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Leavitt path algebras of weighted Cayley graphs $C_n(S, w)$

R MOHAN

Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560 059, India
E-mail: rmohan689@gmail.com

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Abstract. For a postive integer n and a subset S of $\mathbb{Z}_n$, let $\langle S\rangle = \mathbb{Z}_n$, and $w : S \to \mathbb{N}$ be a function. The weighted Cayley graph of the cyclic group $\mathbb{Z}_n$ with respect to S and w is denoted by $C_n(S, w)$. We give an explicit description of the Grothendieck group of the Leavitt path algebras of $C_n(S, w)$. We also give description of Leavitt path algebras of $C_n(S, w)$ in some special cases.

Keywords. Leavitt path algebra; weighted Cayley graph.

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1. Introduction

For a finite group G and a subset $S \subseteq G$, let the associated Cayley graph be denoted by $\mathsf{Cay}(G, S)$. When the given group is $\mathbb{Z}_n$, we write $C_n(S) := \mathsf{Cay}(\mathbb{Z}_n, S)$. The Leavitt path algebras of Cayley graphs of the finite cyclic group $\mathbb{Z}_n$ with respect to the subset $S = \{1, n-1\}$ were initially studied in [8]. It was shown that there are exactly four isomorphism classes represented by the collection $\{L(C_n(1, n-1)) \mid n \in \mathbb{N}\}$.

In subsequent work, [5] contains the computation of the important integers $|K_0(L(C_n(1, j)))|$ and $\det(I_n - A^t_{C_n(1,j)})$, where $A_{(-)}$ denotes the adjacency matrix of a directed graph, and $K_0(-)$ denotes the Grothendieck group of a ring. Also in [5], the collections of K-algebras were described up to isomorphism:

$$\{L(C_n(1, j)) \mid n \in \mathbb{N}\} \quad \text{for } j = 0, 1, 2.$$

The descriptions of all these algebras follow from an application of the powerful tool known as the (restricted) algebraic Kirchberg–Philips theorem.

In [6], the study was extended and a method to compute the Grothendieck group of the Leavitt path algebra $L(C_n(1, j))$ to the case where $0 \le j \le n-1$ and $n \ge 3$ was derived. Specifically, it was shown how to reduce the computation of the Smith normal form of the $n \times n$ matrix $I_n - A^t_{C_n(1,j)}$ to that of calculating the Smith normal form of a $j \times j$ matrix $(M^n_j)^t - I_j$. Further, a description of $K_0(L(C_n(1, j)))$ was also given.

In this paper, we generalize the work done in [6] to study $L(C_n(S, w))$, where S is any nonempty generating subset of $\mathbb{Z}_n$, $w : S \to \mathbb{N}$ is a map and $C_n(S, w)$ is the weighted Cayley graph. In Section 2, we recall the background information required. In Section 3, we present a method to compute the Grothendieck group. Specifically, we find the conditions

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to determine the sign of $\det(I_n - A^t_{C_n(S,w)})$ and also the cardinality of $K_0(L(C_n(S, w)))$. Also we find a method to reduce the computation of the Smith normal form of the $n \times n$ matrix $I_n - A^t_{C_n(S,w)}$ to that of calculating the Smith normal form of a square matrix of smaller size if $0 \notin S$ (Theorem 3.9). In Section 4, we use the method developed in Section 3 to study the following simple cases when $\langle S\rangle = \mathbb{Z}_n$:

Case 1. $|S| = 1$.
Case 2. $|S| = 2$.
Case 3. $|S| = n$.

Moreover, we recover the results studied in [8], [5] and [6] as special cases and get some new results. Among these new results, in particular, we show that $L(K_n) \cong L(1, n)$ where K_n is the unweighted complete n-graph (see 4.1 for Definition) and $L(1, n)$ is the Leavitt algebra. We also show that the main result of [8] holds true if $C_n(1, n-1)$ is replaced by D_n for every $n \in \mathbb{N}$, where D_n denotes the Cayley graph of dihedral group with respect to the usual generating set.

2. Preliminaries

Notation 2.1. Throughout, by K we mean a fixed field. $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}^+$ denotes the set of non-negative integers, $\mathbb{Z}$ denotes the set of integers and $\mathbb{Q}$ denotes the set of rationals. $|S|$ is the cardinality of the set S.

2.1 *Leavitt path algebras and the algebraic KP theorem*

A *graph* $E = (E^0, E^1, s, r)$ consists of two disjoint sets E^0, E^1 and functions $s, r : E^1 \to E^0$. The elements of E^0 are called vertices and the elements of E^1 are called edges. If e is an edge, then $s(e)$ is called its source and $r(e)$ its range. E is called finite if E^0 and E^1 are finite sets.

A vertex v is called a *source* (resp. *sink*) if $r^{-1}(v) = \emptyset$ (resp. if $s^{-1}(v) = \emptyset$). A graph is called *sink-free* (resp. source-free) if it has no sinks (resp. no sources). A non-sink $v \in E^0$ is called *regular* is $s^{-1}(v)$ is finite (v emits only finitely many edges).

A *path* μ in a graph E is either a vertex in E or a finite sequence $e_1e_2 \ldots e_n$ of edges in E such that $r(e_i) = s(e_{i+1})$ for $1 \le i \le n-1$. A path $\mu = e_1e_2 \ldots e_n$ for which $n \le 1$ and $s(e_1) = r(e_n) = v$ is called a closed path based at v. A closed path $\mu = e_1e_2 \ldots e_n$ based at v for which $s(e_i) \neq s(e_j)$ for any $i \neq j$ is called a *cycle* (based at v). A graph which contains no cycles is called *acyclic*.

