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Exact divisibility by powers of the Pell and Associated Pell numbers

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Abstract. Let P_n be the n-th Pell number and Q_n be the n-th Associated Pell number. We obtain certain divisibility and exact divisibility results for the powers of the Pell and Associated Pell numbers by applying the concept of p-adic valuation of the numbers. In particular, we show that $P_n^k \parallel m$ if and only if $P_n^{k+1} \parallel P_{nm}$ for all $m, n \geq 2$ and $k \geq 1$. In addition, $Q_n^k \parallel m$ if and only if $Q_n^{k+1} \parallel Q_{nm}$ for all $m, n \geq 2$ and $k \geq 1$ is also proved.

Keywords. Exact divisibility; Pell number; Associated Pell number; p-adic valuation; order of appearance.

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1. Introduction

The Pell sequence $\{P_n\}$ and Associated Pell sequence $\{Q_n\}$ are defined by means of the binary recurrences $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$ with $P_0 = 0,\ P_1 = 1$ and $Q_{n+1} = 2Q_n + Q_{n-1}$ for $n \geq 1$ with $Q_0 = 1,\ Q_1 = 1$. The Binet form of Pell sequence is given by $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ whereas that of Associated Pell sequence is $Q_n = \frac{\alpha^n + \beta^n}{2}$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. For integers $u, v \geq 1$ and $l \geq 0$, we say that u^l exactly divides v, denoted by $u^l \parallel v$, if $u^l \mid v$ and $u^{l+1} \nmid v$. The exact divisibility properties of the Pell numbers and Associated Pell numbers are not well known unlike the Fibonacci numbers F_n and Lucas numbers L_n. In this paper, we will explore the exact divisibility by powers of the Pell and Associated Pell numbers.

Matijasevic provided a solution to Hilbert's 10-th problem [12–14] via the divisibility relation:

$$F_n^2 \mid F_{nm} \quad \text{if and only if } F_n \mid m. \tag{1.1}$$

Furthermore, Hoggatt and Bicknell-Johnson [4] generalized (1.1) by replacing F_n^2 by F_n^3, and, in addition, they proved that

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$$\text{if } F_n^k \mid m, \quad \text{then } F_n^{k+1} \mid F_{nm} \tag{1.2}$$

for a general k. Then Tangboonduangjit *et al.* [17,27] investigated the results on exact divisibility for a subsequence of $(F_n)_{n\geq 1}$ which was generalized by Onphaeng and Pongsriiam [15]. In a different work, Pongsriiam [19] further extended (1.2) to include the divisibility and exact divisibility for both the Fibonacci and Lucas numbers. Apart from that, the converse of the results in [19] were recently derived by Onphaeng and Pongsriiam [16].

In this paper, we show the divisibility and exact divisibility properties for the Pell and Associated Pell numbers analogous to those in [2,4,16,19,25]. One may refer to [3,5,21,23,26] for other related properties on Fibonacci, Lucas, Pell, and Associated Pell numbers.

2. Preliminaries

We first recall some basic properties of P_n and Q_n in the following lemmas.

Lemma 2.1. *Let m and n be positive integers. Then the following statements hold*:

(*i*) ([8], *Theorem* 8.4, *Theorem* 8.5) $P_n \mid P_m$ *if and only if* $n \mid m$. *Furthermore*, $\gcd(P_m, P_n) = P_{\gcd(m,n)}$.
(*ii*) ([8], *Chapter* 7) $\gcd(P_n, Q_n) = 1$.
(*iii*) ([8], *Chapter* 7) $P_{2n} = 2P_n Q_n$.

From this point on, we let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. In addition, we sometimes apply Lemma 2.1 without reference. Furthermore, if $x \in \mathbb{R}$, then we write $\lfloor x \rfloor$ to denote the largest integer not exceeding x.

Lemma 2.2. *If m and n are positive integers, then*

$$P_{mn} = \sum_{j=1}^{n} \binom{n}{j} P_m^j P_{m-1}^{n-j} P_j = \sum_{j=1}^{m} \binom{m}{j} P_n^j P_{n-1}^{m-j} P_j.$$

