alpha_{[m]}$, where $[m]$ denotes the equivalence class of m in M_α.

Lemma 4.3. *For an equivariant torsion-free coherent sheaf $\mathcal{F}$ on X, the equality*

$$\operatorname{rank}(\mathcal{F}) = \sum_{\lambda \in \mathbb{Z}} e(\mathcal{F}, \alpha, \lambda)$$

holds for all $\alpha \in \Delta(1)$.

Proof. Fix $\alpha \in \Delta(1)$. We may identify M_α with $\mathbb{Z}$ via the association $[m] \longmapsto \langle \alpha, m \rangle$. Note that F_j^α has a natural inclusion in F_{j+1}^α under the multiplication by the character $\chi^\alpha(1)$ of the torus T_α; see (3.6).

By the finiteness of $Gen(\hat{F}^\sigma)$ for all σ, the set

$$S := \{\lambda \in \mathbb{Z} \mid e(\mathcal{F}, \alpha, \lambda) \neq 0\}$$

is finite. Suppose $\lambda_1 < \cdots < \lambda_m$ are the elements of S. By (4.1), we have

$$\operatorname{rank} \mathcal{F} = \dim(F_j^\alpha), \quad \text{for } j \geq \lambda_m. \tag{4.2}$$

Recall the affine subvariety A_α of X_α defined by the ideal generated by

$$\{\chi(m) - 1 \mid m \in S_\alpha^\perp\}.$$

This is, in fact, a nonsingular affine curve. Therefore, as $\mathcal{F}$ is torsion-free, $\Gamma(A_\alpha, \mathcal{F})$ is a free $k[A_\alpha]$-module. Let $\{f_i \in F_{\lambda_i}^\alpha \mid 1 \leq i \leq m\}$ be any collection such that f_i is not in the image of $F_{\lambda_i - 1}^\alpha$ under the multiplication by $\chi^\alpha(1)$. To prove the lemma, in view of (4.2), it is enough to show that $f_1, \ldots, f_m$ are $k[A_\alpha]$-linearly independent.

Assume the contrary, namely, $f_1, \ldots, f_m$ are not $k[A_\alpha]$-linearly independent. Note that $k[A_\alpha]$ is generated as a k-vector space by $\{\chi^\alpha(p) \mid p \in \mathbb{Z}_{\geq 0}\}$. Therefore, we have

$$\sum_{i=1}^{m} c_i \, \chi^\alpha(d_i) \, f_i = 0 \tag{4.3}$$

for some nontrivial $c_i \in k$, and some nonnegative integers d_i. We may assume without loss of generality that $c_m \neq 0$. Moreover, by considering the direct sum decomposition (3.5), we may assume without loss of generality that each summand $\chi^\alpha(d_i) \, f_i$ belongs to the same graded component F_d^α, where $d \geq m$. Then, dividing (4.3) by $c_m \, \chi^\alpha(d - m)$, we obtain that f_m belongs to the image of $F_{\lambda_{m-1}}^\alpha$ which is a contradiction. This concludes the proof. □

5. Equivariant coherent subsheaves of TX

Let σ be an n-dimensional cone in Δ. Let $\alpha_1^\sigma, \ldots, \alpha_n^\sigma$ be the primitive integral generators of the one-dimensional faces of σ. Since X_σ is nonsingular, the vectors $\alpha_1^\sigma, \ldots, \alpha_n^\sigma$ form a basis of the $\mathbb{Z}$-module N. Let

$$\sigma^\vee = \{m \in M \otimes \mathbb{R} \mid \langle m, v \rangle \geq 0 \ \forall \, v \in \sigma\}$$

be the dual cone of σ. Define $m_i^\sigma \in \sigma^\vee \bigcap M$ by

$$\langle m_i^\sigma, \alpha_j^\sigma \rangle = \delta_{ij}, \tag{5.1}$$

where δ_{ij} denotes the Kronecker delta. Note that $m_1^\sigma, \ldots, m_n^\sigma$ form a $\mathbb{Z}$-basis of M.

Set $\mathcal{E} = TX$. Then E^σ is a free $\mathcal{O}_{X_\sigma}$-module of rank n, with generators having T-weights $-m_i^\sigma$, $1 \leq i \leq n$. To be precise, let

$$z_i = \chi(m_i^\sigma) \quad \text{and} \quad \partial_{z_i} = \frac{\partial}{\partial z_i} \quad \text{for } 1 \leq i \leq n.$$

Then E^σ is freely generated over $\mathcal{O}_{X_\sigma}$ by $\{\partial_{z_1}, \ldots, \partial_{z_n}\}$. In the convention followed by Perling [20], T acts on $\chi(m)$ with weight m. Hence, the section ∂_{z_i} has T-weight $-m_i^\sigma$. Note that $\dim E^\sigma_{-m_i^\sigma} = 1$ for $1 \le i \le n$. Consequently, the σ-family $\hat{E}^\sigma$ has the following properties:

$$\begin{array}{ll} \text{(a)} & \dim E^\sigma_m = |\{m_i^\sigma \mid -m_i^\sigma \le_\sigma m\}|. \\ \text{(b)} & \text{For every } m \le_\sigma m', \text{ the multiplication map } \chi^\sigma(m,m') \text{ is injective.} \end{array} \tag{5.2}$$

For a subcone $\tau < \sigma$, the τ–family $\hat{E}^\tau$ satisfies the following:

$$\begin{array}{ll} \text{(a)} & \dim E^\tau_m = |\{m_i^\sigma \mid -m_i^\sigma \le_\tau m\}|, \\ \text{(b)} & \text{For every } m \le_\tau m', \text{ the multiplication map } \chi^\tau(m,m') \text{ is injective.} \end{array} \tag{5.3}$$

By (5.2),

$$e(TX, \alpha, \lambda) = \begin{cases} 1 & \text{if } \lambda = -1, \\ n-1 & \text{if } \lambda = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

for every $\alpha \in \Delta(1)$.

