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Chow group of 1-cycles of the moduli of parabolic bundles over a curve

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Abstract. We study the Chow group of 1-cycles of the moduli space of semistable parabolic vector bundles of fixed rank, determinant and a generic weight over a nonsingular projective curve over $\mathbb{C}$ of genus at least 3. We show that, the Chow group of 1-cycles remains isomorphic as we vary the generic weight. As a consequence, we can give an explicit description of the Chow group in the case of rank 2 and determinant $\mathcal{O}(x)$, where $x \in X$ is a fixed point, which extends the earlier result of Choe and Hwang (*Math. Z.* **253** (2006) 253–281, Main theorem).

Keywords. Chow groups; moduli space; parabolic bundle.

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1. Introduction

Let X be a nonsingular projective curve of genus $g \geq 3$ over $\mathbb{C}$. Let $\mathcal{M}(r, \mathcal{L})$ denote the moduli space of isomorphism classes of stable vector bundles of rank r and fixed determinant $\mathcal{L}$ over X. Let us moreover fix a set of n distinct closed points S over, referred to as *parabolic points*, and let $\mathcal{M}(r, \mathcal{L}, \alpha)$ denote the moduli space of isomorphism classes of parabolic stable vector bundles of rank r, fixed determinant $\mathcal{L}$ of the underlying bundle, *full* flags along the parabolic points, and generic weight α over X (cf. Section 2.4 for the definition). The Chow groups of these moduli spaces are interesting objects to study. Not much is known about the explicit desctiption of these Chow groups (see, e.g. the Introduction in [3] for a list of known results in case of moduli of vector bundles). Here our aim is to study the Chow group of 1-cycles for the moduli space $\mathcal{M}(r, \mathcal{L}, \alpha)$. We outline the content of the paper and our strategy of proof below:

In Section 2, we briefly recall the notions necessary for our discussions, like (parabolic) semistability and stability of (parabolic) vector bundles, their moduli spaces, Chow groups and so on. In Section 3, we begin our study of the Chow group of 1-cycles for the moduli of Parabolic bundles over X. The extra data of parabolic structure makes it necessary to study the effect of varying the weights. Below, we denote $\mathcal{M}(r, \mathcal{L}, \alpha)$ by just $\mathcal{M}_\alpha$, since the r and $\mathcal{L}$ will remain fixed. The main result of Section 3 is the following.

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Theorem 1.1 **(Theorem 3.12).** *For any two generic weights α and β, there exists a canonical isomorphism*

$$\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\beta).$$

Our strategy for proving this is to first prove the result for generic weights α and β which are separated by a single hyperplane (called *walls*, cf. Section 2) inside the set of all possible weights (denoted by V_m, cf. Section 2), and choosing the weight γ which is the point of intersection of the hyperplane and the line joining α and β. We note that γ is not a generic weight, and hence $\mathcal{M}_\gamma$ is only a normal projective variety. To relate the Chow groups for $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$, our approach is to use [2, Theorem 4.1] (equivalently [1, Theorem 3.1]), which says that there exist maps

$$\mathcal{M}_\alpha \xrightarrow{\ \phi_\alpha\ } \mathcal{M}_\gamma \xleftarrow{\ \phi_\beta\ } \mathcal{M}_\beta$$

which act as resolution of singularities for $\mathcal{M}_\gamma$. Next, we consider the fibre product $\mathcal{N} := \mathcal{M}_\alpha \underset{\mathcal{M}_\gamma}{\times} \mathcal{M}_\beta$, which is actually a common blow-up of $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ along suitable subvarieties (cf. discussion in the Introduction in [1] above Section 2). We use the Blow-up formula for Chow groups as in [8, Theorem 9.27] to first relate both $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha)$ and $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\beta)$ with $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N})$, and then through a series of manipulations, we finally get our required isomorphism. To conclude for all generic weights is straightforward from here.

As a consequence, we can give an explicit description of the Chow group when rank is 2 and determinant is $\mathcal{O}_X(x)$ for some closed point $x \in X$. We do this in Section 4, where we prove the following result:

Theorem 1.2 **(Theorem 4.6).** *In case of rank* 2 *and determinant $\mathcal{O}_X(x)$, for any generic weight α, we have*

$$\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \mathrm{CH}_0^{\mathbb{Q}}(X), \quad \textit{where } n = |S|.$$

The main idea here is to show that for a sufficiently small generic weight α (as in Proposition 4.2), $\mathcal{M}_\alpha$ has the structure of a $(\mathbb{P}^1)^n$-bundle over $\mathcal{M}$; we show this in Lemma 4.3. This, together with projective bundle formula for Chow groups as in [8, Theorem 9.25], enable us to explicitly write down $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha)$ in terms of $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M})$, and the latter is known to be isomorphic to $\mathrm{CH}_0^{\mathbb{Q}}(X)$ by [3, Main Theorem]. The case for arbitrary generic weights then follows from Theorem 1.1.

We make a remark that the condition $g \geq 2$ is taken to ensure that the moduli of Parabolic stable bundles is nonempty (cf. [1] for a discussion of an example where the moduli space of parabolic stable bundles is empty for suitable choice of weights when $g \leq 1$).

2. Preliminaries

2.1 *Semistability and stability of vector bundles*

Let X be a nonsingular projective curve over $\mathbb{C}$. Let E be a holomorphic vector bundle of rank r over X.

Here onwards, by a *variety* we will always mean an irreducible quasi-projective variety.

DEFINITION 2.1 (Semistability and stability)

The degree of E, denoted $\deg(E)$, is defined as the degree of the line bundle $\det(E) := \wedge^r E$. The slope of E, denoted $\mu(E)$, is defined as

$$\mu(E) := \frac{\deg(E)}{r}.$$

A bundle E is called semistable (resp. stable), if for any sub-bundle $F \subset E$, $0 < \operatorname{rank}(F) < r$, we have

$$\mu(F) \underset{(\text{resp.} <)}{\leq} \mu(E).$$

It is easy to check that if $gcd(r, \deg(E)) = 1$, then the notion of semistability and stability coincide for a vector bundle E.

2.2 *Moduli space of vector bundles*

We briefly recall the notion of the moduli space of vector bundles over X. If E is a semistable bundle of rank r, then there exists a Jordan–Hölder filtration for E given by

$$E = E_k \supsetneq E_{k-1} \supsetneq \cdots \supsetneq E_1 \supsetneq 0.$$

The filtration is not unique, but the associated graded object $\operatorname{gr}(E) := \bigoplus_{i=1}^k E_i/E_{i-1}$ is unique upto isomorphism. Two vector bundles E and E' are called S-equivalent if $\operatorname{gr}(E) \cong \operatorname{gr}(E')$. When E, E' are stable, being S-equivalent is same as being isomorphic as vector bundles over X.