Let $E = (E^0, E^1, s, r)$ be a graph. The *adjacency matrix* A_E of E is the $|E^0| \times |E^0|$ matrix whose entries are given by

$$A_E(v, w) = |\{e \in E^1 \mid s(e) = v,\ r(e) = w\}| \text{ for any } v, w \in E^0.$$

DEFINITION 2.2

Let E be a graph and $H \subseteq E^0$.

(1) H is *hereditary* if whenever $v \in H$ and $w \in E^0$ for which there exists a path μ such that $s(\mu) = v$ and $r(\mu) = w$, then $w \in H$.

(2) H is *saturated* if whenever $v \in E^0$ is regular such that $\{r(e) \mid e \in E^1, s(e) = v\} \subseteq H$, then $v \in H$.
(3) E satisfies *condition (L)* if every cycle in E has an exit.

DEFINITION 2.3

Let K be a field and let $E = (E^0, E^1, s, r)$ be a graph. The *Leavitt path algebra* of E is the K-algebra presented by generators $E^0 \sqcup E^1 \sqcup \overline{E^1}$, where $\overline{E^1} := \{e^* \mid e \in E^1\}$ and the following relations hold.

(V) $\forall\, v, w \in E^0$, $vw = \delta_{vw} v$,
(E) $\forall\, e \in E^1$, $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$,
(CK1) $\forall\, e, f \in E^1$, $e^* f = \delta_{ef} r(e)$,
(CK2) $\forall\, v \in E^0$ if $0 < |s^{-1}(v)| < \infty$, then $v = \sum_{e \in s^{-1}(v)} ee^*$,

where δ_{ij} is the Kronecker delta. We denote the Leavitt path algebra by $L(E)$ if underlying field K is fixed.

Remark 2.4. It is easy to see that $L(E)$ is unital if and only if $|E^0|$ is finite, in which case $\sum_{v \in E^0} v$ acts as the unity. In this paper, we focus only on finite graphs.

One of the motivations to define Leavitt path algebra is to study a natural abelian monoid associated to a given graph E called the graph monoid M_E which we define below.

DEFINITION 2.5

Let E be a finite graph with vertex set E^0 and adjacency matrix $A_E = (a_{vw})$. The *graph monoid* M_E of E is the abelian monoid presented by the generating set E^0 and the following relations:

$$v = \sum_{w \in E^0} a_{vw} w.$$

Recall that for a unital K-algebra R, the *V-monoid of* R, denoted by $\mathcal{V}(R)$, is the set of isomorphism classes of finitely generated projective left R-modules. We denote the elements of $\mathcal{V}(R)$ using brackets, for example, $[R] \in \mathcal{V}(R)$ represents the isomorphism class of the left regular module ${}_R R$. Then $\mathcal{V}(R)$ is an abelian monoid, with operation $\oplus$, and zero element $[0]$, where 0 is the zero R-module. Also, the moniod $(\mathcal{V}(R), \oplus)$ is conical; that is, the sum of any two nonzero elements of $\mathcal{V}(R)$ is nonzero, or rephrased, $\mathcal{V}(R)^* = \mathcal{V}(R) - [0]$ is a semigroup under $\oplus$. The group completion of $\mathcal{V}(R)$ is denoted by $K_0(R)$ and called the *Grothendieck group* of R.

Theorem 2.6 [10, Theorem 3.5]. *As monoids,* $\mathcal{V}(L(E)) \cong M_E$ *and* $[L(E)] \leftrightarrow \sum_{v \in E^0} [v]$ *under this isomorphism.*

DEFINITION 2.7

A unital K-algebra A is called *purely infinite simple* in case A is not a division ring, and A has the property that for every nonzero element x of A, there exists $b, c \in A$ for which $bxc = 1_A$.

The finite graphs E for which the Leavitt path algebra $L(E)$ is purely infinite simple have been explicitly described in [4].

Theorem 2.8. *$L(E)$ is purely infinite simple if and only if E is sink-free, satisfies Condition (L), and only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

In other words, the graph E satisfies the following properties: every vertex in E connects to every cycle of E; every cycle in E has an exit; and E contains at least one cycle.

It is shown in [9, Corollary 2.2], that if A is a unital purely infinite simple K-algebra, then the semigroup $(\mathcal{V}(A)^*, \oplus)$ is in fact a group, and moreover, that $\mathcal{V}(A)^* \cong K_0(A)$, the Grothendieck group of A. For unital Leavitt path algebras, the converse is true as well: if $\mathcal{V}(L(E))^*$ is a group, then $L(E)$ is purely infinite simple. This converse is not true for general K-algebras.

Theorem 2.9. *If $L(E)$ is unital purely infinite simple, then*

$$K_0(L(E)) \cong \mathcal{V}(L(E))^* \cong M_E^*.$$

The following important theorem will be used to yield a number of key results in the following sections.

