Proc. Indian Acad. Sci. (Math. Sci.) (2021) 131:19
https://doi.org/10.1007/s12044-021-00611-0

Note on a problem of Ramanujan

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MS received 26 August 2020; revised 5 October 2020; accepted 6 October 2020

Abstract. For fixed positive real numbers ω, ω', it is known that the number of lattice points $(u, v), u \geq 0, v \geq 0$ satisfying $0 \leq u\omega + v\omega' \leq \eta$ is given by $\frac{1}{2}(\frac{\eta^2}{\omega\omega'} + \frac{\eta}{\omega} + \frac{\eta}{\omega'}) + O_\varepsilon(\eta^{1-\frac{1}{\alpha_0}+\varepsilon})$, where $\alpha_0 \geq 1$ is a constant. In this paper, we explicitly compute α_0 for certain values of ω/ω'. In particular, in Ramanujan's case (i.e., when $\omega = \log 2$ and $\omega' = \log 3$), we show that $\alpha_0 = 2^{18} \log 3$ is admissible. This improves an earlier result of the paper (Ramachandra K, Sankaranarayanan A and Srinivas K, *Hardy Ramanujan J.* **19** (1996) 2–56), where it was shown that $\alpha_0 = 2^{40} \log 3$ holds.

Keywords. Lattice points; irrationality measure; diophantine approximation.

Mathematics Subject Classification. 11J82, 11K60, 11P21.

1. Introduction

Lattice point problems refer to the problem of counting points with integer coordinates in a closed domain. It is well known that the number of lattice points N inside or on a simple closed curve C with the area A and length $L \geq 1$ is given by $|N - A| < L$, as a first approximation. The problem is then to optimize the quantity $|N - A|$. The *circle problem* and the *divisor problem* are two classical lattice point problems which have been well studied and their investigations enriched mathematics with new ideas and powerful techniques. For the benefit of readers, we shall briefly touch upon these problems before stating the main result of this paper. Interested reader may refer to [2,7,9] for details.

The circle problem. Let $R(x)$ be the number of lattice points which lie on or inside the circle $u^2 + v^2 = x$. C. F. Gauss was the first to show that

$$R(x) = \pi x + O(x^\alpha), \quad \text{with } \alpha = 1/2$$

Sierpinski [13] showed that $\alpha \leq 1/3$, J. G. van der Corput in 1923 proved that $\alpha \leq 1/3 - \varepsilon$. Since then, a lot of progress has been made on improving the value of α. More recently, Huxley [6] showed that one can take $\alpha \leq 131/416$. In the other direction, Landau [10]

Published online: 16 June 2021

and Hardy [3] showed that $\alpha \geq 1/4$. A beautiful analysis of this problem and for recent results, the readers may refer to Chapter 13 of [7].

The divisor problem. Let $D(x)$ be the number of lattice points which lie in the region $u > 0, v > 0,\ uv \leq x$, i.e., bounded by the axes and the rectangular hyperbola $uv = x$ excluding those on the axes (since there are infinitely many of them on the axes). Note that $D(x) = \sum_{mn \leq x} 1 = \sum_{n \leq x} d(n)$, where $d(n)$ is the divisor function (counts the number of divisors of n). In 1849, P. G. L. Dirichlet showed that

$$D(x) = x \log x + (2\gamma - 1)x + O(x^{\theta}), \quad \text{with } \theta = 1/2.$$

The value of θ was subsequently improved and the latest is by Huxley [6] obtained $\theta \leq 131/416$ is admissible. The analysis of the circle problem and the divisor problem are similar in the sense that both require estimation of certain exponential sums, which in turn reduces to finding optimal exponent pairs. In the other direction, the result of Landau [10] and Hardy [3] gives an omega result with $\theta \geq 1/4$ (see Chapter 13 of [7] for more details and recent developments).

In this paper, we shall discuss the lattice point problem inside a right-angled triangle and indicate an improvement in the error term in certain cases.

Ramanujan's lattice point problem. The famous (first) letter of S. Ramanujan to G. H. Hardy, written in 1913 contains, among other important results, the following statement:

The number of numbers of the form $2^u 3^v$ *less than n is* $\frac{\log 2n \ \log 3n}{2 \log 2 \ \log 3}$.