Let $\mathcal{F}$ be any equivariant coherent subsheaf of TX. Then, $\mathcal{F}$ is torsion-free. Therefore, if $e(\mathcal{F}, \alpha, \lambda) \neq 0$, then $d(\mathcal{F}, \alpha, \lambda) \neq 0$, and consequently $E^\alpha_m \neq 0$, where $\langle \alpha, m\rangle = \lambda$ and $E^\alpha = \Gamma(TX_\alpha)$. As a result, we have

$$e(\mathcal{F}, \alpha, \lambda) \neq 0 \implies \lambda \ge -1. \tag{5.5}$$

Hence, for an equivariant coherent subsheaf $\mathcal{F}$ of TX, the rank formula of Lemma 4.3 takes the form

$$\operatorname{rank}(\mathcal{F}) = \sum_{\lambda \in \mathbb{Z}_{\ge -1}} e(\mathcal{F}, \alpha, \lambda), \tag{5.6}$$

where α is an arbitrary element of $\Delta(1)$.

Let D_α denote the torus invariant Weil divisor of X corresponding to $\alpha \in [\Delta(1)]$.

Theorem 5.1 [14, Corollary 1.2.18]. *The first Chern class of $\mathcal{F}$ has the expression*

$$c_1(\mathcal{F}) = -\sum_{\alpha \in \Delta(1), \lambda \in \mathbb{Z}} \lambda\, e(\mathcal{F}, \alpha, \lambda)\, D_\alpha.$$

Let $H = \sum a_\alpha D_\alpha$ be a polarization of X; in other words, H is an ample Cartier divisor on X. Let P be the polytope of X in $M \otimes \mathbb{R}$ associated to H. The convex polytope P basically encodes the linear system of H. It has a facet P_α corresponding to each α. The facet P_α lies in the hyperplane $\{v \in M \otimes \mathbb{R} \mid \langle v, \alpha\rangle = -a_\alpha\}$. Now P is easily determined from these supporting hyperplanes, and has the explicit formula

$$P = \{v \in M \otimes \mathbb{R} \mid \langle v, \alpha \rangle \ge -a_\alpha \ \forall \alpha \in [\Delta(1)]\}.$$

By [9, Corollary on p. 112], the intersection product

$$D_\alpha \cdot H^{n-1} = (n-1)!\,\mathrm{Vol}(P_\alpha), \tag{5.7}$$

where the volume is measured with respect to the lattice $S_{\alpha^\perp}$ (see (3.3)).

Using Theorem 5.1, (5.7), (5.4) and (5.5), we now have

$$\deg TX = (n-1)! \sum_{\alpha\in\Delta(1)} \mathrm{Vol}(P_\alpha) \tag{5.8}$$

and

$$\deg\mathcal{F} = -(n-1)! \sum_{\alpha\in\Delta(1),\lambda\in\mathbb{Z}_{\geq -1}} \lambda\, e(\mathcal{F},\alpha,\lambda)\mathrm{Vol}(P_\alpha). \tag{5.9}$$

Lemma 5.2. *An upper bound for the slope of an equivariant coherent subsheaf $\mathcal{F}$ of TX of rank r is given by*

$$\mu(\mathcal{F}) \leq \frac{(n-1)!}{r} \sum_{\alpha\in\Delta(1)} \mathrm{Vol}(P_\alpha).$$

Proof. Fix an $\alpha \in \Delta(1)$ and an n-dimensional cone σ containing α. Consider the summand

$$-(n-1)!\,\mathrm{Vol}(P_\alpha) \sum_{\lambda\in\mathbb{Z}_{\geq -1}} \lambda\, e(\mathcal{F},\alpha,\lambda)$$

of (5.9) corresponding to α. As the sum is decreasing in the λ's, the optimal choice would be $\lambda = -1$. However if $e(\mathcal{F},\alpha,-1) \neq 0$, then there is an element of the form $g(z)\partial z_\alpha$ in F^σ with $g(z) \in k[S_\sigma]$. Moreover all such elements are multiples of one another when restricted to the dense torus $X_{\{0\}}$. Therefore, we have $e(\mathcal{F},\alpha,-1) \leq 1$. Hence,

$$-(n-1)!\,\mathrm{Vol}(P_\alpha) \sum_{\lambda\in\mathbb{Z}_{\geq -1}} \lambda\, e(\mathcal{F},\alpha,\lambda) \leq (n-1)!\,\mathrm{Vol}(P_\alpha).$$

This completes the proof of the lemma. □

The above upper bound is not very sharp. We will use finer estimates in what follows.