Proof. Applying the Binet formula for Pell numbers, we get

$$\begin{aligned} \alpha P_m + P_{m-1} &= \frac{1}{2\sqrt{2}}(\alpha(\alpha^m - \beta^m) + \alpha^{m-1} - \beta^{m-1}) \\ &= \frac{1}{2\sqrt{2}}\left(\alpha^m\left(\alpha + \frac{1}{\alpha}\right) - \beta^m\left(\alpha + \frac{1}{\beta}\right)\right) = \alpha^m \end{aligned}$$

and similarly,

$$\begin{aligned} \beta P_m + P_{m-1} &= \frac{1}{2\sqrt{2}}(\beta(\alpha^m - \beta^m) + \alpha^{m-1} - \beta^{m-1}) \\ &= \frac{1}{2\sqrt{2}}\left(\alpha^m\left(\beta + \frac{1}{\alpha}\right) - \beta^m\left(\beta + \frac{1}{\beta}\right)\right) = \beta^m. \end{aligned}$$

Putting the values of α^m and β^m into $P_{mn} = \frac{(\alpha^m)^n - (\beta^m)^n}{2\sqrt{2}}$ and then applying the binomial expansion, we obtain the first equality. Furthermore, since $P_{mn} = P_{nm}$, we can interchange the role of m and n to obtain the second equality. □

Lemma 2.3. *Let m and n be positive integers. Then the following statements hold*:

(*i*) $P_{mn} = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{m-2j-1} Q_n^{m-2j-1} P_n^{2j+1} 2^j$,
(*ii*) $Q_{mn} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{m-2j} Q_n^{m-2j} P_n^{2j} 2^j$.

Proof. By the Binet formula, we obtain that

$$Q_n + \sqrt{2}P_n = \frac{1}{2}(\alpha^n + \beta^n + \alpha^n - \beta^n) = \alpha^n,$$
$$Q_n - \sqrt{2}P_n = \frac{1}{2}(\alpha^n + \beta^n - (\alpha^n - \beta^n)) = \beta^n.$$

Substituting α^n and β^n in $P_{mn} = \frac{(\alpha^n)^m - (\beta^n)^m}{2\sqrt{2}}$, we obtain

$$P_{mn} = \frac{1}{2\sqrt{2}}((Q_n + \sqrt{2}P_n)^m - (Q_n - \sqrt{2}P_n)^m),$$

and applying the binomial expansion, we get

$$\begin{aligned} P_{mn} &= \frac{1}{2\sqrt{2}} \sum_{i=0}^{m} \binom{m}{i} \sqrt{2}^i P_n^i Q_n^{m-i} (1 - (-1)^i) \\ &= \frac{1}{2\sqrt{2}} \sum_{i=0}^{m} \binom{m}{m-i} \sqrt{2}^i P_n^i Q_n^{m-i} (1 - (-1)^i). \end{aligned} \tag{2.1}$$

If i is even, then $1 - (-1)^i = 0$. So we consider the sum in (2.1) when i is odd, say $i = 2j + 1$. Then (2.1) becomes

$$P_{mn} = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{m-2j-1} P_n^{2j+1} Q_n^{m-2j-1} 2^j,$$

which is the same as (i). The proof of (ii) is similar. □

Lemma 2.4 [15]. *Let* k, ℓ, m, s *be positive integers and* $s^k \mid m$. *Then* $s^{k+\ell} \mid \binom{m}{j} s^j$ *for all* $1 \le j \le m$ *satisfying* $2^{j-\ell+1} > j$. *In particular,* $s^{k+1} \mid \binom{m}{j} s^j$ *for all* $1 \le j \le m$, *and* $s^{k+2} \mid \binom{m}{j} s^j$ *for all* $3 \le j \le m$.

DEFINITION 2.5

For each $n \in \mathbb{N}$, the order of appearance of n in the Pell sequence, denoted by $\alpha(n)$,is defined as the smallest positive integer k such that $n \mid P_k$. In addition, if p is a prime and $n \in \mathbb{N}$, the p-adic valuation (or p-adic order) of n, denoted by $\nu_p(n)$, is defined as the nonnegative integer k such that $p^k \parallel n$.

Remark. The order of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest $k \in \mathbb{N}$ such that $n \mid F_k$ and it is well-known that $n \mid F_m$ if and only if $z(n) \mid m$, and $n \mid P_m$ if and only if $\alpha(n) \mid m$. More general results by Ballot [1], Sanna [24], and Young [29] also lead to formulas for $\nu_p(P_n)$ and $\nu_p(Q_n)$ as follows.

Marques [9] obtained a formula for $z(F_n^k)$ and for $z(L_n^k)$ in some special cases which were later completed by Pongsriiam [20], and were used in the proof of the converse of exact divisibility results in [16]. For other formulas concerning $z(n)$, see for example in the recent articles by Marques [10], Marques and Trojovský [11], Trojovský [28], Pongsriiam [22], and Khaochim and Pongsriiam [6,7]. Here we recall the formula of $\alpha(P_n^k)$ obtained by Patel, Dutta, and Ray [18].