Theorem 2.10 ((Restricted) algebraic KP theorem) [7, Corollary 2.7]. *Suppose E and F are finite graphs for which the Leavitt path algebras $L(E)$ and $L(F)$ are purely infinite simple. Suppose that there is an isomorphism $\varphi : K_0(L(E)) \to K_0(L(F))$ for which $\varphi([L(E)]) = [L(F)]$, and suppose also that the two integers $\det(I_{|E^0|}-A_E^t)$ and $\det(I_{|F^0|}-A_F^t)$ have the same sign. Then $L(E) \cong L(F)$ as K-algebras.*

Example 2.11 (*Leavitt algebras*). For any integer $m \geq 2$, $L(1, m)$ is the free associative K-algebra in $2m$ generators $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$, subject to the relations

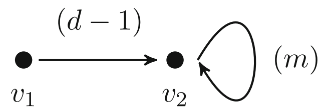
$$y_i x_j = \delta_{i,j} 1_K \quad \text{and} \quad \sum_{i=1}^{m} x_i y_i = 1_K.$$

These algebras were first defined and investigated in [12] in the context of finding counter-examples for the invariant basis number problem, and formed the motivating examples for the more general notion of Leavitt path algebra. It is easy to see that for $m > 2$, if R_m is the graph having one vertex and m loops, then $L(R_m) \cong L(1, m)$. From Theorem 2.8, it follows that $L(R_m)$ is unital purely infinite simple and hence $K_0(L(R_m)) \cong M_{R_m}^*$ is the cyclic group $\mathbb{Z}_{m-1}$, where the regular module $[L(R_m)]$ in $K_0(L(R_m))$ corresponds to 1 in $\mathbb{Z}_{m-1}$.

Unital purely infinite simple Leavitt path algebras $L(E)$ whose corresponding K_0 groups are cyclic and for which $\det(I_{|E^0|} - A_E^t) \leq 0$ are relatively well-understood, and arise as matrix rings over the Leavitt algebras $L(1, m)$, as follows: Let $d \geq 2$, and consider the graph R_m^d having two vertices v_1, v_2; $d - 1$ edges from v_1 to v_2; and m loops at v_2.

It is shown in [1] that $L(R_m^d)$ is isomorphic to the matrix algebra $M_d(L(1, m))$. By standard Morita equivalence theory, we have that $K_0(M_d(L(1, m))) \cong K_0(L(1, m))$.

Moreover, the element $[M_d(L(1,m))]$ of $K_0(M_d(L(1,m)))$ corresponds to the element d in $\mathbb{Z}_{m-1}$. In particular,

$R_m^d =$ 

the element $[M_{m-1}(L(1,m))]$ of $K_0(M_{m-1}(L(1,m)))$ corresponds to $m-1 \equiv 0$ in $\mathbb{Z}_{m-1}$. Finally, an easy computation yields that $\det(I_2 - A^t_{R^d_m}) = -(m-1) \leq 0$ for all m, d. Therefore, by invoking the algebraic KP theorem, the previous discussion immediately yields the following.

PROPOSITION 2.12

Suppose that E is a graph for which $L(E)$ is unital purely *infinite simple. Suppose that M_E^* is isomorphic to the cyclic group $\mathbb{Z}_{m-1}$, via an isomorphism which takes the element $\sum_{v\in E^0}[v]$ of M_E^* to the element d of $\mathbb{Z}_{m-1}$. Finally, suppose that* $\det(I_{|E^0|} - A_E^t) \leq 0$. *Then $L(E) \cong M_d(L(1,m))$.*

2.1.1. *Computation of Grothendieck group.* Let E be a finite directed graph for which $|E^0| = n$. We view $I_n - A_E^t$ both as a matrix, and as a linear transformation $I_n - A_E^t : \mathbb{Z}^n \to \mathbb{Z}^n$, via left multiplication (viewing elements of $\mathbb{Z}^n$ as column vectors). As discussed in [1, Section 3], we have as follows.

PROPOSITION 2.13

If $L(E)$ is purely infinite simple, *then*

$$M_E^* \cong K_0(L(E)) \cong \mathbb{Z}^n/\mathrm{Im}(I_n - A_E^t) = \mathrm{Coker}(I_n - A_E^t).$$

Under this isomorphism $[v_i] \mapsto \vec{b_i} + \mathrm{Im}(I_n - A_E^t)$, where $\vec{b_i}$ is the element of $\mathbb{Z}^n$ which is 1 in the i-th coordinate and 0 elsewhere.

Let $M \in M_n(\mathbb{Z})$ and view M as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$ via left multiplication on columns. The cokernel of M is a finitely generated abelian group, having at most n summands; as such, by the invariant factors version of the fundamental theorem of finitely generated Abelian groups, we have

$$\mathrm{Coker}(M) \cong \mathbb{Z}_{s_l} \oplus \mathbb{Z}_{s_{l+1}} \oplus \cdots \oplus \mathbb{Z}_{s_n},$$

for some $1 \leq l \leq n$, where either $n = l$ and $s_n = 1$ (i.e., Coker(M) is a trivial group), or there are (necessarily unique) nonnegative integers $s_l, s_{l+1}, \ldots, s_n$, for which the nonzero values $s_l, s_{l+1}, \ldots, s_r$ satisfy $s_j \geq 2$ for $1 \leq j \leq r$ and $s_i | s_{i+1}$ for $l \leq i \leq r-1$, and $s_{r+1} = \cdots = s_n = 0$. Coker$(M)$ is a finite group if and only if $r = n$. In case $l > 1$, we define $s_1 = s_2 = \cdots = s_{l-1} = 1$. Clearly then we have

$$\mathrm{Coker}(M) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_l} \oplus \cdots \oplus \mathbb{Z}_{s_n},$$

since any additional direct summands are isomorphic to the trivial group $\mathbb{Z}_1$.