The moduli space of S-equivalence classes of vector bundles of rank r and determinant $\mathcal{L}$ on X, denoted $\mathcal{M}(r, \mathcal{L})$, is a normal projective variety of dimension $(r^2 - 1)(g - 1)$; its singular locus is given by the strictly semistable bundles (bundles which are semistable but not stable).

In the case when $gcd(r, \deg(\mathcal{L})) = 1$, $\mathcal{M}(r, \mathcal{L})$ is the isomorphism class of stable vector bundles on X. It is a nonsingular projective variety; moreover, it is a fine moduli space.

When r, $\mathcal{L}$ are fixed, we shall denote the moduli space by $\mathcal{M}$, when there is no scope for confusion.

2.3 *Parabolic bundles and stability*

DEFINITION 2.2 (Parabolic bundles, parabolic data)

Let us fix a set S of n distinct closed points on X. A *parabolic vector bundle of rank r on X* is a holomorphic vector bundle E on X with a parabolic structure along points of S. By this, we mean a collection of weighted flags of the fibres of E over each point $p \in S$:

$$E_p = E_{p,1} \supsetneq E_{p,2} \supsetneq \cdots \supsetneq E_{p,s_p} \supsetneq E_{p,s_{p+1}} = 0, \tag{2.1}$$
$$0 \leq \alpha_{p,1} < \alpha_{p,2} < \cdots < \alpha_{p,s_p} < 1, \tag{2.2}$$

where s_p is an integer between 1 and r. The real number $\alpha_{p,i}$ is called the weight attached to the subspace $E_{p,i}$. The multiplicity of the weight $\alpha_{p,i}$ is the integer $m_{p,i} := \dim(E_{p,i}) - \dim(E_{p,i-1})$. Thus $\sum_i m_{p,i} = r$. We call the flag to be *full* if $s_p = r$, or equivalently $m_{p,i} = 1$ for all i.

Let $\alpha := \{(\alpha_{p,1}, \alpha_{p,2}, \ldots, \alpha_{p,s_p}) \mid p \in S\}$ and $m := \{(m_{p,1}, \ldots, m_{p,s_p}) \mid p \in S\}$. We call the tuple $(r, \mathcal{L}, m, \alpha)$ as the parabolic data for the parabolic bundle E, where $\mathcal{L} := \det(E)$. Usually we denote the parabolic bundle as E_* to distinguish from the underlying vector bundle E.

DEFINITION 2.3 (Parabolic degree and slope)

The degree of a parabolic bundle E_* is defined as $\deg(E)$, E being the underlying vector bundle. The *Parabolic degree* of E_*, denoted $\mathrm{Pardeg}(E_*)$, is defined as

$$\mathrm{Pardeg}(E_*) := \deg(E) + \sum_{p\in S}\sum_{i=1}^{s_p} m_{p,i}\alpha_{p,i}.$$

The parabolic slope of E_* is defined as

$$\mathrm{Par}\,\mu(E_*) := \frac{\mathrm{Pardeg}(E_*)}{\mathrm{rank}(E)}.$$

DEFINITION 2.4 (Parabolic semistability and stability).

Any vector sub-bundle $F \hookrightarrow E$ obtains a parabolic structure in a canonical way: For each $p \in S$, the flag at F_p is obtained intersecting F_p with the flag at E_p, and the weight attached to the subspace $F_{p,j}$ is α_k, where k is the largest integer such that $F_{p,j} \subseteq E_{p,k}$ (for more details, see [7, Definition 1.7].) We call the resulting parabolic bundle to be a parabolic sub-bundle, and denote it by F_*.

A parabolic bundle E_* is called *parabolic semistable (resp. parabolic stable)*, if for every proper sub-bundle $F \subset E$ we have

$$\mathrm{Par}\,\mu(F_*) \underset{(\mathrm{resp.} <)}{\leq} \mathrm{Par}\,\mu(E_*).$$

2.4 *Generic weights and walls*

We briefly recall the notion of generic weights and walls. For more details, we refer to [1,2].

Fix a set S, rank r, line bundle $\mathcal{L}$ on X and multiplicities m as defined above. Let $\Delta^r := \{(a_1 \ldots a_r) \mid 0 \leq a_1 \leq \cdots \leq a_r < 1\}$, and define $W := \{\alpha : S \to \Delta^r\}$. Note that the elements of W determine both weights and the multiplicities at the parabolic points, and hence a parabolic data. Conversely, given any parabolic data $(r, \mathcal{L}, m, \alpha)$, we can associate a map $S \to \Delta^r$, by repeating each weight $\alpha_{p,i}$ according to its multiplicty $m_{p,i}$. This leads to a natural notion of when a given weight α is compatible with the multiplicity m. The set of all weights compatible with m is a product of $|S|$-many simplices. We denote by V_m the set of all weights compatible with m.

Suppose V_l, V_m are the set of weights which are compatible with the flag structure corresponding to the multiplicities l, m respectively. We will say $V_l > V_m$ if the flag structure corresponding to l is a refinement of the flag structure corresponding to m.

Let $\alpha \in V_m$. If a parabolic bundle E_* with data $(r, \mathcal{L}, m, \alpha)$ is parabolic semistable but not parabolic stable, then it would contain a parabolic sub-bundle with same parabolic slope. It is easy to see that this gives a linear condition on V_m, i.e. such weights belong to the intersection of a hyperplane with V_m.

There can be only finitely many such hyperplanes (see [1,2]); call them $H_1, \ldots, H_l$.

DEFINITION 2.5 (Walls and generic weights)

We call each of the intersections $H_i \cap V_m$ a wall in V_m. There are only finitely many such walls.

We call the connected components of $V_m \backslash \cup_{1 \leq i \leq l} H_i$ as chambers, and weights belonging to these chambers are called generic.

Clearly, for weights in $V_m \backslash \cup_{1 \leq i \leq l} H_i$, a parabolic bundle is parabolic semistable iff it is parabolic stable.

2.5 *Moduli of parabolic bundles*

Again, we briefly recall the notion of moduli space of parabolic semistable bundles over X. The construction is analogous to Section 2.2; for details, we refer to [7].

For a parabolic semistable bundle E_* with fixed parabolic data $(r, \mathcal{L}, m, \alpha)$, there exists a Jordan-Holder filtration, and an associated graded object $gr_\alpha(E_*)$ analogous to Section 2.2. Again, we call two parabolic semistable bundles to be S-equivalent if their associated graded objects are isomorphic. Let $\mathcal{M}(r, \mathcal{L}, m, \alpha)$ denote the moduli space of S-equivalence classes of parabolic semistable bundles over X with parabolic data $(r, \mathcal{L}, m, \alpha)$. It is a normal projective variety, with singular locus given by the strictly semistable bundles. When $r, \mathcal{L}, m$ are fixed, we will denote the moduli space by $\mathcal{M}_\alpha$ if no confusion occurs.