Lemma 2.6 [18]. *If P_n is the n-th Pell number and α is the order of appearance, then $\alpha(P_n^{k+1}) = nP_n^k$, for every $k \geq 0$ and $n \geq 1$.*

Lemma 2.7. *For each $n \in \mathbb{N}$, the p-adic valuation of P_n is given by*

$$\nu_2(P_n) = \nu_2(n)$$

and if $p \neq 2$, then

$$\nu_p(P_n) = \begin{cases} \nu_p(n) + \nu_p(P_{\alpha(p)}), & \text{if } \alpha(p) \mid n; \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, $\nu_2(Q_n) = 0$, since Q_n is always odd.

3. Main results

In this section, we will provide some exact divisibility properties related to the sequences (P_n) and (Q_n). We first apply Lemma 2.6 to obtain the following theorem.

Theorem 3.1. *Let m and n be positive integers. Then the following statements hold*:

(*i*) $P_n^2 \mid P_{mn}$ *if and only if* $P_n \mid m$.
(*ii*) $P_n^3 \mid P_{mn}$ *if and only if* $P_n^2 \mid m$.

Proof. By the second equality in Lemma 2.2, we obtain that

$$P_{mn} \equiv mP_nP_{n-1}^{m-1} \pmod{P_n^2}.$$

From this we see that

$$P_n^2 \mid P_{mn} \Leftrightarrow P_n^2 \mid mP_nP_{n-1}^{m-1} \Leftrightarrow P_n \mid mP_{n-1}^{m-1} \Leftrightarrow P_n \mid m,$$

where the last equivalence is obtained from the fact that $\gcd(P_n, P_{n-1}^{m-1}) = 1$. Similarly, by Lemma 2.2, we have

$$P_{mn} \equiv mP_nP_{n-1}^{m-1} + m(m-1)P_n^2P_{n-1}^{m-2} \pmod{P_n^3}$$

$$\equiv mP_nP_{n-1}^{m-2}(P_{n-1}+(m-1)P_n) \pmod{P_n^3}.$$

Since $\gcd(P_n, (P_{n-1}+(m-1)P_n)) = \gcd(P_n, P_{n-1}) = 1$, we obtain

$$P_n^3 \mid P_{mn} \Leftrightarrow P_n^3 \mid mP_nP_{n-1}^{m-2} \Leftrightarrow P_n^2 \mid m.$$

This completes the proof. □

We can generalize Theorem 3.1 to higher powers of P_n as shown in Theorem 3.2.

Theorem 3.2. *For all $k \geq 1$ and $m, n \geq 2$, we have $P_n^k \mid m$ if and only if $P_n^{k+1} \mid P_{nm}$.*

Proof. Let $P_n^k \mid m$. Then by Lemma 2.4, $P_n^{k+1} \mid \binom{m}{j}P_n^j$, for all $1 \leq j \leq m$. Then by Lemma 2.2, we obtain $P_n^{k+1} \mid P_{nm}$. Conversely, let us assume that $P_n^{k+1} \mid P_{nm}$. Then $\alpha(P_n^{k+1}) \mid nm$. Applying Lemma 2.6 gives us the desired result. Alternatively, to prove that $P_n^k \mid m$, we can show that $\nu_p(P_n^k) \leq \nu_p(m)$ for all primes p dividing P_n. This method will appear again in the proof of Theorem 3.5. If $p = 2$ and $p \mid P_n$, then we obtain from the assumption $P_n^{k+1} \mid P_{nm}$ that

$$\begin{aligned}0 \leq \nu_2(P_{nm}) - \nu_2(P_n^{k+1}) &= \nu_2(nm) - (k+1)\nu_2(n)\\ &= \nu_2(n) + \nu_2(m) - (k+1)\nu_2(n)\\ &= \nu_2(m) - k\nu_2(n) = \nu_2(m) - \nu_2(P_n^k),\end{aligned}$$

which implies $\nu_2(P_n^k) \leq \nu_2(m)$. Similarly, if $p \neq 2$ and $p \mid P_n$, then

$$\begin{aligned}0 \leq \nu_p(P_{nm}) - \nu_p(P_n^{k+1}) &= \nu_p(nm) + \nu_p(P_{\alpha(p)})\\ &\quad - (k+1)(\nu_p(n) + \nu_p(P_{\alpha(p)}))\\ &= \nu_p(m) - k(\nu_p(n) + \nu_p(P_{\alpha(p)}))\\ &= \nu_p(m) - \nu_p(P_n^k),\end{aligned}$$

which leads to $\nu_p(P_n^k) \leq \nu_p(m)$. Therefore, $\nu_p(P_n^k) \leq \nu_p(m)$ for all p dividing P_n, as required. This completes the proof. □