Lemma 5.3. *Suppose $\mathcal{F}$, $\mathcal{G}$ are equivariant coherent subsheaves of TX having the same rank. If $\mathcal{F}$ is a proper subsheaf of $\mathcal{G}$, then $\mu(\mathcal{F}) < \mu(\mathcal{G})$.*

Proof. For subsheaves $\mathcal{F} \subseteq \mathcal{G}$ of TX, we have $F^\alpha_m \subseteq G^\alpha_m$. If, in addition, $\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{G})$, then Lemma 4.3 implies that generators of $\mathcal{F}$ are associated with bigger values of λ. Therefore, we have

$$\sum_{\alpha\in\Delta(1),\lambda\in\mathbb{Z}} \lambda\, e(\mathcal{G},\alpha,\lambda)\,\mathrm{Vol}(P_\alpha) \leq \sum_{\rho\in\Delta(1),\lambda\in\mathbb{Z}} \lambda\, e(\mathcal{F},\alpha,\lambda)\,\mathrm{Vol}(P_\alpha).$$

Thus, when $\mathcal{F} \subseteq \mathcal{G} \subseteq TX$, and $\operatorname{rank}(\mathcal{F}) = \operatorname{rank}(\mathcal{G})$, by (5.9), we have

$$\deg \mathcal{F} \leq \deg \mathcal{G}. \tag{5.10}$$

Now, suppose $\mathcal{F}$ is a proper subsheaf of $\mathcal{G}$. Then there exists an n-dimensional cone σ and an $m \in Gen(\hat{G}^\sigma)$ such that $F_m^\sigma \subsetneq G_m^\sigma$. Then there exist at least one $\alpha_i \in \sigma \cap \Delta(1)$ such that $e(\mathcal{F}, \alpha_i, \langle \alpha_i, m\rangle) < e(\mathcal{G}, \alpha_i, \langle \alpha_i, m\rangle)$. The lemma follows. □

6. Hirzebruch surfaces

In this section, we study semi-stability of tangent bundle for smooth projective toric surface. The following lemma is very crucial in computing degree of rank 1 subsheaves of the tangent bundle.

Lemma 6.1. *To any equivariant rank one coherent subsheaf $\mathcal{F}$ of TX on a complete nonsingular toric surface X, one can associate an integral vector*

$$\vec{\lambda} := (\lambda_1, \ldots, \lambda_p), \tag{6.1}$$

where

(1) $p = |\Delta(1)|$ *and* $\Delta(1) = \{\alpha_1, \ldots, \alpha_p\}$
(2) $e(\mathcal{F}, \alpha_i, \lambda_i) = 1$ *for each* i,
(3) *each* $\lambda_i \in \mathbb{Z}_{\geq -1}$, *and*
(4) $(\lambda_i, \lambda_j) \neq (-1, -1)$ *if* α_i, α_j *form a cone in* Δ.

Proof. Given a rank one subsheaf $\mathcal{F}$ of TX, for each $\alpha_i \in \Delta(1)$, by (5.6) and (5.5), there exists a unique $\lambda_i \in \mathbb{Z}_{\geq -1}$ such that $e(\mathcal{F}, \alpha_i, \lambda_i) = 1$.

Now, suppose α_i, α_j generate a cone σ. Denote the characters $\chi(m_i^\sigma)$, $\chi(m_j^\sigma)$ by z_i and z_j respectively. Then $\lambda_i = -1$ implies that $F_m^\sigma \neq 0$ for some $m = -m_i^\sigma + km_j^\sigma$, where $k \geq 0$. Therefore, F^σ has a generator of the form $z_j^k \partial_{z_i}$. Similarly, F^σ has a generator of the form $z_i^l \partial_{z_j}$ if $\lambda_j = -1$. Thus $\operatorname{rank}(\mathcal{F})$ would exceed one if $(\lambda_i, \lambda_j) = (-1, -1)$. □

However, not every vector of type (6.1) corresponds to an equivariant rank one coherent subsheaf of TX. To illustrate this, we consider the example of a Hirzebruch surface. The Hirzebruch surface $X = \mathbb{F}_m$ is a projective toric surface which may be obtained by the projectivization of the bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m)$ on $\mathbb{P}^1$. $\mathbb{F}_m$ has fan Δ with $\Delta(1) = \{\alpha_1, \ldots, \alpha_4\}$, where

$$[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & m & -1 \end{bmatrix}.$$

We denote the 2-dimensional cone generated by α_i and $\alpha_{i+1 (\text{mod } 4)}$ by σ_i.

Assume that $m \geq 1$. Consider a collection (6.1) such that $\lambda_1 = \lambda_3 = -1$. Denote

$$t_1 = \chi(m_1^{\sigma_1}) = \chi((1, 0)), \quad t_2 = \chi(m_2^{\sigma_1}) = \chi((0, 1)).$$

As $\lambda_1 = -1$, there is an element s_1 of the form $t_2^k \partial_{t_1}$ in F^{σ_1}. Note that $m_2^{\sigma_2} = (m, 1)$ and $m_3^{\sigma_2} = (-1, 0)$. (Here, it is convenient to continue to use the notation introduced in the proof of Lemma 6.1 as opposed to the last section.) Denote

$$z_2 = \chi(m_2^{\sigma_2}) = t_1^m t_2, \quad z_3 = \chi(m_3^{\sigma_2}) = t_1^{-1}.$$

This implies that

$$t_1 = z_3^{-1}, \quad t_2 = z_2 z_3^m.$$

Therefore, we have the Jacobian

$$\frac{\partial(t_1, t_2)}{\partial(z_2, z_3)} = \begin{bmatrix} 0 & -z_3^{-2} \\ z_3^m & m z_2 z_3^{m-1} \end{bmatrix} = \begin{bmatrix} 0 & -t_1^2 \\ t_1^{-m} & m t_1 t_2 \end{bmatrix}.$$

Thus we have

$$\partial_{z_2} = t_1^{-m} \partial_{t_2}, \quad \partial_{z_3} = -t_1^2 \partial_{t_1} + m t_1 t_2 \partial_{t_2}.$$

Again, as $\lambda_3 = -1$, there must be an element s_2 in F^{σ_2} of the form $z_2^l \partial_{z_3}$. The restrictions of s_1 and s_2 to the dense torus $X_{\{0\}}$ are independent. Therefore, $\mathcal{F}$ must have rank 2, which is a contradiction.