In other words, Ramanujan asserts that the number of integer points $u \geq 0,\ v \geq 0$ satisfying the inequality $u \log 2 + v \log 3 \leq \log n$ is approximately

$$\frac{\log^2 n}{2 \log 2 \ \log 3} + \frac{\log n}{2 \log 2} + \frac{\log n}{2 \log 3} + \frac{1}{2}.$$

Hardy formulated the problem in a general set up in Ch. 5 of [4]. Let ω, ω' be positive real numbers such that $N(\eta) := N(\eta, \omega, \omega')$ is the number of solutions of

$$u \geq 0,\ \ v \geq 0, \quad 0 < u\omega + v\omega' \leq \eta,$$

i.e., $N(\eta)$ counts the number of lattice points inside and on the right triangle with vertices at $(0, 0)$, $(\eta/\omega, 0)$ and $(0, \eta/\omega')$. Then

$$\begin{aligned} N(\eta) &= \sum_{u \leq \frac{\eta}{\omega}} \sum_{v \leq \frac{\eta - u\omega}{\omega'}} 1 = \sum_{u \leq \frac{\eta}{\omega}} \left(1 + \left[\frac{\eta - u\omega}{\omega'}\right]\right) \\ &= \sum_{u \leq \frac{\eta}{\omega}} \left(\frac{\eta - u\omega}{\omega'} + \frac{1}{2}\right) - \sum_{u \leq \frac{\eta}{\omega}} \left\{\frac{\eta - u\omega}{\omega'}\right\} \\ &= M(\eta) + R(\eta) \quad \text{(say)}, \end{aligned}$$

where $[x]$ denotes greatest integer less than or equal to x and $\{x\} = x - [x] - 1/2$.

Evaluation of the first sum gives

$$M(\eta) = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'}.$$

By using the arithmetic nature of θ, Hardy (Ch. 5 of [4]) proved the following:

$$R(\eta) = O\left(\frac{\eta}{q_m}\right) + O(q_m), \tag{1}$$

where q_m is the denominator of the m-th convergent to the continued fraction for θ. From (1), he derived

$$R(\eta) = o(\eta)$$

for all irrational θ. However, in Ramanujan's case (i.e., when $\omega = \log 2$ and $\omega' = \log 3$), a judicious choice of q_m, namely ($A \log \eta < q_m \leq \frac{\eta}{\log \eta} \leq q_{m+1}$, $A > 0$ is constant, allowed him to derive

$$R(\eta) = O\left(\frac{\eta}{\log \eta}\right).$$

Furthermore, he managed to show that

$$R(\eta) = o\left(\frac{\eta}{\log \eta}\right)$$

by using a theorem of Pillai (see page 80 of [4]). On the other hand, Hardy and Littlewood [5] refined the method further and proved the following.

Theorem [4]. *Let* $q_{m+1} = O(q_m^{\alpha_0})$, *where* α_0 *is a constant satisfying* $1 \leq \alpha_0 \ll 1$. *If* $\alpha_0 = 1$, *then* $R(\eta) = O(\log \eta)$; *otherwise* $R(\eta) = O_\varepsilon(\eta^{1-\alpha_0^{-1}+\varepsilon})$ *for every* $\varepsilon > 0$.

In Theorem 2.1 of [11], it was pointed out that $\alpha_0 = 1$ is admissible for all quadratic irrationals θ and in Ramanujan's case $\alpha_0 = 2^{40} \log 3$ is admissible.