For generic weight α, $\mathcal{M}_\alpha =$ moduli space of isomorphism classes of parabolic stable bundles on X, is a nonsingular projective variety; moreover, it is a fine moduli space by [2, Proposition 3.2].

2.6 *Chow groups*

For a variety Y over $\mathbb{C}$, let $Z_k(Y)$ denote the free abelian group generated by the irreducible k-dimensional closed subvarieties of Y. The Chow group of k-cycles, denoted $\mathrm{CH}_k(Y)$, is given by

$$\mathrm{CH}_k(Y) := \frac{Z_k(Y)}{\sim},$$

where $\sim$ denotes 'rational equivalence'. We refer to [8, Section 9] and [4] for the details regarding Chow groups and the related notions (proper pushforward and flat pullback of cycles, intersection product, Chern class of vector bundles etc.)

Let $\mathrm{CH}_k^{\mathbb{Q}}(Y) := \mathrm{CH}_k(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$; this is a $\mathbb{Q}$-vector space. By a slight abuse of notation, throughout the rest of the discussion, we will address $\mathrm{CH}_k^{\mathbb{Q}}(Y)$ as 'Chow group' as well, since no confusion will arise.

We recall a few results from [4] which we will require in Section 3.

Theorem 2.6 [4, Theoerm 3.3]. *Let E be a vector bundle of rank r on Y, with projection $\pi : E \to Y$. The flat pull-back*

$$\pi^* : \mathrm{CH}_{k-r}(Y) \to \mathrm{CH}_k(E)$$

is an isomorphism for all k.

DEFINITION 2.7 [4, Definition 3.3]

Let s denote the zero section of the bundle E above. Hence $\pi \circ s = \mathrm{Id}_Y$. Then there exist *Gysin homomorphisms*:

$$s^* : \mathrm{CH}_k(E) \to \mathrm{CH}_{k-r}(Y),$$
$$s^*(W) := (\pi^*)^{-1}(W),$$

where $r = \mathrm{rank}(E)$.

Lemma 2.8 [4, *Example* 3.3.2]. *If s is the zero section of a vector bundle E of rank r on Y, then*

$$s^* s_*(Z) = c_r(E) \cap Z \quad \textit{for all } Z \in \mathrm{CH}_*(Y),$$

where $c_r(E)$ denotes the r-th Chern class of E (cf. [4, Section 3.2]).

Let us also recall the concept of excess normal bundle, as described in [4, §6.1]. Let Y be a nonsingular variety, and $X \overset{i}{\hookrightarrow} Y$ be a nonsingular closed subvariety of codimension d. Let $\tilde{Y}$ be the blow-up of Y along X, and $\tilde{X}$ be its exceptional divisor. We have a fibre square:

$$\begin{array}{ccc} \tilde{X} & \overset{j}{\hookrightarrow} & \tilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \overset{i}{\hookrightarrow} & Y \end{array} .$$

Since the ideal sheaf of $\tilde{X}$ in $\tilde{Y}$ is generated by the ideal sheaf of X in Y, we have an inclusion of bundles $N_{\tilde{X}/\tilde{Y}} \hookrightarrow g^*(N_{X/Y})$, where $N_{X/Y}$ (resp. $N_{\tilde{X}/\tilde{Y}}$) denotes the normal bundle of X (resp. normal bundle of $\tilde{X}$). Moreover, since $\tilde{X}$ is a nonsingular hypersurface in $\tilde{Y}$, $N_{\tilde{X}/\tilde{Y}} \simeq \mathcal{O}(\tilde{X})|_{\tilde{X}}$, and in case of blow-ups along smooth locus, $\tilde{X} = \mathbb{P}(N_{X/Y})$, and $\mathcal{O}(\tilde{X})|_{\tilde{X}} \simeq \mathcal{O}_{\mathbb{P}(N_{X/Y})}(-1)$.

DEFINITION 2.9

The quotient bundle $g^*(N_{X/Y})/N_{\tilde{X}/\tilde{Y}}$ is called the excess normal bundle.

We have the following excess intersection formula:

PROPOSITION 2.10 [4, Proposition 6.7(a)]

Let Y be a nonsingular variety, and $X \overset{i}{\hookrightarrow} Y$ be a nonsingular closed subvariety of codimension d, with normal bundle N. Let $\tilde{Y}$ be the blow-up of Y along X, and $\tilde{X}$ be its exceptional divisor. Let $E := g^*(N)/\mathcal{O}_N(-1)$ be the excess normal bundle. Then for all $Z \in \mathrm{CH}_k(X)$,

$$f^* i_*(Z) = j_*(c_{d-1}(E) \cap g^*(Z)).$$

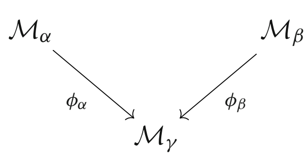
3. Relation between Chow groups of 1-cycles of moduli of parabolic bundles for arbitrary generic weights

Fix a set S of parabolic points, rank r and determinant $\mathcal{L}$. We assume that we are working with full flags, i.e. $m_{p,i} = 1$ for all p, i. Consider V_m, the set of weights compatible with m, as in Section 2.4. Recall that V_m is cut out by finitely many walls. Moreover, as the flags are full, V_m contains a generic weight by [2, Proposition 3.2].

Let $\alpha, \beta \in V_m$ be two generic weights in adjacent chambers separated by a single wall. Let γ be the weight which is the point of intersection of the wall and the line joining α and β. Then $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are nonsingular projective varieties, while $\mathcal{M}_\gamma$ is normal projective variety, with the singular locus $\Sigma_\gamma \subset \mathcal{M}_\gamma$ given by the class of strictly semistable bundles. Note that since γ lies on only one hyperplane in W, Σ_γ is nonsingular ([1, Section 3.1]).

Let us recall the following theorem:

Theorem 3.1 [2, Theorem 3.1]. *There are integers n_α, n_β and canonical projective morphisms*



so that

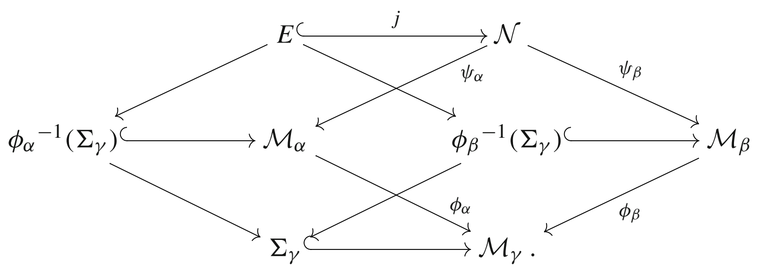
(*a*) *ϕ_α and ϕ_β are isomorphisms along $\mathcal{M}_\gamma \setminus \Sigma_\gamma$,*
(*b*) *along Σ_γ, ϕ_α and ϕ_β are $\mathbb{P}^{n_\alpha}$ and $\mathbb{P}^{n_\beta}$-bundles respectively, and*
(*c*) codim $\Sigma_\gamma = 1 + n_\alpha + n_\beta$.