However, it can be verified that the vector

$$\vec{\lambda} = (0, -1, 0, -1) \tag{6.2}$$

corresponds to a rank one coherent subsheaf of TX on every $\mathbb{F}_m$. An analogous, but more general example is worked out in Theorem 8.1.

Theorem 6.2. *The tangent bundle of the Hirzebruch surface $\mathbb{F}_m$ is unstable with respect to every polarization if $m \geq 2$.*

Proof. Let D_i denote the divisor of $X = \mathbb{F}_m$ corresponding to α_i, $1 \leq i \leq 4$. The D_i's generate the Picard group of X. It is easy to check that $H = \sum a_i D_i$ is ample if and only if $a := a_1 + a_3 - m a_2 > 0$ and $b := a_2 + a_4 > 0$. The polytope P associated to H has vertices $A = (-a_1, -a_2)$, $B = (a_3 - m a_2, -a_2)$, $C = (m a_4 + a_3, a_4)$ and $D = (-a_1, a_4)$. The facets (edges) are $P_{\alpha_1} = AD$, $P_{\alpha_2} = AB$, $P_{\alpha_3} = BC$ and $P_{\alpha_4} = CD$. Their volumes are b, a, b and $a + mb$ respectively.

The slope of a rank one coherent subsheaf $\mathcal{F}$ associated to the collection (6.1) is

$$\mu(\mathcal{F}) = \deg(\mathcal{F}) = - \sum_{\alpha_i \in \Delta(1)} \lambda_i \mathrm{Vol}(P_{\alpha_i}). \tag{6.3}$$

It follows that the slope is a decreasing function of each λ_i. Since there is no rank one equivariant coherent subsheaves of TX with $\lambda_1 = -1 = \lambda_3$, the rank one equivariant coherent subsheaf with maximum slope corresponds to the collection (6.2). The slope of this subsheaf is

$$\mu(\mathcal{F}) \; = \; \mathrm{Vol}(P_{\alpha_2}) + \mathrm{Vol}(P_{\alpha_4}) \; = \; 2a + mb.$$

Note that the rank one subsheaves with $\vec{\lambda} = (-1, 0, 0, 0)$, $(0, -1, 0, 0)$, $(0, 0, -1, 0)$ and $(0, 0, 0, -1)$, if they exist, have slopes b, a, b and $a + mb$ respectively; and all of them are less than $2a + mb$.

On the other hand, by (5.8), the slope of TX is

$$\mu(TX) = \frac{1}{2} \sum_{i=1}^{4} \mathrm{Vol}(P_{\alpha_i}) = a + \frac{m+2}{2} b.$$

Since a and b are positive, the theorem follows. □

It is known that $T\mathbb{F}_0$ and $T\mathbb{F}_1$ are semistable with respect to the anti-canonical polarization, namely when each $a_i = 1$. We can deduce more about $\mathbb{F}_1$ from the above calculations.

COROLLARY 6.3

For a polarization with $2a < b$, *for example, when* $0 < 2a_1 < a_4$ *and* $a_2 = a_3 = 0$, *the tangent bundle* $T\mathbb{F}_1$ *is stable. On the other hand, if* $2a > b$, *for example, when* $0 < a_4 < 2a_1$ *and* $a_2 = a_3 = 0$, *then* $T\mathbb{F}_1$ *is unstable.*

7. Projective spaces

The method of the previous section can be adapted to give an alternative proof of the well-known result that the tangent bundle of the complex projective space $\mathbb{P}^n$ is stable with respect to the anti-canonical polarization (cf. [10, Section 1.4]).

Theorem 7.1. *$T\mathbb{P}^n$ is stable with respect to the anti-canonical polarization for every n.*

Proof. Let Δ be the fan of $\mathbb{P}^n$. Then $\Delta(1) = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$, where

$$\{\alpha_1 = (1, 0, \ldots, 0, 0), \ldots, \alpha_n = (0, 0, \ldots, 0, 1)\}$$

is the standard basis of $\mathbb{R}^n$ and

$$\alpha_{n+1} = -\sum_i \alpha_i = (-1, -1, \ldots, -1).$$

Δ has $n + 1$ cones of dimension n which may be enumerated as

$$\sigma_i = \langle \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{n+1} \rangle, \quad 1 \leq i \leq n + 1.$$

Let P denote the polytope of $\mathbb{P}^n$ with respect to the anti-canonical polarization. For any pair (i, j) there is an automorphism A_{ij} of the lattice N that interchanges α_i and α_j keeping the other α's fixed. This implies that every facet P_{α_i} of P has the same volume, say V_n. Hence, by (5.9), the slope of an equivariant rank r coherent subsheaf $\mathcal{F}$ of the tangent bundle is

$$\mu(\mathcal{F}) = -(n-1)!\,\frac{V_n}{r} \sum_{\alpha \in \Delta(1), \lambda \in \mathbb{Z}_{\geq -1}} \lambda\, e(\mathcal{F}, \alpha, \lambda). \tag{7.1}$$

We claim that, if $r < n$, then there can be at most r many α_i's with $e(\mathcal{F}, \alpha_i, -1) = 1$. This would imply that $\mu(\mathcal{F}) \leq (n-1)!\, V_n$. However, by (5.8),

$$\mu(T\mathbb{P}^n) = (n-1)! \frac{(n+1)V_n}{n} > (n-1)!\, V_n.$$

Thus $\mu(\mathcal{F}) < \mu(T\mathbb{P}^n)$ for every proper sub sheaf $\mathcal{F}$ of $T\mathbb{P}^n$.