In the present work, we evaluate the error term $R(\eta)$ in terms of irrationality measure of θ. This gives improvement in several cases. Our main result is the following:

Theorem 1. *Let* a, b *be positive integers such that* a *and* b *are multiplicatively independent. Put* $\omega = \log a$, $\omega' = \log b$ *and* $\theta = \omega/\omega'$. *Let* μ *be an irrationality measure of* θ *and* $\beta_0 + 1 \geq \mu \geq 1$. *Then for every* $\varepsilon > 0$, *the number of lattice points* (u, v) *with* $u \geq 0$, $v \geq 0$ *and satisfying* $0 < u\omega + v\omega' \leq \eta$ *is*

$$\frac{1}{2}\left(\frac{\eta^2}{\omega\omega'} + \frac{\eta}{\omega} + \frac{\eta}{\omega'}\right) + O_\varepsilon(\eta^{1-\frac{1}{\beta_0}+\varepsilon}), \tag{2}$$

where

$$\beta_0 = \begin{cases} 378166.906\, \frac{(\log a)(\log b)}{(\log 3)^2} & \text{if } \min(a, b) \geq 3, \\ 1415045.37\, \frac{(\log 2)(\log 3)}{(\log 7)^2} & \text{if } a = 2, b = 3, \\ 155167.947\, \frac{\log b}{\log 2} & \text{if } a = 2, b > 4, \\ 756333.813\, \frac{(\log a)(\log 2)}{(\log 3)^2} & \text{if } a \geq 3, b = 2 \end{cases}$$

holds for all sufficiently large η. *In particular, for Ramanujan's case we have*

$$\beta_0 = 1415045.37\,\frac{(\log 2)(\log 3)}{(\log 7)^2} \leq (259030.116)\log 3 \leq 2^{18}\log 3.$$

2. Preliminaries

We recall that the notation $f = O(g)$ and $f \ll g$ are equivalent to the assertion that the inequality $|f| \leq cg$ holds for some constant $c > 0$. Two positive rational numbers are multiplicatively independent if the quotient of their logarithms is irrational. Throughout this paper, μ denotes the irrationality measure of θ.

DEFINITION 1

A real number γ has an irrationality measure $\mu = \mu(\gamma)$, if μ is the infimum of $\rho \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $q_0(\varepsilon) > 0$ such that

$$\left|\gamma - \frac{p}{q}\right| \geq \frac{1}{q^{\rho+\varepsilon}} \tag{3}$$

holds for all integers $q > q_0(\varepsilon)$.

From Dirichlet's approximation theorem, it is easily seen that $\mu \geq 2$ for every irrational number γ. The celebrated theorems of Khinchin [8] and of Roth [12] assert that $\mu = 2$ for almost all real (in the sense of the Lebesgue measure) and all irrational algebraic numbers γ, respectively.

Lemma 1 ([1], *Corollary* 2.4). *Let* a, b *be mutiplicatively independent positive integers greater than* 1. *Let* A *and* B *be real numbers such that* $A \geq \max\{a, e\}$, $B \geq \max\{b, e\}$. *For positive integers* x *and* y, *set*

$$X' = \frac{x}{\log A} + \frac{y}{\log B},\ E = 1 + \min\left\{\frac{\log A}{\log a}, \frac{\log B}{\log b}\right\},$$
$$\log E^* = \max\{\log E, \log E - 0.946\log\log\log E + 3.965\},$$
$$\log X = \max\{\log X' + \log E, 265\log E, 150\log E^*\}.$$

Suppose moreover that $3 \leq E \leq \min\{A^{1/2}, B^{1/3}\}$. *Then*

$$\log|y\log a - x\log b| \geq -8550(\log A)(\log B)(\log X)(\log E^*)(\log E)^{-3}. \tag{4}$$

Remark 1. Gouillon in [1] considered E to be atleast 2. We replace it by $E \geq 3$ to avoid any ambiguity with the quantity $\log\log\log E$ occurring in the definition of E^*, which is not defined if E is smaller than e.

Remark 2. As $E \geq 3$, we have $\log E^* \leq 7 + \log E$. Therefore, inequality (4) can be written as

$$\log|y\log a - x\log b| \geq -8550(\log A)(\log B)(\log X)(7+\log E)(\log E)^{-3}. \quad (5)$$

3. Proof of the theorem

Let $\theta = \frac{\omega}{\omega'} = \frac{\log a}{\log b}$. Let $\frac{p_m}{q_m}$ be the m-th convergent for θ and let $\beta_0 + 1$ be an upper bound for the irrationality measure for θ. We will first show that

$$q_{m+1} \ll q_m^{\beta_0}$$

holds for all but finitely many $m \in \mathbb{N}$. To see this, we have

$$|\theta - \tfrac{p}{q}| > \frac{1}{q^{\beta_0+1+\varepsilon}}$$

for all $q \geq q_0(\varepsilon)$ by definition of $\beta_0 + 1$.