Since Σ_γ is nonsingular and ${\phi_\alpha}^{-1}(\Sigma_\gamma), {\phi_\beta}^{-1}(\Sigma_\gamma)$ are projective bundles, they are nonsingular closed subvarieties of $\mathcal{M}_\alpha, \mathcal{M}_\beta$ respectively.

Let $\mathcal{N} := \mathcal{M}_\alpha \underset{\mathcal{M}_\gamma}{\times} \mathcal{M}_\beta$. Let ψ_α and ψ_β denote the natural maps from $\mathcal{N}$ to $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ respectively. Then according to the discussion in the end of Section 1 in [1], $\mathcal{N}$ is the common blow-up with exceptional divisor a $(\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta})$-bundle over Σ_γ.

Call the exceptional divisor E, with $j : E \hookrightarrow \mathcal{N}$ the inclusion.

We have the following diagram:



Here E is a $\mathbb{P}^{n_\beta}$-bundle over ${\phi_\beta}^{-1}(\Sigma_\gamma)$ via $\psi_\alpha\,|_E$, and a $\mathbb{P}^{n_\alpha}$-bundle over ${\phi_\beta}^{-1}(\Sigma_\gamma)$ via $\psi_\beta\,|_E$.

Remark 3.2. From the diagram above, we note that $E \cong {\phi_\alpha}^{-1}(\Sigma_\gamma) \underset{\Sigma_\gamma}{\times} {\phi_\beta}^{-1}(\Sigma_\gamma)$, since

$$\begin{aligned} E = \mathcal{N} \underset{\mathcal{M}_\beta}{\times} {\phi_\beta}^{-1}(\Sigma_\gamma) &= (\mathcal{M}_\alpha \underset{\mathcal{M}_\gamma}{\times} \mathcal{M}_\beta) \underset{\mathcal{M}_\beta}{\times} {\phi_\beta}^{-1}(\Sigma_\gamma) \\ &\cong \mathcal{M}_\alpha \underset{\mathcal{M}_\gamma}{\times} {\phi_\beta}^{-1}(\Sigma_\gamma) \\ &\cong {\phi_\alpha}^{-1}(\Sigma_\gamma) \underset{\Sigma_\gamma}{\times} {\phi_\beta}^{-1}(\Sigma_\gamma). \quad [\because {\phi_\beta}^{-1}(\Sigma_\gamma) \text{ maps to } \Sigma_\gamma] \end{aligned}$$

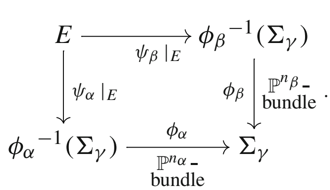
Lemma 3.3. *The varieties* ${\phi_\alpha}^{-1}(\Sigma_\gamma)$ *and* ${\phi_\beta}^{-1}(\Sigma_\gamma)$ *are rational* (*i.e. birational to* $\mathbb{P}^n$ *for some n*); *hence* $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(\Sigma_\gamma)) \cong \mathbb{Q} \cong \mathrm{CH}_0^{\mathbb{Q}}({\phi_\beta}^{-1}(\Sigma_\gamma))$.

Proof. By equation (5) in [1], Σ_γ is the product of two smaller dimensional moduli, which are rational (by [2, Theorem 6.1]), so Σ_γ is itself rational.

Since ${\phi_\alpha}^{-1}(\Sigma_\gamma)$ and ${\phi_\beta}^{-1}(\Sigma_\gamma)$ are projective bundles over Σ_γ, they are also rational. This proves the first assertion.

Moreover, by [4, Example 16.1.11], the Chow groups of 0-cycles is a birational invariant; and $\mathrm{CH}_0(\mathbb{P}^n) \cong \mathbb{Z}\,\forall\, n$, so we get the second assertion as well. □

Recall the fibre diagram from Remark 3.2:



Therefore, if we choose a point $p \in \Sigma_\gamma$, then by base changing to $\{p\}$, the diagram above transforms to

$$\begin{array}{ccc} \mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} & \xrightarrow{p_2} & \mathbb{P}^{n_\beta} \cong {\phi_\beta}^{-1}(p) \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow \\ \mathbb{P}^{n_\alpha} \cong {\phi_\alpha}^{-1}(p) & \longrightarrow & \{p\} \end{array},$$

where p_1, p_2 denote the first and second projections respectively.

Let ${\phi_\alpha}^{-1}(p) \cong \mathbb{P}^{n_\alpha} \overset{\iota_\alpha}{\hookrightarrow} {\phi_\alpha}^{-1}(\Sigma_\gamma)$, ${\psi_\alpha}^{-1}({\phi_\alpha}^{-1}(p)) \cong \mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} \overset{\tilde{\iota}}{\hookrightarrow} E$ denote the inclusions. We have the fibre diagram

$$\begin{array}{ccccc} \mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} & \xrightarrow{\cong} & {\psi_\alpha}^{-1}({\phi_\alpha}^{-1}(p)) & \overset{\tilde{\iota}}{\hookrightarrow} & E \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \psi_\alpha|_E} \\ \mathbb{P}^{n_\alpha} & \xrightarrow{\cong} & {\phi_\alpha}^{-1}(p) & \overset{\iota_\alpha}{\hookrightarrow} & {\phi_\alpha}^{-1}(\Sigma_\gamma) \end{array}.$$

Choose a point $x \in \mathbb{P}^{n_\alpha} \cong {\phi_\alpha}^{-1}(p)$. In the following, under slight abuse of notation, we will think of the element $[x] \in \mathrm{CH}_0^{\mathbb{Q}}(\mathbb{P}^{n_\alpha})$ as an element of $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(p))$, and we will think of the element $[\{x\} \times \mathbb{P}^{n_\beta}] \in \mathrm{CH}_{n_\beta}^{\mathbb{Q}}(\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta})$ as an element of $\mathrm{CH}_{n_\beta}^{\mathbb{Q}}({\psi_\alpha}^{-1}({\phi_\alpha}^{-1}(p)))$.

Lemma 3.4. *We have* $(\psi_\alpha|_E)^*((\iota_\alpha)_*[x]) = \tilde{\iota}_*[\{x\} \times \mathbb{P}^{n_\beta}]$.