To prove the claim, assume that there are at least $(r+1)$ many α_i's with $e(\mathcal{F}, \alpha_i, -1) = 1$. Since $r + 1 \leq n$, there exists an n-dimensional cone σ containing $(r+1)$ of these α_i's. Note that the corresponding $(r+1)$ many $\partial_{z_i^\sigma}$'s (up to multiplication by characters) are all generators of F^σ, contradicting the fact that $\operatorname{rank}(\mathcal{F}) = r$. $\square$

8. Unstable Fano examples in higher dimensions

If X is a Fano toric variety, then the polytope P corresponding to the anti-canonical polarization is a reflexive polytope (cf. [4]). This implies that P has integral vertices and the origin is the unique integral point which is in the interior of P. When X is a surface, it is easy to tabulate such polytopes. One easily checks that the tangent bundle of a nonsingular Fano toric surface is semistable with respect to the anti-canonical polarization. However, it is known that the tangent bundle of any nonsingular Fano surface is semistable with respect to the anti-canonical polarization (cf. [8]).

The product of two Fano varieties $X_1 \times X_2$ is Fano. If TX_1 and TX_2 are both semistable with respect to their respective anti-canonical polarizations, then the same holds for $T(X_1 \times X_2)$ (cf. [21]). This yields more examples of semistable Fano toric varieties. However, there are a lot of Fano toric varieties with unstable tangent bundle in higher dimensions, as the following result shows. For $n = 3$, the result below had appeared in [21].

Theorem 8.1. *Suppose X is the Fano toric variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(m))$, where $0 < m \leq n-1$ and $n \geq 3$. Then TX is unstable with respect to the anti-canonical polarization.*

Proof. Let Δ be the fan of X. Then $\Delta(1) = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}\}$, where $\{\alpha_1 = (1, 0, \ldots, 0, 0), \ldots, \alpha_n = (0, 0, \ldots, 0, 1)\}$ is the standard basis of $\mathbb{R}^n$ and

$$\alpha_{n+1} = -\alpha_n, \quad \alpha_{n+2} = -\sum_{j=1}^{n} \alpha_j + (m+1)\alpha_n = (-1, -1, \ldots, -1, m).$$

Δ has $2n$ cones of dimension n. They may be divided into two groups: upper and lower. The upper cones contain α_n but not α_{n+1}, and vice versa for the lower cones. There are n upper cones, each missing one of the remaining α_i's, and similarly for lower cones.

We verify that there is an equivariant subsheaf $\mathcal{F}$ of TX of rank 1 associated to the vector

$$\vec{\lambda} = (0, \ldots, 0, -1, -1),$$

i.e. $\lambda_n = \lambda_{n+1} = -1$, $\lambda_j = 0$ for other j (see Lemma 6.1). Note that for this $\vec{\lambda}$, in every n-dimensional cone σ, there is exactly one α_i, say α_{i_0}, such that $e(\mathcal{F}, \alpha_i, -1) \neq 0$. The number i_0 is n for upper cones, and $n+1$ for lower cones.

Consider the upper cone σ generated by $\alpha_1, \ldots, \alpha_n$. Denote $\chi(m_i^\sigma)$ by t_i for $1 \leq i \leq n$. Then $\lambda_n = -1$ implies that F^σ is generated by an element of the form $g(t_1, \ldots, t_{n-1})\partial_{t_n}$. Set $g(t_1, \ldots, t_{n-1}) = 1$.

Now consider any other upper cone τ. Assume τ is missing the ray α_j, where $j \leq n$. Let $z_i^\tau := \chi(m_i^\tau)$, where $1 \leq i \leq n+2$ and $i \neq j, n+1$. Then F^τ is generated by an element of the form $f(z)\partial_{z_n^\tau}$. We have

$$z_i^\tau = \begin{cases} t_i {t_j}^{-1} & \text{if } i \neq n, n+2, \\ t_j^m t_n & \text{if } i = n, \\ t_j^{-1} & \text{if } i = n+2. \end{cases}$$

It follows easily that $\partial_{t_n} = t_j^m \partial_{z_n^\tau} = (z_{n+2}^\tau)^{-m} \partial_{z_n^\tau}$. Note that $\lambda_n = -1$ means F^τ should be generated by an element of the form $h(z_1^\tau, \ldots, \hat{z_n^\tau}, \ldots, z_{n+2}^\tau)\partial_{z_n^\tau}$. Set $h = (z_{n+2}^\tau)^{-m}$.