We note that if P, Q are invertible in $M_n(\mathbb{Z})$ (hence their determinant is ± 1), then Coker$(M) \cong$ Coker(PMQ). In other words, if $N \in M_n(\mathbb{Z})$ is a matrix which is constructed by performing any sequence of $\mathbb{Z}$-elementary row (or column) operations starting with M, then $\text{Coker}(M) \cong \text{Coker}(N)$ as abelian groups.

DEFINITION 2.14

Let $M \in M_n(\mathbb{Z})$, and suppose $\text{Coker}(M) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_n}$ as described above. The *Smith Normal Form* of M ((SNF(M), in short)) is the $n \times n$ diagonal matrix $\text{diag}(s_1, s_2, \ldots, s_r, 0, \ldots, 0)$.

For any matrix $M \in M_n(\mathbb{Z})$, the Smith normal form of M exists and is unique. If $D \in M_n(\mathbb{Z})$ is a diagonal matrix with entries $d_1, d_2, \ldots, d_n$, then clearly $\text{Coker}(D) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}$. We also note the following.

PROPOSITION 2.15

Let $M \in M_n(\mathbb{Z})$, and let S denote the Smith normal form of M. Suppose the diagonal entries of S are $s_1, s_2, \ldots, s_n$. Then

$$\text{Coker}(M) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_n}.$$

In particular, if there are no zero entries in the Smith normal form of M, then $|\text{Coker}(M)| = s_1 s_2 \ldots s_n = |\det(S)| = |\det(M)|$.

Proposition 2.15 yields the following.

PROPOSITION 2.16

Let E be a finite graph with $|E^0| = n$ and adjacency matrix A_E. Suppose that $L(E)$ is purely infinite simple. Let S be the Smith normal form of the matrix $I_n - A_E^t$, with diagonal entries $s_1, s_2, \ldots, s_n$. Then

$$K_0(L(E)) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_n}.$$

Moreover, if $K_0(L(E))$ is finite, then an analysis of the Smith normal form of the matrix $I_n - A_E^t$ yields

$$|K_0(L(E))| = \left|\det(I_n - A_E^t)\right|,$$

Conversely, $K_0(L(E))$ is infinite if and only if $\det(I_n - A_E^t) = 0$ *and in this case* $\text{rank}(K_0(L(E)) = \text{nullity}(I_n - A_E^t)$.

We record the following theorem which will be used in computations of Smith normal forms in later sections.

Theorem 2.17 (Determinant divisors theorem) [14, Theorem II.9]. *Let $M \in M_n(\mathbb{Z})$. Define $\alpha_0 := 1$, and for each $1 \leq i \leq n$, define the i-th determinant divisor of M to be the integer*

$$\alpha_i := \textit{the greatest common divisor of the set of all } i \times i \textit{ minors of } M.$$

Let $s_1, s_2, \ldots, s_n$ denote the diagonal entries of the Smith normal form of M, and assume that each s_i is nonzero. Then

$$s_i = \frac{\alpha_i}{\alpha_{i-1}}$$

for each $1 \le i \le n$.

2.2 *Weighted Cayley graphs and circulant matrices*

Let E be a graph and $w : E^1 \to \mathbb{N}$ be a function, the *weighted graph* of E associated to w is a new graph $E_w = (E^0, E_w^1, s_w, r_w)$, where $E_w^1 := \{e_1 \ldots e_{w(e)} \mid e \in E^1\}$, $s_w(e_i) = s(e)$ and $r_w(e_i) = r(e)$.

Recall that given a group G, and a subset $S \subseteq G$, the *associated Cayley graph* $\mathsf{Cay}(G, S)$ is the directed graph $E(G, S)$ with vertex set $\{v_g \mid g \in G\}$, and in which there is an edge $e(g, h)$ from v_g to v_h in case there exists (a necessarily unique) $s \in S$ with $h = gs$ in G. Thus, in $\mathsf{Cay}(G, S)$, at every vertex v_g, the number of edges emitted is $|S|$. The identity of G is in S if and only if $\mathsf{Cay}(G, S)$ contains a loop at every vertex.

DEFINITION 2.18

Let G be a group, $S \subseteq G$ and $w : S \to \mathbb{N}$ be a map. Then w induces a map (also denoted by w) from the set of edges of $\mathsf{Cay}(G, S)$ to $\mathbb{N}$ by $e(g, h) \mapsto w(s)$ whenever $h = gs$. The weighted graph of $\mathsf{Cay}(G, S)$ associated to the map w is called the *weighted Cayley graph* (or *w-Cayley graph*) and denoted by $\mathsf{Cay}(G, S, w)$

In particular, $\mathsf{Cay}(G, S)$ is a special case of $\mathsf{Cay}(G, S, w)$ when w is the constant map $w(e) = 1$ for every edge e. In this case, we say $\mathsf{Cay}(G, S)$ is unweighted.

Remark 2.19. $\mathsf{Cay}(G, S, w)$ is strongly connected if and only if $\langle S \rangle = G$. In particular, $\mathsf{Cay}(\langle S \rangle, S, w)$ is a connected component of $\mathsf{Cay}(G, S, w)$, where $\langle S \rangle$ is the subgroup generated by S.