Proof. This follows from [4, Proposition 1.7], since $\psi_\alpha|_E$ is flat, being a projective bundle map, and ι_α is proper, being a closed immersion. □

Now, since $\mathcal{N}$ is the blow-up over $\mathcal{M}_\alpha$ along ${\phi_\alpha}^{-1}(\Sigma_\gamma)$, hence by [8, Theorem 9.27] there is an isomorphism of Chow groups:

$$\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(\Sigma_\gamma)) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \underset{\sim}{\xrightarrow{g_\alpha}} \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N}) \tag{3.1}$$

$$\text{given by } (W_0 \quad W_1) \longmapsto j_*(c_1(h_\alpha)^{n_\beta - 1} \cap (\psi_\alpha|_E)^*(W_0)) + {\psi_\alpha}^*(W_1), \tag{3.2}$$

where $h_\alpha := \mathcal{O}_E(1)$ (E thought of as a $\mathbb{P}^{n_\beta}$-bundle over ${\phi_\alpha}^{-1}(\Sigma_\gamma)$), and $\cap$ denotes the intersection product.

Similarly, there exists an isomorphism defined similarly to g_α above:

$$\mathrm{CH}_0^{\mathbb{Q}}({\phi_\beta}^{-1}(\Sigma_\gamma)) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\beta) \underset{\sim}{\xrightarrow{g_\beta}} \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N}) \tag{3.3}$$

$$\text{given by } (Z_0 \quad Z_1) \longmapsto j_*(c_1(h_\beta)^{n_\alpha - 1} \cap (\psi_\beta|_E)^*(Z_0)) + {\psi_\beta}^*(Z_1) \tag{3.4}$$

where $h_\beta := \mathcal{O}_E(1)$ (E thought of as a $\mathbb{P}^{n_\alpha}$-bundle over ${\phi_\beta}^{-1}(\Sigma_\gamma)$), and $\cap$ denotes the intersection product.

We make a few useful remarks regarding g_α and g_β.

Remark 3.5. We note that the second projection with respect to g_α, namely $p_2 \circ g_\alpha^{-1} : CH_1^{\mathbb{Q}}(\mathcal{N}) \twoheadrightarrow \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha)$ is given by $(\psi_\alpha)_*$, since by [8, Corollary 9.15], $\psi_{\alpha *} \circ {\psi_\alpha}^* = \mathrm{Id}_{\mathcal{M}_\alpha}$, and $\psi_{\alpha *}$ sends the terms coming from $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(\Sigma_\gamma))$ to 0, which follows from the definition of pushforward of cycles, since the image of such terms under ψ_α has strictly smaller dimension than the source.

Thus for all $Z \in \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N})$, $Z = (Z - ({\psi_\alpha}^* \circ \psi_{\alpha *})(Z)) + {\psi_\alpha}^*(\psi_{\alpha *}(Z))$, and by description of g_α in (3.4), we get that $Z - ({\psi_\alpha}^* \circ \psi_{\alpha *})(Z)$ is the first projection with respect to g_α, i.e.

$$(p_1 \circ g_\alpha^{-1})(Z) = Z - ({\psi_\alpha}^* \circ \psi_{\alpha *})(Z).$$

Of course, analogous statement is true for g_β as well.

Remark 3.6. Again, identifying $\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}$ and ${\psi_\alpha}^{-1}({\phi_\alpha}^{-1}(p))$, it is easy to see that the pull-back bundle $\tilde{\imath}^*(h_\alpha) \cong \mathcal{O}_{\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}}(1)$.

Remark 3.7. By Lemma 3.3, we have $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(\Sigma_\gamma)) \cong \mathbb{Q}$, hence the class $(\imath_\alpha)_*([x])$ will be a $\mathbb{Q}$-basis.

Lemma 3.8. *We have* $g_\alpha((\imath_\alpha)_*([x])) = j_*(\tilde{\imath}_*[\{\mathrm{x}\} \times l])$, *where l is a line in* $\mathbb{P}^{n_\beta}$.

Proof. By (3.2),

$$\begin{aligned} g_\alpha((\imath_\alpha)_*([x])) &= j_*(c_1(h_\alpha)^{n_\beta - 1} \cap (\psi_\alpha|_E)^*((\imath_\alpha)_*([x]))) \\ &= j_*(c_1(h_\alpha)^{n_\beta - 1} \cap (\tilde{\imath}_*[\{x\} \times \mathbb{P}^{n_\beta}])) \qquad (\text{Lemma } 3.4) \end{aligned} \tag{3.5}$$

By Remark 3.6 and projection formula applied to $\tilde{\imath}$ (cf. [4, Proposition 2.5]),

$$\begin{aligned} c_1(h_\alpha)^{n_\beta - 1} \cap (\tilde{\imath}_*[\{x\} \times \mathbb{P}^{n_\beta}]) &= \tilde{\imath}_*(c_1(\tilde{\imath}^*(h_\alpha))^{n_\beta - 1} \cap [\{x\} \times \mathbb{P}^{n_\beta}]) \\ &= \tilde{\imath}_*(c_1(\mathcal{O}_{\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}}(1))^{n_\beta - 1} \cap [\{x\} \times \mathbb{P}^{n_\beta}]). \end{aligned} \tag{3.6}$$

But $\mathcal{O}_{\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}}(1)|_{\{x\} \times \mathbb{P}^{n_\beta}} = \mathcal{O}_{\{x\} \times \mathbb{P}^{n_\beta}}(1)$, which corresponds to the divisor of a hyperplane section H (say), and so by definition of intersection product,

$$c_1(\mathcal{O}_{\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}}(1)) \cap [\{x\} \times \mathbb{P}^{n_\beta}] = [\{x\} \times H].$$

Repeating this $n_\beta - 1$ times, we get

$$c_1(\mathcal{O}_{\mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta}}(1))^{n_\beta - 1} \cap [\{x\} \times \mathbb{P}^{n_\beta}] = [\{x\} \times l],$$

where l is a line in $\mathbb{P}^{n_\beta}$. Hence from (3.5) and (3.6) we finally get $g_\alpha((\imath_\alpha)_*([x])) = j_*(\tilde{\imath}_*[\{x\} \times l])$, as claimed. □

PROPOSITION 3.9

Let $Z := g_\alpha(\imath_{\alpha *}([x])) \in \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N})$, *then* $Z \neq ({\psi_\beta}^* \circ {\psi_\beta}_*)(Z)$.