The calculations for the lower cones are quite analogous. Finally, it is enough to consider the transition between the upper cone σ and the lower cone δ generated by $\alpha_1, \dots, \alpha_{n-1}, \alpha_{n+1}$. The coordinates on X_δ are $t_1, \dots, t_{n-1}$ and $w = \chi(m^\delta_{n+1}) = t_n^{-1}$. As $\lambda_{n+1} = -1$, we assume that F^δ is generated by ∂_w. Note that $\partial_w = -t_n^2\partial_{t_n}$ and t_n is invertible on $X_\sigma \cap X_\delta$. This confirms the existence of the desired rank one sheaf $\mathcal{F}$.

Consider the anti-canonical divisor H on X. The associated polytope is combinatorially equivalent to a prism $S^{n-1} \times I$, where S^{n-1} and I denote a simplex of dimension $n-1$ and an interval respectively. The top and bottom facets, $P_{\alpha_{n+1}}$ and P_{α_n}, are $(n-1)$-dimensional simplices and lie on the hyperplanes $x_n = 1$ and $x_n = -1$ respectively. The top facet $P_{\alpha_{n+1}}$ has the vertices

$$(-1, -1, \dots, -1, 1),\ (n + m - 1, -1, \dots, -1, 1),$$
$$(-1, n + m - 1, \dots, -1, 1), \dots, (-1, -1, \dots, n + m - 1, 1).$$

The bottom facet P_{α_n} has the vertices

$$(-1, -1, \dots, -1, -1),\ (n - m - 1, -1, \dots, -1, -1),$$
$$(-1, n - m - 1, \dots, -1, -1), \dots, (-1, -1, \dots, n - m - 1, -1).$$

We have

$$\mathrm{Vol}(P_{\alpha_{n+1}}) = \frac{(n+m)^{n-1}}{(n-1)!} \quad \text{and} \quad \mathrm{Vol}(P_{\alpha_n}) = \frac{(n-m)^{n-1}}{(n-1)!}.$$

Consequently, using (5.9), we have

$$\mu(\mathcal{F}) = (n+m)^{n-1} + (n-m)^{n-1}.$$

The prism P has n side facets corresponding to the rays $\alpha_1, \dots, \alpha_{n-1}$ and α_{n+2}. Each side facet, in turn, is a prism of height 2 bounded by a facet of $P_{\alpha_{n+1}}$ and P_{α_n} on top and bottom respectively. Each of these side facets have volume

$$\frac{(n+m)^{n-2} + (n-m)^{n-2}}{(n-2)!}.$$

Therefore, using (5.8), we have

$$\mu(TX) = \frac{(n+m)^{n-1} + (n-m)^{n-1} + n(n-1)((n+m)^{n-2} + (n-m)^{n-2})}{n}.$$

It is now easy to verify $\mu(\mathcal{F}) > \mu(TX)$ using $(a^{n-2} - b^{n-2})(a-b) > 0$, where $a = n+m$ and $b = n - m$. Here, we have used $n \geq 3$. The theorem follows. □

9. Fano toric 4-folds with small Picard number

Steffens has studied the stability of the tangent bundle of all Fano 3-folds in [21]. Moreover, Nakagawa [17,18] has identified all Fano toric 4-folds that admit Einstein–Kähler metrics.

We study which Fano toric 4-folds with Picard number ≤ 2 have semi-stable tangent bundle.

A list of Fano toric 4-folds is given by Batyrev [5], see also earlier work of Kleinschmidt [11]. There are 10 classes of these varieties with Picard number ≤ 2, which are

(1) $\mathbb{P}^4$,
(2) $B_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(3))$,
(3) $B_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2))$,
(4) $B_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1))$,
(5) $B_4 = \mathbb{P}^1 \times \mathbb{P}^3$,
(6) $B_5 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1))$,
(7) $C_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(2))$,
(8) $C_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^2 \oplus \mathcal{O}_{\mathbb{P}^2}(1))$,
(9) $C_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}^2(1))$,
(10) $C_4 = \mathbb{P}^2 \times \mathbb{P}^2$.

However, we will first need to fine-tune our tools to deal with subsheaves of higher rank. We will describe a generalization of Lemma 6.1 for rank r subsheaves.

Lemma 9.1. *To any equivariant rank r coherent subsheaf $\mathcal{F}$ of TX on a complete nonsingular toric variety X, one can associate a unique $r \times p$ matrix of integers $\Lambda_{r\times p} := (\lambda_{ij})$, where*

(1) $p = |\Delta(1)|$,
(2) $e(\mathcal{F}, \alpha_j, \lambda_{ij}) \neq 0$ *for each* $1 \leq j \leq p$,
(3) $-1 \leq \lambda_{1j} \leq \lambda_{2j} \leq \cdots \leq \lambda_{rj}$ *for every* j,
(4) $\sum e(\mathcal{F}, \alpha_j, \lambda_{ij}) = r$ *for every j, where the sum is over any maximal set of row indices i such that corresponding λ_{ij}'s in the column j are distinct,*
(5) $(\lambda_{ij_1}, \ldots, \lambda_{ij_{r+1}}) \neq (-1, \ldots, -1)$ *if* $\alpha_{j_1}, \ldots, \alpha_{j_{r+1}}$ *form a cone in* Δ, *and*
(6) $(\lambda_{1j}, \lambda_{2j}) \neq (-1, -1)$ *for any* j.

Proof. We are tabulating which $e(\mathcal{F}, \alpha_j, \lambda) \neq 0$ such that each entry λ_{ij} in the j-th column contributes 1 to $e(\mathcal{F}, \alpha_j, \lambda_{ij})$. This implies that if $e(\mathcal{F}, \alpha_j, \lambda) = k$, then k entries of the j-th column have entry λ.