Notation 2.20. For a positive integer n, let $G = \mathbb{Z}_n$, and S be any non-empty subset of G. We denote the w-Cayley graph $\mathsf{Cay}(G, S, w)$ simply by $C_n(S, w)$.

In other words, if $S = \{s_1, s_2, \ldots s_k\}$ then the w-Cayley graph $C_n(S, w)$ is the directed graph with vertex set $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and edge set $\{e_l(i, s_j) \mid 0 \le i \le n-1, 1 \le j \le k, 1 \le l \le w(s_j)\}$ for which $s(e_l(i, s_j)) = v_i$, and $r(e_l(i, s_j)) = v_{i+s_j}$ where indices are interpreted modulo n. Therefore, $C_n(S, w)$ is a finite graph.

DEFINITION 2.21

For a positive integer n, let $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{Q}^n$. Consider the shift operator $T : \mathbb{Q}^n \to \mathbb{Q}^n$, defined by $T(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2})$. The *circulant matrix* circ(c), associated with $\mathbf{c}$ is the $n \times n$ matrix C whose k-th row is $T^{k-1}(\mathbf{c})$, for $k = 1, 2, \ldots .n$. Thus C is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}.$$

In other words, a circulant matrix is obtained by taking an arbitrary first row, and shifting it cyclically one position to the right in order to obtain successive rows. The (i, j) element of C is c_{j-i}, where subscripts are taken modulo n.

Note that $A_{C_n(S,w)}$ is the $n \times n$ matrix with the (i, j)-th entry is $w(s)$ if $i+s = j$ modulo n, for some $s \in S$, otherwise 0. Hence $A_{C_n(S,w)}$ is a circulant matrix with non-negative integer entries. In the case of unweighted Cayley graph $C_n(S)$, the adjacency matrix is binary circulant matrix.

DEFINITION 2.22

For $\mathbf{c} \in \mathbb{Q}^n$, let $C = \mathrm{circ}(\mathbf{c})$. The *representer polynomial* of C is defined to be the polynomial $P_C(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \mathbb{Q}[x]$.

Lemma 2.23. *Let* $C = \mathrm{circ}(\mathbf{c})$ *be a circulant matrix. Then the eigenvalues of* C *equal* $P_C(\zeta_n^k) = c_0 + c_1\zeta_n^k + \cdots + c_{n-1}\zeta_n^{k(n-1)}$ *for* $k = 0, 1, \dots, n-1$, *where* $\zeta_n = e^{\frac{2\pi i}{n}}$, *the primitive* n*-th root of unity. Further*,

$$\det(C) = \prod_{l=0}^{n-1}\left(\sum_{k=0}^{n-1} c_j \zeta_n^{lk}\right).$$

For a proof of Lemma 2.23, we refer the reader to [11, Theorem 6]. Note that the n-th cyclotomic polynomial, denoted by

$$\Phi_n(x) = \prod_{\substack{1 \le a < n \\ \gcd(a,n)=1}} (x - \zeta_n^a),$$

is an element of $\mathbb{Z}[x]$. Also, $x^n - 1 = \prod_{d|n} \Phi_d(x)$. Since $\Phi_n(x)$ is the minimal polynomial of ζ_n, $f(\zeta_n) = 0$ for some $f(x) \in \mathbb{Z}[x]$ implies $\Phi_n(x)$ divides $f(x)$. By applying Lemma 2.23, we get as follows.

Lemma 2.24. *Let* $C = \mathrm{circ}(\mathbf{c})$. *Then the following are equivalent*:

(*a*) C *is singular*,
(*b*) $P_C(\zeta_n^k) = 0$ *for some* $k \in \mathbb{Z}$,
(*c*) *The polynomials* $P_C(x)$ *and* $x^n - 1$ *are not relatively prime.*

3. Leavitt path algebras of $C_n(S, w)$

Theorem 3.1. *Let* G *be a finite group,* S *its generating set and* $w : S \to \mathbb{N}$ *a weight function. Let* $W = \sum_{s \in S} w(s)$. *Then the following are equivalent*:

(*a*) $L(\mathsf{Cay}(G, S, w))$ *is purely infinite simple*,

(*b*) $W \geq 2$,
(*c*) $L(\mathsf{Cay}(G, S, w))$ *does not have invariant basis number.*

Proof.

(1) $\Rightarrow$ (2) $\Leftarrow$ (3). Let $|G| = n$. If $W = 1$, then $|S| = 1$. Setting $S = \{g\}$, we have G is cyclic group generated by g. Hence $\mathsf{Cay}(G, S, w)$ is the graph C_n which is cycle of length n, which does not satisfy condition L, which contradicts (1). By [2, Theorem 3.8 and 3.10], $L(C_n) \cong M_n(K[x, x^{-1}])$ which has invariant basis number.
(2) $\Rightarrow$ (1). Let $W \geq 2$. In $\mathsf{Cay}(G, S, w)$, the number of edges emitted at each vertex v_g is W. So there is at least two edges emitted from each vertex. This also implies condition (L). Since $\langle S\rangle = G$, $\mathsf{Cay}(G, S, w)$ is strongly connected. Hence for any vertex v_g there is a non-trivial path connecting v_g to v_1 and vice versa. Therefore $\mathsf{Cay}(G, S, w)$ contains a cycle and there is no non-trivial hereditary subset of vertices.
(2) $\Rightarrow$ (3). For a finite graph E, $L(E)$ has invariant basis number if and only if for each pair of positive integers m and n,

$$m \sum_{v\in E^0} [v] = n \sum_{v\in E^0} [v] \text{ in } M_E \Rightarrow m = n.$$