Proof. Let $j_\beta : {\phi_\beta}^{-1}(\Sigma_\gamma) \hookrightarrow \mathcal{M}_\beta$ be the inclusion, so that we have the following blow-up diagram:

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & \mathcal{N} \\ {\scriptstyle \mathbb{P}^{n_\alpha}\text{-bundle}}\ \downarrow {\scriptstyle \psi_\beta|_E} & & \downarrow {\scriptstyle \psi_\beta} \\ {\phi_\beta}^{-1}(\Sigma_\gamma) & \overset{j_\beta}{\hookrightarrow} & \mathcal{M}_\beta \end{array} .$$

Let $E = \mathbb{P}(N)$, where N denotes the normal bundle of the embedding j_β. Let $\mathcal{Q} := \dfrac{(\psi_\beta|_E)^*(N)}{\mathcal{O}_E(-1)}$ be the excess normal bundle of rank n_α as defined in Definition 2.9, with $\pi : \mathcal{Q} \to E$ being the projection.

Let $W := \tilde{\imath}_*[\{x\} \times l]$. Then, according to Lemma 3.8, we want to show that

$$({\psi_\beta}^* \circ {\psi_\beta}_*)(j_* W) \neq j_* W.$$

On the contrary, suppose they are equal. We have

$$\begin{aligned} \text{LHS} &= ({\psi_\beta}^* \circ \underbrace{{\psi_\beta}_*)(j_*}\, W) \\ &= \underbrace{{\psi_\beta}^*(j_{\beta *}}(\psi_\beta|_E)_* W) \quad [\because \psi_\beta \circ j = j_\beta \circ (\psi_\beta|_E)] \\ &= j_*(c_{n_\alpha}(\mathcal{Q}) \cap (\psi_\beta|_E)^*(\psi_\beta|_E)_*(W)), \quad \text{(Proposition 2.10)}. \end{aligned}$$

So, we would get $j_*(c_{n_\alpha}(\mathcal{Q}) \cap (\psi_\beta|_E)^*(\psi_\beta|_E)_*(W)) = j_* W$. Thus, denoting $Z := c_{n_\alpha}(\mathcal{Q}) \cap (\psi_\beta|_E)^*(\psi_\beta|_E)_*(W) - W$, we have $j_*(Z) = 0$; moreover, $(\psi_\beta|_E)_*(Z) = 0$ (cf. [4, Example 3.3.3]). This implies that $Z = 0$ by [4, Proposition 6.7(c)], i.e.,

$$W = c_{n_\alpha}(\mathcal{Q}) \cap (\psi_\beta|_E)^*(\psi_\beta|_E)_*(W). \tag{3.7}$$

Moreover, by Lemma 2.8,

$$c_{n_\alpha}(\mathcal{Q}) \cap (\psi_\beta|_E)^*(\psi_\beta|_E)_*(W) = s^* s_*((\psi_\beta|_E)^*(\psi_\beta|_E)_*(W)),$$

where $s : E \to \mathcal{Q}$ denotes the zero section of the bundle map $\mathcal{Q} \xrightarrow{\pi} E$, and s^* is defined as in Definition 2.7.

Therefore, from (3.7), we would finally get

$$W = s^* s_*((\psi_\beta|_E)^*(\psi_\beta|_E)_*(W)). \tag{3.8}$$

Let us write down the following square:

$$\begin{array}{ccccc} \mathbb{P}^{n_\alpha} \times \mathbb{P}^{n_\beta} & \overset{\cong}{\longrightarrow} & {\psi_\beta}^{-1}({\phi_\beta}^{-1}(p)) & \overset{\tilde{\imath}}{\hookrightarrow} & E \\ \downarrow {\scriptstyle p_2} & & \downarrow & & {\scriptstyle \text{flat}}\ \downarrow {\scriptstyle \psi_\beta|_E} \\ \mathbb{P}^{n_\beta} & \overset{\cong}{\longrightarrow} & {\phi_\beta}^{-1}(p) & \underset{\text{proper}}{\overset{\imath}{\hookrightarrow}} & {\phi_\beta}^{-1}(\Sigma_\gamma) \end{array} .$$

Recalling $W := \tilde{\imath}_* [\{x\} \times l]$, we get from the diagram above:

$$\begin{aligned}(\psi_\beta|_E)^*(\psi_\beta|_E)_*(W) &= (\psi_\beta|_E)^* \underbrace{(\psi_\beta|_E)_*(\tilde{\imath}_*}[\{x\} \times l]) \\ &= \underbrace{(\psi_\beta|_E)^*((\imath_*} \circ p_{2*})[\{x\} \times l]) \\ &= (\tilde{\imath}_* \circ p_2^*)(p_{2*}[\{x\} \times l]) \\ &= (\tilde{\imath}_* \circ p_2^*)([l]) \\ &= \tilde{\imath}_*([\mathbb{P}^{n_\alpha} \times l]). \end{aligned} \tag{3.9}$$

Therefore, (3.8) becomes

$$W = s^* \circ s_*(\tilde{\imath}_*[\mathbb{P}^{n_\alpha} \times l]) \tag{3.10}$$

$$\implies \pi^*(W) = s_*(\tilde{\imath}_*([\mathbb{P}^{n_\alpha} \times l])) \qquad [\because s^* = (\pi^*)^{-1}]. \tag{3.11}$$

But applying π_* to both sides, we see that $\pi_*\pi^*(W) = 0$, since by definition of push-forward, π_* takes a cycle to zero if the dimension of its image under π is strictly smaller than the dimension of the source, and clearly $\dim(\pi(\pi^{-1}(W)) < \dim W$.

On the other hand, since $\pi \circ s = \mathrm{Id}_E$,

$$\pi_* \circ s_*(\tilde{\imath}_*[\mathbb{P}^{n_\alpha} \times l]) = \tilde{\imath}_*([\mathbb{P}^{n_\alpha} \times l]).$$

Thus we would get $\tilde{\imath}_*([\mathbb{P}^{n_\alpha} \times l]) = 0$. But from (3.10) we would get $W = 0$, which would give, by Lemma 3.8, that $g_\alpha(\imath_{\alpha*}([x])) = 0$, which is a contradiction since g_α is an isomorphism. Hence the claim is proved. □

3.1 *Proof of the main theorem*

Before stating the main theorem, we make the following remark:

Remark 3.10. Let V, W be two $\mathbb{Q}$-vector spaces with an isomorphism $\varphi : \mathbb{Q}\langle e\rangle \oplus V \xrightarrow{\sim} \mathbb{Q}\langle f\rangle \oplus W$. Consider the composite map

$$\begin{aligned}\mathbb{Q}\langle e\rangle \hookrightarrow \mathbb{Q}\langle e\rangle \oplus V &\xrightarrow{\varphi} \mathbb{Q}\langle f\rangle \oplus W \\ e &\mapsto (\psi(e), \phi(e)),\end{aligned}$$

and suppose $\psi(e) \neq 0$, i.e. the composition below is nonzero:

$$\mathbb{Q}\langle e\rangle \hookrightarrow \mathbb{Q}\langle e\rangle \oplus V \xrightarrow{\varphi} \mathbb{Q}\langle f\rangle \oplus W \overset{p_1}{\twoheadrightarrow} \mathbb{Q}\langle f\rangle.$$

Then clearly $\varphi|_V$ induces an isomorphism $V \cong W$, since the composition has to be an isomorphism, as it is a nonzero map between two 1-dimensional spaces, and hence going modulo $\mathbb{Q}\langle e\rangle$ and $\mathbb{Q}\langle f\rangle$ on both sides of φ, we get our claim.