Then the proof is almost immediate. Condition (3) is a choice of order made for the sake of uniqueness. Condition (4) follows from Theorem 4.3. Condition (5) ensures that rank of $\mathcal{F}$ does not exceed r. Condition (6) follows from the dependence of relevant generators over $k[S_{\{0\}}]$. It is equivalent to saying $e(\mathcal{F}, \alpha_j, -1) \leq 1$ for each j. □

We use the associated matrix Λ below to give a proof of the semi-stability of the toric Fano 4-fold $B_5 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ from the above list.

Theorem 9.2. *Suppose X is the Fano toric 4-fold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Then TX is strictly semistable with respect to the anti-canonical polarization.*

Proof. Let Δ be the fan of X. Then $\Delta(1) = \{\alpha_1 = (1, 0, 0, 0), \alpha_2 = (0, 1, 0, 0), \alpha_3 = (0, 0, 1, 0), \alpha_4 = (-1, -1, -1, 0), \alpha_5 = (0, 0, 0, 1), \alpha_6 = (1, 0, 0, -1)\}$ (see [5]).

We have $\mathrm{Vol}(P_{\alpha_1}) = 8$, $\mathrm{Vol}(P_{\alpha_2}) = \frac{56}{3}$, $\mathrm{Vol}(P_{\alpha_3}) = \frac{56}{3}$, $\mathrm{Vol}(P_{\alpha_4}) = \frac{56}{3}$, $\mathrm{Vol}(P_{\alpha_5}) = \frac{32}{3}$ and $\mathrm{Vol}(P_{\alpha_6}) = \frac{32}{3}$ with respect to anti-canonical divisor. Then

$$\mu(TX) = \frac{3!}{4}\sum_{i=1}^{6}\mathrm{Vol}(P_{\alpha_i}) = 128$$

We denote the 4-dimensional cones by σ_i and we have 8 of them. They are

$$\begin{aligned}
\sigma_1 &= \langle \alpha_2, \alpha_3, \alpha_4, \alpha_6\rangle\\
\sigma_2 &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_5\rangle\\
\sigma_3 &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_6\rangle\\
\sigma_4 &= \langle \alpha_1, \alpha_2, \alpha_4, \alpha_5\rangle\\
\sigma_5 &= \langle \alpha_1, \alpha_2, \alpha_4, \alpha_6\rangle\\
\sigma_6 &= \langle \alpha_1, \alpha_3, \alpha_4, \alpha_5\rangle\\
\sigma_7 &= \langle \alpha_1, \alpha_3, \alpha_4, \alpha_6\rangle\\
\sigma_8 &= \langle \alpha_2, \alpha_3, \alpha_4, \alpha_5\rangle
\end{aligned}$$

Denote

$$\begin{aligned}
&t_1 = \chi((1,0,0,0)), \quad t_2 = \chi((0,1,0,0)), \quad t_3 = \chi((0,0,1,0)),\\
&t_4 = \chi((0,0,0,1)).
\end{aligned}$$

First consider rank 1 equivariant subsheaves of TX. There is a possible such subsheaf $\mathcal{F}$ associated to the vector $\vec{\lambda} = (0, 0, 0, 0, -1, -1)$.

The other choices for $\vec{\lambda}$ that satisfy the conditions of Lemma 6.1 have at most one -1 entry. Then it is easy to check using Lemma 5.3 that $\mathcal{F}$ has the possible maximum slope among rank 1 subsheaves. The slope of $\mathcal{F}$ is

$$\mu(\mathcal{F}) = \frac{3!}{1}(\mathrm{Vol}(P_{\alpha_5}) + \mathrm{Vol}(P_{\alpha_6})) = 128.$$

As this equals $\mu(TX)$, we need to check the existence of $\mathcal{F}$.

First consider the 4-dimensional cones containing α_5, these are $\sigma_2, \sigma_4, \sigma_6, \sigma_8$. Define

$$x_i := \chi(m_i^{\sigma_2}), \quad y_i := \chi(m_i^{\sigma_4}), \quad z_i := \chi(m_i^{\sigma_6}) \quad \text{and} \quad w_i := \chi(m_i^{\sigma_8}),$$

where $1 \leq i \leq 4$.

Then

$$\begin{aligned}
&x_1 = t_1, && x_2 = t_2, && x_3 = t_3, && x_4 = t_4,\\
&y_1 = t_1 t_3^{-1}, && y_2 = t_2 t_3^{-1}, && y_3 = t_3^{-1}, && y_4 = t_4,\\
&z_1 = t_1 t_2^{-1}, && z_2 = t_2^{-1} t_3, && z_3 = t_2^{-1}, && z_4 = t_4,\\
&w_1 = t_1^{-1} t_2, && w_2 = t_1^{-1} t_3, && w_3 = t_1^{-1}, && w_4 = t_4.
\end{aligned} \tag{9.1}$$

As $\lambda_5 = -1$, there is an element generated by (meaning, a multiple of) ∂_{x_4} in F^{σ_2}. Similarly, there are elements generated by ∂_{y_4}, ∂_{z_4} and ∂_{w_4} in F^{σ_4}, F^{σ_6} and F^{σ_8}, respectively.