COROLLARY 3.3

For all $k \geq 1$ and $m, n \geq 2$, we obtain $P_n^k \parallel m$ if and only if $P_n^{k+1} \parallel P_{nm}$.

Proof. Assume that $P_n^k \parallel m$. Then $P_n^k \mid m$. By Theorem 3.2, we obtain $P_n^{k+1} \mid P_{nm}$. So to prove that $P_n^{k+1} \parallel P_{nm}$, it is enough to show that $P_n^{k+2} \nmid P_{nm}$. Applying Theorem 3.2 again but replacing k by $k+1$, we see that $P_n^{k+2} \mid P_{nm}$ implies $P_n^{k+1} \mid m$, which contradicts the assumption $P_n^k \parallel m$. So $P_n^{k+2} \nmid P_{nm}$, as required. For the converse part, suppose $P_n^{k+1} \parallel P_{nm}$. So $P_n^{k+1} \mid P_{nm}$, and hence $P_n^k \mid m$ by Theorem 3.2. If $P_n^{k+1} \mid m$, then again, we would have $P_n^{k+2} \mid P_{nm}$, contradicting $P_n^{k+1} \parallel P_{nm}$. Therefore $P_n^{k+1} \nmid m$. Hence $P_n^k \parallel m$ and the proof is complete. □

Theorem 3.4. *Let $m, r \geq 1$. If r is even, then $Q_m^2 \nmid Q_{mr}$. If r is odd, then*

(i) $Q_m^2 \mid Q_{mr}$ *if and only if* $Q_m \mid r$,
(ii) $Q_m^3 \mid Q_{mr}$ *if and only if* $Q_m^2 \mid r$.

Proof. By the Binet formula, we get $\alpha^m = Q_m + \sqrt{2}P_m$ and $\beta^m = Q_m - \sqrt{2}P_m$. Hence proceeding in the same way as in Lemma 2.3, we obtain

$$Q_{mr} = \frac{(\alpha^m)^r + (\beta^m)^r}{2} = \frac{1}{2}\sum_{j=0}^{r}\binom{r}{j}Q_m^{r-j}P_m^j\sqrt{2}^j(1+(-1)^j). \tag{3.1}$$

If j is odd, then $1+(-1)^j = 0$. So we consider (3.1) when j is even, say $j = 2i$. Then

$$Q_{mr} = \sum_{i=0}^{\lfloor \frac{r}{2} \rfloor}\binom{r}{2i}Q_m^{r-2i}P_m^{2i}2^i. \tag{3.2}$$

Now, it can be said that except the last term, all the terms in the right-hand side of (3.2) are divisible by Q_m^2. If r is even, then the last term is $2^{r/2}P_m^r$ and so $Q_{mr} \equiv 2^{r/2}P_m^r \pmod{Q_m^2}$. Since $\gcd(Q_m, 2) = 1 = \gcd(Q_m, P_m)$, we obtain $Q_m^2 \nmid Q_{mr}$. Suppose r is odd. Then all terms in (3.2), except the last term, are divisible by Q_m^3. Therefore,

$$Q_{mr} \equiv rQ_mP_m^{r-1}2^{\frac{r-1}{2}} \pmod{Q_m^3}.$$

From this, we see that $Q_m^3 \mid Q_{mr} \Leftrightarrow Q_m^2 \mid r$ and $Q_m^2 \mid Q_{mr} \Leftrightarrow Q_m \mid r$. □

By Theorem 3.4, it is natural to consider the divisibility of Q_{nm} by Q_n^{k+1} only when m is odd as shown in the next theorem.