PROPOSITION 3.11

For generic weights α, β in adjacent chambers, the map $g_\beta^{-1} \circ g_\alpha$, when restricted to $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha)$, induces isomorphism

$$\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \xrightarrow[\sim]{g_\beta^{-1} \circ g_\alpha} \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\beta).$$

Proof. Using Lemma 3.3, let us write $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(\Sigma_\gamma)) = \mathbb{Q}\langle e\rangle$ and $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\beta}^{-1}(\Sigma_\gamma)) = \mathbb{Q}\langle f\rangle$, where e, f are some basis elements. Recall the maps g_α, g_β from (3.1) and (3.3).

Consider the composition

$$\mathbb{Q}\langle e\rangle \hookrightarrow \mathbb{Q}\langle e\rangle \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \xrightarrow[\sim]{g_\beta^{-1} \circ g_\alpha} \mathbb{Q}\langle f\rangle \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\beta) \overset{p_1}{\twoheadrightarrow} \mathbb{Q}\langle f\rangle, \tag{3.12}$$

where p_1 is the first projection.

According to the remark above, we will be done if we can show that the composition in (3.12) is nonzero.

Consider the first projection $p_1 \circ g_\beta^{-1} : \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N}) \twoheadrightarrow \mathbb{Q}\langle f\rangle$ with respect to g_β. By Remark 3.5, we know its description, namely

$$(p_1 \circ g_\beta^{-1})(Z) = Z - ({\psi_\beta}^* \circ {\psi_\beta}_*)(Z) \ \forall Z \in \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{N}).$$

Thus taking $Z = g_\alpha(\iota_*([x]))$, by Proposition 3.9, we get that

$$(p_1 \circ g_\beta^{-1})(Z) \neq 0.$$

Since $\iota_{\alpha *}([x])$ is a basis for $\mathrm{CH}_0^{\mathbb{Q}}({\phi_\alpha}^{-1}(\Sigma_\gamma)) = \mathbb{Q}\langle e\rangle$ by Remark 3.7, it is clear that the composite map in (3.12) is nonzero. Hence we are done. □

Theorem 3.12. *For any two generic weights α and β, there exists a canonical isomorphism*

$$\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\beta).$$

Proof. By [1, Lemma 2.7 and Remark 2.9], the moduli spaces corresponding to weights in the same chamber are isomorphic. Moreover, we can order the finitely many chambers in such a way that any two consecutive chambers are separated by a single wall. Combining this fact with Proposition 3.11, we get our claim. □

4. Consequence: Description of the Chow group for moduli of parabolic bundles of rank 2 and determinant $\mathcal{O}_X(x)$

From now on, we consider the case when the rank is 2 and determinant is $\mathcal{O}_X(x)$ for some closed point $x \in X$, and full flags. In rank 2 case, having full flags amounts to giving a 1-dimensional subspace of each fibre over the parabolic points.

Let us recall the following result due to Choe and Hwang [3]

Theorem 4.1 [3, Main Theorem]. *Under the above hypotheses, there is a canonical isomorphism* $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}) \cong \mathrm{CH}_0^{\mathbb{Q}}(X)$.

4.1. *When the generic weight is small enough.* We recall the following proposition from [2].

PROPOSITION 4.2 [2, Proposition 5.2]

Suppose E be a vector bundle of rank r and degree d on X. Define the following quantities:

$$\epsilon_{\pm}(d,r) = \inf\left\{\pm\left(\frac{d}{r}-\frac{d'}{r'}\right) \mid d', r' \in \mathbb{Z}, 1 \le r' < r, \quad and \quad \pm\left(\frac{d}{r}-\frac{d'}{r'}\right) > 0\right\}$$
$$\epsilon(d,r) = \min\{\epsilon_{\pm}(d,k) \mid k = 1,\ldots,r\}.$$

Furthermore, suppose $\sum_{p\in S}\sum_{1=i}^{s_p} m_{p,i}\alpha_{p,i} < \epsilon(d,r)/2$.

(*i*) *If E is stable as a vector bundle, then E_* is parabolic stable.*

(*ii*) *If E_* is parabolic stable, then E is a semistable vector bundle.*

We call a weight α *small* if it satisfies the condition of the proposition above.

Lemma 4.3. *We can choose a sufficiently small generic weight α (made precise in the proof), so that there exists a canonical morphism $\mathcal{M}_\alpha \to \mathcal{M}$ making $\mathcal{M}_\alpha$ into a $(\mathbb{P}^1)^n$-bundle over $\mathcal{M}$, where $n = |S|$.*

Proof. Since α is generic, E_* is in fact parabolic stable, so by proposition 4.2 (ii), E is semistable, hence stable as well, since we are in coprime case. Hence there is a map

$$g : \mathcal{M}_\alpha \to \mathcal{M}$$

by forgetting the parabolic structure.

Actually, the map g is a composition of of maps of the form $\mathcal{M}_\alpha \to \mathcal{M}_{\alpha'}$, where $\mathcal{M}_{\alpha'}$ is the moduli space of stable parabolic bundles whose parabolic structure is obtained by forgetting the data for one of the parabolic points.

To ensure that the resulting parabolic bundle with the modified parabolic structure still remains parabolic stable, we can do the following: from Section 2.4, the set of all possible weights W as defined in Section 2.4 is a product of simplices, and the 'origin' $\{(0,0)_{p\in S}\} \in W$ corresponds to the trivial parabolic data, and clearly for the trivial parabolic structure, the parabolic stable bundles are simply the usual stable vector bundles. Since we are in the case that degree and rank is coprime, the origin is also a generic weight, i.e. no wall can pass through the origin. Now, if we visualize W in the Euclidean space and give it a metric, then since there are only finitely many walls, we can choose a constant c very small so that $c << \epsilon(2,1)$ (cf. Proposition 4.2) and moreover we can choose a weight α satisfying $\sum_{p\in S}(\alpha_{p,1}+\alpha_{p,2}) < c$ and lying very close to the 'origin' of W, avoiding all the walls. Then forgetting the parabolic structure at one parabolic point will still ensure that the modified weight α' is still very close to origin and don't lie on any walls (i.e. α' is also generic weight), and moreover there are no walls separating α and α'. Hence the resulting parabolic bundles with modified parabolic structure will remain parabolic stable.If the weights are chosen carefully as per the above discussion, the map $\mathcal{M}_\alpha \to \mathcal{M}_{\alpha'}$ will be a $\mathbb{P}^1$-bundle by [2, Theorem 4.2], which proves the following.