But, using the Jacobians of the transformations between the t and other coordinates, we have

$$\partial_{x_4} = \partial_{y_4} = \partial_{z_4} = \partial_{w_4} = \partial_{t_4}.$$

Thus the generators agree on the dense torus.

Secondly, consider cones containing α_6 as a generator, which are $\sigma_1, \sigma_2, \sigma_5, \sigma_7$. Call now $p_i := \chi(m_i^{\sigma_1})$, $q_i := \chi(m_i^{\sigma_3})$, $r_i := \chi(m_i^{\sigma_5})$ and $s_i := \chi(m_i^{\sigma_7})$, where $1 \le i \le 4$. Similar to the above computations, we find that

$$\begin{array}{llll} p_1 = t_1^{-1}t_2t_4^{-1}, & p_2 = t_1^{-1}t_3t_4^{-1}, & p_3 = t_1^{-1}t_4^{-1}, & p_4 = t_4^{-1}, \\ q_1 = t_1t_4, & q_2 = t_2, & q_3 = t_3, & q_4 = t_4^{-1}, \\ r_1 = t_1t_3^{-1}t_4, & r_2 = t_2t_3^{-1}, & r_3 = t_3^{-1}, & r_4 = t_4^{-1}, \\ s_1 = t_1t_2^{-1}t_4, & s_2 = t_2^{-1}t_3, & s_3 = t_2^{-1}, & s_4 = t_4^{-1}. \end{array} \tag{9.2}$$

As $\lambda_6 = -1$, there are elements generated by $\partial_{p_4}, \partial_{q_4}, \partial_{r_4}$ and ∂_{s_4} in F^{σ_1}, F^{σ_3}, F^{σ_5} and F^{σ_7} respectively. Using various Jacobians, we have

$$\partial_{p_4} = \partial_{q_4} = \partial_{r_4} = \partial_{s_4} = t_1t_4\partial_{t_1} - t_4^2\partial_{t_4}.$$

However, on the dense torus, the generators ∂_{t_4} and $t_1t_4\partial_{t_1} - t_4^2\partial_{t_4}$ are linearly independent. Hence the rank of $\mathcal{F}$ must be at least 2, leading to a contradiction. We conclude that $\mu(\mathcal{F}) < 128 = \mu(TX)$ for every rank 1 equivariant subsheaf of TX.

Next we consider rank 2 equivariant subsheaves of TX with maximum possible slope. By condition (5) of Lemma 9.1, a subsheaf $\mathcal{F}$ with $\lambda_{1j} = -1$ for 3 values of j is only possible when 2 of the j's are 5 and 6. Then, using condition (6) of Lemma 9.1, the slope of $\mathcal{F}$ has the following bound:

$$\mu(\mathcal{F}) \le \frac{3!}{2}(\mathrm{Vol}(P_{\alpha_5}) + \mathrm{Vol}(P_{\alpha_6}) + \max_{1\le j\le 4}\mathrm{Vol}(P_{\alpha_j})) = 120.$$

Thus there is no destabilizing subsheaf of rank 2.

There is a rank 3 equivariant subsheaf $\mathcal{F}$ of TX with associated matrix $\Lambda_{3\times 6} = (\lambda_{ij})$ such that $\lambda_{1j} = -1$ for $1 \le j \le 4$, and all other $\lambda_{ij} = 0$. For every 4-dimensional cone σ_l, F^{σ_l} is generated over $k[S_{\sigma_l}]$ by $\partial_{\chi(m_i^{\sigma_l})}$, $1 \le i \le 3$. The generators for different cones σ_l and σ_m are related by Jacobians of the corresponding transition maps which are naturally regular on $X_{\sigma_l \cap \sigma_m}$. Moreover, as $\chi(m_4^{\sigma_l})$ is either t_4 or t_4^{-1}, it follows easily that $\partial_{\chi(m_i^{\sigma_l})}$, $1 \le i \le 3$, is always a combination of $\partial_{t_1}, \partial_{t_2}$ and ∂_{t_3}. Thus the subsheaf $\mathcal{F}$ indeed exists. It has the maximum slope among rank 3 subsheaves, and the slope is

$$\mu(\mathcal{F}) = \frac{3!}{3}(\mathrm{Vol}(P_{\alpha_1}) + \mathrm{Vol}(P_{\alpha_2}) + \mathrm{Vol}(P_{\alpha_3}) + \mathrm{Vol}(P_{\alpha_4})) = 128 = \mu(TX).$$

This concludes the proof. □

Here is the classification of all nonsingular Fano toric 4-folds with Picard number ≤ 2 according to stability with respect to the anticanonical polarization.

Theorem 9.3. *Suppose X is one of the Fano toric 4-folds with Picard number ≤ 2. Then*

(1) *$T\mathbb{P}^4$ is stable,*
(2) *TB_4 and TC_4 are polystable,*
(3) *TB_5 is strictly semistable,*
(4) *TB_1, TB_2 and TB_3 are unstable,*
(5) *TC_1, TC_2 and TC_3 are unstable.*

Proof.

(1) This is well-known. See Theorem 7.1 for an alternative proof.
(2) B_4 and C_4 admit Einstein–Kähler metrics (see [18, Theorem 3.4]). Hence, these are polystable.
(3) See Theorem 9.2.
(4) These are special cases of Theorem 8.1.
(5) TC_1 and TC_2 are destabilized by rank 2 subsheaves, and TC_3 is destabilized by a rank 1 subsheaf. These may be verified by following a similar approach as in the proof of Theorem 9.2.

□

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