In $M_{\mathsf{Cay}(G,S,w)}$, for each v_g, we have $[v_g] = \sum_{s\in S} w(s)[v_{gs}]$ and hence

$$\sum_{g\in G}[v_g] = \sum_{g\in G}\sum_{s\in S} w(s)[v_{gs}] = \sum_{s\in S}\sum_{g\in G} w(s)[v_{gs}] = \sum_{s\in S} w(s) \sum_{g\in G}[v_{gs}] = W \sum_{g\in G}[v_{gs}].$$

Since G is a finite group we have $G = \{gs \mid g \in G\}$ and hence $\sum_{g\in G}[v_{gs}] = \sum_{g\in G}[v_g]$. Hence we have $\sum_{g\in G}[v_g] = W \sum_{g\in G}[v_g]$. If $W \geq 2$, then $L(\mathsf{Cay}(G, S, w))$ does not have invariant basis number. □

COROLLARY 3.2 [13, Proposition 4.1, Theorem 4.2]

Let G be a finite group, S its generating set. Then the following are equivalent:

(1) $L(\mathsf{Cay}(G, S))$ *is purely infinite simple*,
(2) $L(\mathsf{Cay}(G, S))$ *does not have invariant basis number*,
(3) $|S| \geq 2$.

Proof. In this case, $W = |S|$. □

From here on, we work with the following assumption.

Assumption 3.3. Let $n \in \mathbb{N}$, $S = \{s_1, s_2, \dots s_k\} \subseteq \mathbb{Z}_n$. $s_1 < s_2 < \cdots < s_k$. Further set $W := \sum_{s_j\in S} w(s_j)$.

Theorem 3.4. *Let $\langle S\rangle = \mathbb{Z}_n$ and $W \geq 2$. Then in the group $M^*_{C_n(S,w)}$, the order of $\sum_{i=0}^{n-1}[v_i]$ divides $W-1$. Further if $\gcd(W-1, n) = 1$, then order of $\sum_{i=0}^{n-1}[v_i]$ is $W-1$.*

Proof. Let $S = \{s_1, s_2, \ldots, s_k\}$. Then in $M^*_{C_n(S,w)}$, we have the following relations:

$$[v_i] = \sum_{s_j \in S} w(s_j)[v_{i+s_j}].$$

Let $\sigma = \sum_{i=0}^{n-1}[v_i]$. Then using the defining relations in $M^*_{C_n(S),w}$, we have

$$\sigma = \sum_{i=0}^{n-1}[v_i] = \sum_{i=0}^{n-1}\left(\sum_{s_j \in S} w(s_j)[v_{i+s_j}]\right) = \sum_{s_j \in S} w(s_j)\left(\sum_{i=0}^{n-1}[v_{i+s_j}]\right)$$
$$= \sum_{s_j \in S} w(s_j)\left(\sum_{i=0}^{n-1}[v_i]\right) = \left(\sum_{s_j \in S} w(s_j)\right)\sigma = W\sigma.$$

Thus, in the group $M^*_{C_n(S,w)}$, we have $(W-1)\sigma = 0$. This proves the first part of the theorem.

By Theorem 2.13, $M^*_{C_n(S,w)} \cong \mathrm{Coker}(I_n - A^t_{C_n(S,w)})$, and under the isomorphism $[v_i] \mapsto \vec{b_i} + \mathrm{Im}(I_n - A^t_{C_n(S,w)})$, where $\vec{b_i}$ is the element of $\mathbb{Z}^n$ which has 1 in the i-th coordinate and 0 elsewhere.

Hence for a natural number d, $d\sigma = 0$ in $M^*_{C_n(S,w)}$ if and only if $d\vec{v} \in \mathrm{Im}(I_n - A^t_{C_n(S,w)})$, where $\vec{v} = (1, 1, \ldots, 1)^t$. This is equivalent to $\vec{u} - A^t\vec{u} = d\vec{v}$ for some $\vec{u} = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Z}^n$, which in turn equivalent to

$$u_l - \sum_{s_j \in S} w(s_j)u_{n-s_j+l} = d \qquad 0 \le l \le n-1$$

Adding all the above equations, we get

$$\sum_{l=0}^{n-1} u_l - \sum_{l=0}^{n-1}\sum_{s_j \in S} w(s_j)u_{n-s_j+l} = nd, \quad \text{LHS} = \sum_{l=0}^{n-1} u_l - \sum_{s_j \in S}\sum_{l=0}^{n-1} w(s_j)u_{n-s_j+l}$$
$$= \sum_{l=0}^{n-1} u_l - \sum_{s_j \in S} w(s_j)\sum_{l=0}^{n-1} u_{n-s_j+l}$$
$$= \left(1 - \sum_{s_j \in S} w(s_j)\right)\sum_{l=0}^{n-1} u_l$$
$$= (1-W)\sum_{l=0}^{n-1} u_l.$$

Thus $W-1$ divides nd. If $\gcd(W-1, n) = 1$, then $W-1$ divides d. In particular, when $\gcd(W-1, n) = 1$ order of $\sum_{i=0}^{n-1}[v_i]$ is $W-1$. □

Assumption 3.5. In what follows, we always assume that $\langle S \rangle = \mathbb{Z}_n$ and $W \ge 2$.