Theorem 4.4 [2, Theorem 4.2]. *Suppose that* $\alpha \in V_l$, $\beta \in V_m$, $V_l > V_m$ (*cf. Section* 2.4), *and that* α *and* β *are generic and not separated by any walls. Then there exists a fibration* $\psi : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ *with fiber a* (*possibly twisted*) *product of flag varieties*, *and this fibration is zariski locally trivial.*

The proof of the above theorem in the case of rank 2 amounts to the following: for simplicity, first let $n = 1$, i.e. only one parabolic point. Recall that $\mathcal{M}$ is a fine moduli space, since $\deg(\mathcal{L}) = 1$ ([2, Proposition 3.2]). Consider the *universal* (*or Poincaré* bundle over $X \times \mathcal{M}$, whose fibre over each $(p, [E])$ is given by E_p. Restrict the bundle over $\{p\} \times \mathcal{M}$. Call the resulting bundle $\mathcal{E}$. For each $[E] \in \mathcal{M}$, the fibre of $\mathbb{P}(\mathcal{E})$ over $(p, [E])$ is $\mathbb{P}(E_p)$, i.e. lines in E_p. Hence the fibre of $\mathbb{P}(\mathcal{E})$ over $(p, [E])$ gives the set of all possible full flags at E_p. Moreover, by Proposition 4.1(i), the parabolic bundle E_* resulting from the weight α and parabolic point p will be automatically parabolic stable. In other words, for each $[E] \in \mathcal{M}$, each point in the fibre of $\mathbb{P}(\mathcal{E})$ over $(p, [E])$ corresponds to a unique point $[E_*] \in \mathcal{M}_\alpha$. This way we identify $\mathcal{M}_\alpha$ with $\mathbb{P}(\mathcal{E})$.

In general, if the parabolic data consists of n distinct set of closed points $S = \{p_1, \ldots, p_n\}$ and generic weight α as in Proposition 4.2, for each $i = 1, \ldots, n$ let $\mathcal{E}_i$ denote the restriction of the universal bundle over $X \times \mathcal{M}$ to $\{p_i\} \times \mathcal{M}$. Then an analogous argument as above shows that $\mathcal{M}_\alpha$ is isomorphic to the fibre product of $\mathbb{P}(\mathcal{E}_i)$'s over $\mathcal{M}$, i.e.,

$$\mathcal{M}_\alpha \cong \mathbb{P}(\mathcal{E}_1) \times_{\mathcal{M}} \mathbb{P}(\mathcal{E}_2) \times_{\mathcal{M}} \times \cdots \times_{\mathcal{M}} \mathbb{P}(\mathcal{E}_n).$$

□

PROPOSITION 4.5

For small enough generic weights α,

$$\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}), \quad \textit{where } n = |S|.$$

Proof. For each $1 \le i \le n$, $n = |S|$, let $\mathcal{F}_i := \mathbb{P}(\mathcal{E}_1) \times_{\mathcal{M}} \mathbb{P}(\mathcal{E}_2) \times_{\mathcal{M}} \times \cdots \times_{\mathcal{M}} \mathbb{P}(\mathcal{E}_i)$. By Lemma 4.3, we have $\mathcal{M}_\alpha \cong \mathcal{F}_n$, so we have the following fibre diagram:

$$\begin{array}{ccc} \mathcal{M}_\alpha & \longrightarrow & \mathbb{P}(\mathcal{E}_n) \\ \downarrow & & \downarrow \\ \mathcal{F}_{n-1} & \longrightarrow & \mathcal{M} \end{array} .$$

The left and right vertical arrows above are $\mathbb{P}^1$-bundles, and hence by [8, Theorem 9.25], there exist isomorphisms of Chow groups:

$$\begin{aligned} \mathrm{CH}_1^{\mathbb{Q}}(\mathbb{P}(\mathcal{E}_n)) &\cong \mathrm{CH}_0^{\mathbb{Q}}(\mathcal{M}) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}) \\ \text{and } \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{F}_n) &\cong \mathrm{CH}_0^{\mathbb{Q}}(\mathcal{F}_{n-1}) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{F}_{n-1}). \end{aligned} \tag{4.1}$$

Iterating the same for $\mathcal{F}_{n-1}, \mathcal{F}_{n-2}$, and so on, we get from (4.1):

$$
\begin{aligned}
\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{F}_n) &\cong \mathrm{CH}_0^{\mathbb{Q}}(\mathcal{F}_{n-1}) \oplus (\mathrm{CH}_0^{\mathbb{Q}}(\mathcal{F}_{n-2}) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{F}_{n-2}))\\
&\ \vdots\\
&\cong \bigoplus_{i=1}^{n-1} \mathrm{CH}_0^{\mathbb{Q}}(\mathcal{F}_i) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{F}_1)\\
&\cong \bigoplus_{i=1}^{n-1} \mathrm{CH}_0^{\mathbb{Q}}(\mathcal{F}_i) \oplus \mathrm{CH}_0^{\mathbb{Q}}(\mathcal{M}) \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}) \quad \text{[8, Theorem 9.25]}.
\end{aligned}
$$

Now, by [6, Theorem 1.2], $\mathcal{M}$ is rational, and hence any projective bundle over it must also be rational; so each $\mathcal{F}_i$ must be rational. By [4, Example 16.1.11], the Chow group of 0-cycles is a birational invariant, hence it follows that $\mathrm{CH}_0^{\mathbb{Q}}(\mathcal{M}) \cong \mathbb{Q}$, and $\mathrm{CH}_0^{\mathbb{Q}}(\mathcal{F}_i) \cong \mathbb{Q}\ \forall i$.

Hence we conclude that

$$
\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}). \tag{4.2}
$$

□

We are now able to generalize the result of Choe and Hwang (Theorem 4.1) for the moduli of parabolic bundles.

Theorem 4.6. *In case of rank* 2 *and determinant* $\mathcal{O}_X(x)$, *for any generic weight* α, *we have*

$$
\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \mathrm{CH}_0^{\mathbb{Q}}(X), \ \textit{where } n = |S|.
$$

Proof. Combining Proposition 4.5 and Theorem 4.1, we get $\mathrm{CH}_1^{\mathbb{Q}}(\mathcal{M}_\alpha) \cong \mathbb{Q}^n \oplus \mathrm{CH}_0^{\mathbb{Q}}(X)$ for small weights α. But using Theorem 3.12, we can conclude that the same result holds true for arbitrary generic weights as well. □

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