On the other hand, by the properties of continued fraction expansion, we have

$$|\theta - \tfrac{p_m}{q_m}| \leq \frac{1}{q_m q_{m+1}}.$$

If $q_{m+1} \gg q_m^{\beta_0}$ is satisfied for infinitely many $m \in \mathbb{N}$, then

$$|\theta - \tfrac{p_m}{q_m}| \leq \frac{1}{q_m^{\beta_0+1}}$$

holds true for infinitely many $m \in \mathbb{N}$. This is clearly a contradiction.

Now as $q_{m+1} \ll q_m^{\beta_0}$ by Hardy–Littlewood theorem, we have

$$R(\eta) = O(\eta^{1-\frac{1}{\beta_0}+\varepsilon}). \quad (6)$$

Below, we shall compute the values of β_0 by invoking Lemma 1. From (3), it follows that

$$\log\left|\theta - \frac{p}{q}\right| > \log c - (\beta_0 + 1 + \varepsilon)\log q.$$

which is equivalent to

$$\log|q\log a - p\log b| > -(\beta_0 + \varepsilon)(c' + \log\max\{p, q\}) \quad (7)$$

for some constant c' which may depend on a, b and ε.

Our main tool is application of Lemma 1. In order to find optimal value of β_0, we shall divide our problem into a few cases depending on a and b.

Case 1: $\min(a, b) \geq 3$. In this case, we choose $A = a^2$ and $B = b^3$ and observe that

$$E = 1 + \min\left\{\frac{\log A}{\log a}, \frac{\log B}{\log b}\right\}$$

satisfies

$$3 \leq E = 1 + \min\{2, 3\} = 3 \leq \min\{a, b\}.$$

Then from Lemma 1, it follows that

$$\log|q\log a - p\log b| \geq -378166.906\,\frac{(\log a)(\log b)}{(\log 3)^2}(\log\max\{p, q\} + \log 3),$$

whenever $\max\{p, q\}$ is sufficiently large.

Case 2: $a = 2$ and $b = 3$ (*Ramanujan's case*). Here we choose $A = 2^6$ and $B = 3^6$ and note that E satisfies

$$3 \leq E = 1 + \min\{6, 6\} = 7 < \min\{8, 9\}.$$

Now from Lemma 1, it follows that

$$\log|q\log a - p\log b| \geq -1415045.37\,\frac{(\log 2)(\log 3)}{(\log 7)^2}(\log\max\{p, q\} + \log 7),$$

whenever $\max\{p, q\}$ is sufficiently large.

Case 3: $a = 2$ *and* $b > 4$. Here, we take $A = 2^4$ and $B = b^3$ and note that E satisfies

$$3 \leq E = 1 + \min\{4, 3\} = 4 \leq \min\{4, b\}.$$

Then from Lemma 1, it follows that

$$\log|q\log a - p\log b| \geq -155167.947\,\frac{\log b}{\log 2}(\log\max\{p, q\} + 2\log 2),$$

whenever $\max\{p, q\}$ is sufficiently large.

Case 4: $a \geq 3$ *and* $b = 2$. We choose $A = a^2$ and $B = 2^6$. As E satisfies

$$3 \leq E = 1 + \min\{2, 6\} = 3 \leq \min\{a, 4\},$$

we obtain from Lemma 1,

$$\log|q\log a - p\log b| \geq -756333.813\,\frac{(\log a)(\log 2)}{(\log 3)^2}(\log\max\{p, q\} + \log 3),$$

whenever $\max\{p, q\}$ is sufficiently large.

In each of the cases these lower bounds are of the form (7), with explicit values β_0 and c'. This establishes the upper bounds for the irrationality measure of θ in each of the cases. This completes the proof of the theorem.

Acknowledgements

The work is partly supported by SERB MATRICS grant, project no. MTR/2017/001006. The authors are grateful to the referee for pointing out some inaccuracies in an earlier draft of this paper.

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COMMUNICATING EDITOR: B Sury