As we noted in 2.23, for a circulant matrix C,

$$\det(C) = \prod_{l=0}^{n-1} \left(\sum_{j=0}^{n-1} c_j \zeta_n^{lj} \right),$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$, the primitive n-th root of unity. For $C_n(S, w)$, the adjacency matrix $A_{C_n(S,w)}$ is circulant. Also, $I_n - A^t_{C_n(S,w)}$ is circulant (with integer entries). Let $S = \{s_1, s_2, \ldots, s_k\}$. Then

$$\det(I_n - A^t_{C_n(S)}) = \det(I_n - A_{C_n(S)}) = \prod_{l=0}^{n-1} \left(1 - \sum_{s_j \in S} w(s_j) \zeta_n^{ls_j} \right).$$

PROPOSITION 3.6

Let $S_0 := \{j \in S \mid j \equiv 0 \pmod 2\}$, $S_1 := \{j \in S \mid j \equiv 1 \pmod 2\}$, $W_0 := \sum_{s_j \in S_0} w(s_j)$, *and* $W_1 := \sum_{s_j \in S_1} w(s_j)$. *Then* $\det(I_n - A^t_{C_n(S,w)}) > 0$ *if and only if* n *is even and* $1 + W_1 < W_0$.

Proof. Let $P(x) = 1 - \sum_{s_j} w(s_j) x^{s_j}$ be the representer polynomial of $I_n - A^t_{C_n(S,w)}$. Let $z_l = P(\zeta_n^l) = 1 - \sum_{s_j} w(s_j) \zeta^{ls_j}$. It is easy to see that $z_0 = 1 - \sum_{s_j \in S} w(s_j) = 1 - W < 0$ and $z_{n-l} = \overline{z_l}$ for all l. Thus $\det(I_n - A^t_{C_n(S)}) > 0$ if and only if n is even and $z_{\frac{n}{2}} < 0$. Since

$$z_{\frac{n}{2}} = 1 - \sum_{j \text{ even}} w(s_j) + \sum_{j \text{ odd}} w(s_j),$$

Thus $z_{\frac{n}{2}} < 0$ iff $1 + W_1 < W_0$. □

PROPOSITION 3.7

Let $P(x) \in \mathbb{Z}[x]$ *be the representer polynomial associated with the circulant matrix* $I_n - A^t_{C_n(S,w)}$. *Then* $K_0(L(C_n(S, w)))$ *is infinite if and only if* $P(x)$ *and* $x^n - 1$ *are relatively prime.*

Proof. Follows from Lemma 2.24 and Proposition 2.16. □

In order to compute the Grothendieck group of the Leavitt path algebra of $C_n(S, w)$, we look at the generating relations for $M^*_{C_n(S,w)}$,

$$[v_i] = \sum_{s_j \in S} w(s_j)[v_{i+s_j}],$$

where $0 \leq i \leq n - 1$ (subscripts are modulo n) and $S = \{s_1, s_2, \ldots, s_k\}$, ($s_l < s_m$ for $l < m$). Any statement about $[v_0]$ in $M^*_{C_n(S,w)}$ has an analogous statement for $[v_k]$ for $0 \leq k \leq n - 1$, by symmetry of relations.

DEFINITION 3.8

The *companion matrix* of the monic polynomial $p(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + t^n$, the $n \times n$ matrix defined as

$$T(p) = \begin{pmatrix} 0\ 0 \ldots 0 & -c_0 \\ 1\ 0 \ldots 0 & -c_1 \\ 0\ 1 \ldots 0 & -c_2 \\ \vdots\ \vdots\ \ddots\ \vdots & \vdots \\ 0\ 0 \ldots 1 & -c_{n-1} \end{pmatrix}.$$

Let a linear recursive sequence is of the form

$$u_{n+k} - c_{n-1}u_{n+k-1} - \cdots - c_0 u_n = 0 \quad (n \geq 0),$$

where $c_0, c_1, \ldots, c_{n-1}$ are constants. The *characteristic polynomial* of the above linear recursive sequence is defined as $p(t) = t^n - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0$ whose companion matrix is

$$T(p) = \begin{pmatrix} 0\ 0 \ldots 0 & c_0 \\ 1\ 0 \ldots 0 & c_1 \\ 0\ 1 \ldots 0 & c_2 \\ \vdots\ \vdots\ \ddots\ \vdots & \vdots \\ 0\ 0 \ldots 1 & c_{n-1} \end{pmatrix}.$$

This matrix generates the sequence in the sense that

$$\left(a_k\ a_{k+1} \ldots a_{k+n-1}\right) T(p) = \left(a_{k+1}\ a_{k+2} \ldots a_{k+n}\right).$$

In particular, the (n, n)-th entry of $T(p)^k$ is u_{n+k-2}. When $0 \notin S$, from the linear recursive relation in $M^*_{C_n(S,w)}$, we have the characteristic polynomial $p(S, w, t) = t^{s_k} - \sum_{s_j \in S} w(s_j)t^{s_k - s_j}$. The companion matrix of $p(S, w, t)$ is denoted by $T_{C_n(S,w)}$, is then the $s_k \times s_k$ matrix

$$T_{C_n(S,w)} = \begin{pmatrix} 0\ 0 \ldots 0 & \\ 1\ 0 \ldots 0 & \\ 0\ 1 \ldots 0 & \\ \vdots\ \vdots\ \ddots\ \vdots & \mathbf{c} \\ & \\ 0\ 0 \ldots 1 & \end{pmatrix},$$

where $\mathbf{c}$ is the last column of $T_{C_n(S,w)}$ which contains entry $w(s_j)$ at positions $s_k - s_j + 1$ and 0 elsewhere.