Theorem 3.5. *Suppose $m, n \geq 1$ and m is odd. Then $Q_n^k \mid m$ if and only if $Q_n^{k+1} \mid Q_{nm}$.*

Proof. The proof is similar to that of Theorem 3.2. Assume that $Q_n^k \mid m$. By virtue of Lemma 2.4, $Q_n^{k+1} \mid \binom{m}{j}Q_n^j$ for $1 \leq j \leq m$. So if we replace j by $m-2j$, we have $Q_n^{k+1} \mid \binom{m}{m-2j}Q_n^{m-2j}$ for $1 \leq m-2j \leq m$ which simplifies to $0 \leq j \leq \frac{m-1}{2}$. Then by virtue of Lemma 2.3, $Q_n^{k+1} \mid Q_{nm}$. Conversely, assume that $Q_n^{k+1} \mid Q_{nm}$. To show that $Q_n^k \mid m$, we prove that $\nu_p(Q_n^k) \leq \nu_p(m)$ for every prime p dividing Q_n. Since Q_n is odd, we consider only $p \mid Q_n$ and $p \geq 3$. If $\alpha(p) \mid n$, then we obtain from Lemmas 2.1 and 2.7 that

$$\begin{aligned}\nu_p(Q_n) = \nu_p\left(\frac{P_{2n}}{2P_n}\right) &= \nu_p(P_{2n}) - \nu_p(2P_n) = \nu_p(P_{2n}) - \nu_p(P_n)\\ &= (\nu_p(2n) + \nu_p(P_{\alpha(p)})) - (\nu_p(n) + \nu_p(P_{\alpha(p)})) = 0,\end{aligned}$$

which is not the case. Similarly, if $\alpha(p) \nmid 2n$, then $\nu_p(Q_n) = \nu_p\left(\frac{P_{2n}}{2P_n}\right) = 0$, which is false. Therefore, $\alpha(p) \mid 2n$ and $\alpha(p) \nmid n$. Thus $\nu_p(Q_n) = \nu_p\left(\frac{P_{2n}}{2P_n}\right) = \nu_p(P_{2n}) =$

$\nu_p(n) + \nu_p(P_{\alpha(p)})$. Now since $Q_n^{k+1} \mid Q_{nm}$, we obtain

$$\begin{aligned} 0 \geq \nu_p(Q_n^{k+1}) - \nu_p(Q_{nm}) &= \nu_p(Q_n^k) + \nu_p(Q_n) - \nu_p(P_{2nm}) + \nu_p(P_{nm}) \\ &= \nu_p(Q_n^k) + \nu_p(P_{nm}) + \nu_p(n) + \nu_p(P_{\alpha(p)}) - \nu_p(2mn) - \nu_p(P_{\alpha(p)}) \\ &= \nu_p(Q_n^k) + \nu_p(P_{nm}) - \nu_p(m) \geq \nu_p(Q_n^k) - \nu_p(m) \end{aligned}$$

which implies $\nu_p(Q_n^k) \leq \nu_p(m)$, as required. So the proof is complete. □

COROLLARY 3.6

Suppose $m, n \geq 1$ and m is odd. Then $Q_n^k \parallel m$ if and only if $Q_n^{k+1} \parallel Q_{nm}$.

Proof. Let $Q_n^k \parallel m$. So $Q_n^k \mid m$ and $Q_n^{k+1} \nmid m$. By virtue of Theorem 3.5, $Q_n^{k+1} \mid Q_{nm}$. The only thing left to prove is $Q_n^{k+2} \nmid Q_{nm}$. Now by Lemma 2.4, we have $Q_n^{k+2} \mid \binom{m}{j} Q_n^j$ for $3 \leq j \leq m$ which can be rewritten as $Q_n^{k+2} \mid \binom{m}{m-2j} Q_n^{m-2j}$ for $0 \leq j \leq \frac{m-3}{2}$. Then by virtue of Lemma 2.3,

$$Q_{nm} \equiv m Q_n P_n^{m-1} 2^{\frac{m-1}{2}} \pmod{Q_n^{k+2}}.$$

Since $Q_n^{k+1} \nmid m$, we obtain $Q_n^{k+2} \nmid Q_{nm}$ by virtue of Theorem 3.5. Conversely, let $Q_n^{k+1} \parallel Q_{nm}$. Since $Q_n^{k+1} \mid Q_{nm}$, we obtain by Theorem 3.5 that $Q_n^k \mid m$. If $Q_n^{k+1} \mid m$, then we again apply Theorem 3.5 to get $Q_n^{k+2} \mid Q_{nm}$, which contradicts the assumption $Q_n^{k+1} \parallel Q_{nm}$. Hence $Q_n^{k+1} \nmid m$ which gives the desired result that $Q_n^k \parallel m$. □

Theorem 3.7. *Let m and r be positive integers. If r is odd, then $Q_m \nmid P_{mr}$. Suppose r is even. Then the following statements hold*:

(*i*) $Q_m \mid P_{mr}$.
(*ii*) $Q_m^2 \mid P_{mr}$ *if and only if* $Q_m \mid r$.
(*iii*) $Q_m^3 \mid P_{mr}$ *if and only if* $Q_m^2 \mid r$.

Proof. Proceeding in the similar manner as in Theorem 3.4, we obtain

$$\begin{aligned} P_{mr} &= \frac{(\alpha^m)^r - (\beta^m)^r}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \sum_{j=0}^{r} \binom{r}{j} Q_m^{r-j} P_m^j \sqrt{2}^j (1 - (-1)^j) \\ &= \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r}{2i+1} Q_m^{r-2i-1} P_m^{2i+1} 2^i. \end{aligned} \tag{3.3}$$

Case 1. *r is odd*. Then all terms except the last term in (3.3) are divisible by Q_m. Therefore $P_{mr} \equiv P_m^r 2^{\frac{r-1}{2}} \pmod{Q_m}$. Since $Q_m > 1$ and $\gcd(Q_m, P_m) = 1 = \gcd(Q_m, 2)$, we see that $Q_m \nmid P_{mr}$.
Case 2. *r is even*. Then all terms except the last term in (3.3) are divisible by Q_m^3. The last term corresponds to $i = (r-2)/2$ and so

$$P_{mr} \equiv r Q_m P_m^{r-1} 2^{\frac{r-2}{2}} \pmod{Q_m^3}. \tag{3.4}$$

Since (3.4) also holds when mod Q_m^3 is replaced by mod Q_m and mod Q_m^2, we see that Q_m divides P_{mr}, $Q_m^2 \mid P_{mr} \Leftrightarrow Q_m^2 \mid rQ_m \Leftrightarrow Q_m \mid r$, and $Q_m^3 \mid P_{mr} \Leftrightarrow Q_m^3 \mid rQ_m \Leftrightarrow Q_m^2 \mid r$. This completes the proof. □

Since $Q_n \nmid P_{nm}$ if m is odd, it is natural to extend Theorem 3.7 under the assumption that m is even as follows.

Theorem 3.8. *Suppose k, m, n are positive integers and m is even. Then $Q_n^k \mid m$ if and only if $Q_n^{k+1} \mid P_{nm}$.*

Proof. Let $Q_n^k \mid m$. Then by Lemma 2.4, we obtain $Q_n^{k+1} \mid \binom{m}{j} Q_n^j$ for all $1 \le j \le m$ and hence $Q_n^{k+1} \mid \binom{m}{m-2j-1} Q_n^{m-2j-1}$ for every $0 \le j \le \frac{m-2}{2}$. So by Lemma 2.3, $Q_n^{k+1} \mid P_{mn}$. Conversely, let us assume that $Q_n^{k+1} \mid P_{nm}$. To show that $Q_n^k \mid m$, we follow the proof of Theorem 3.5. So let p be an odd prime dividing Q_n. As already shown in the proof of Theorem 3.5, we can assume that $\alpha(p) \mid 2n$ and $\alpha(p) \nmid n$. Then $\alpha(p) \mid nm$ because m is even. Since $Q_n^{k+1} \mid P_{nm}$, we have

$$\begin{aligned} 0 \le \nu_p(P_{nm}) - \nu_p(Q_n^{k+1}) &= \nu_p(n) + \nu_p(m) + \nu_p(P_{\alpha(p)}) \\ &\quad - \nu_p(Q_n) - \nu_p(Q_n^k) \\ &= \nu_p(m) - \nu_p(Q_n^k), \end{aligned}$$

which implies $\nu_p(Q_n^k) \le \nu_p(m)$. Therefore $Q_n^k \mid m$, as required. □

COROLLARY 3.9

Suppose k, m, n are positive integers and m is even. Then $Q_n^k \parallel m$ if and only if $Q_n^{k+1} \parallel P_{nm}$.

Proof. Let $Q_n^k \parallel m$. By Theorem 3.8, $Q_n^{k+1} \mid P_{nm}$. If $Q_n^{k+2} \mid P_{nm}$, we apply Theorem 3.8 again to obtain $Q_n^{k+1} \mid m$, which contradicts the assumption $Q_n^k \parallel m$. So $Q_n^{k+1} \parallel P_{nm}$. The converse can be proved similarly. If $Q_n^{k+1} \parallel P_{nm}$, we apply Theorem 3.8 twice to conclude that $Q_n^k \parallel m$. □

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