In $M^*_{C_n(S,w)}$, we observe that by writing the generating relations and then expanding the equation such that the subscripts are kept in *increasing order*, at i-th step we get the coefficients to be the last column of $T^i_{C_n(S,w)}$.

The computation of the Smith normal form of $I_n - A^t_{C_n(S,w)}$ is the key tool for determining the K_0 of the Leavitt path algebra of $C_n(S, w)$. We show that this computation reduces to calculating the Smith normal form of an $s_k \times s_k$ matrix.

Theorem 3.9. *Let $n \in \mathbb{N}$, $S = \{s_1, s_2, \ldots, s_k\} \subsetneq \mathbb{Z}_n$ such that $\langle S \rangle = \mathbb{Z}_n$, $0 \notin S$ and $W \geq 2$. Then* $\mathrm{Coker}(I_n - A^t_{C_n(S,w)}) \cong \mathrm{Coker}(T^n_{C_n(S,w)} - I_n)$.

Proof. Since the Smith normal form of $I_n - A^t_{C_n(S,w)}$ and $A_{C_n(S,w)} - I_n$ are the same, their cokernels are same and we only show that $\mathrm{Coker}(A_{C_n(S,w)} - I_n) \cong \mathrm{Coker}(T^n_{C_n(S,w)} - I_n)$. For simplicity, we write $B = A_{C_n(S,w)} - I_n$ and $T = T_{C_n(S,w)}$. First we observe that

$$B_{pq} = \begin{cases} -1 & \text{if } \ q = p, \\ w(s_j) & \text{if } \ q = p + s_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let P be a $(s_k \times s_k)$ lower triangular matrix given by

$$P_{pq} = \begin{cases} 0 & \text{if } \ q > p, \\ w(s_j) & \text{if } \ p - q = s_k - s_j, \\ 0 & \text{otherwise} \end{cases}$$

and let Q be a $(s_k \times s_k)$ upper triangular matrix given by

$$Q_{pq} = \begin{cases} 0 & \text{if } \ q < p, \\ -1 & \text{if } \ q = p, \\ w(s_j) & \text{if } \ q - p = s_j, \\ 0 & \text{otherwise.} \end{cases}$$

It is direct that P and Q are invertible. Let $R = -Q^{-1}$. Then a direct computation yields $PR = T^{s_k}$, and also $QR = -I_{s_k}$. Let P' be the block matrix $[P \mid 0_{s_k \times (n-s_k)}]$ and Q' be the block matrix $[0_{s_k \times (n-s_k)} \mid Q]$. The $(s_k \times (n - s_k))$ submatrix of B consisting of bottom $s - k$ rows can be written as $P' + Q'$.

The first $(n-s_k)$ reduction steps of the Smith normal form will result in an $(n-s_k) \times (n-s_k)$ identity submatrix in the upper left corner. On the bottom s_k rows, the i-th reduction step adds the i-th column to the sum of $w(s_j)$ times $(i + s_k)$-th columns, then zeros out the i-th column. The matrix that accomplishes this reduction step is

$$P^{-1}TP = \begin{pmatrix} & & \mathbf{r} & & \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $\mathbf{r}$ is the first row contains entry $w(s_j)$ at positions s_k and 0 elsewhere. After i reduction steps, the first $(s_k \times s_k)$ submatrix with nonzero column vectors on the bottom s_k rows will be

$$P \cdot (P^{-1}TP)^i = T^i P.$$

Therefore the first $(n - s_k)$ reduction steps of the Smith normal form will result in the following form:

$$B \sim \begin{pmatrix} I_{(n-s_k)} & 0_{(n-s_k)\times s_k} \\ 0_{s_k \times (n-s_k)} & T^{n-s_k}P + Q \end{pmatrix}.$$

Since $(T^{n-s_k}P + Q)R = T^n - I_{s_k}$,

$$B \sim \begin{pmatrix} I_{(n-s_k)} & 0_{(n-s_k)\times s_k} \\ 0_{k\times(n-s_k)} & T^n - I_{s_k} \end{pmatrix}.$$

Hence $\mathrm{Coker}(B) \cong \mathrm{Coker}(T^n - I_{s_k})$. □

4. Illustrations

As illustrations of the above discussion we consider some simple cases when $W \geq 2$ and $\sum_{i=0}^{n-1}[v_i]$ is the identity in $M^*_{C_n(S,w)}$, which recovers the examples obtained in [5,8], and [6].

4.1 $S = \mathbb{Z}_n$

In this subsection, we only look at the following two simple cases when $\sum_{i=0}^{n-1}[v_i]$ is the identity in $M^*_{C_n(\mathbb{Z}_n,w)}$

DEFINITION 4.1

Let n, l be two positive integers. We define $K_n^{(l)}$ to be the graph with n vertices $v_0, v_1, \ldots v_{n-1}$, in which there is exactly one edge from v_i to v_j for each $0 \leq i \neq j \leq n-1$ and l loops at each vertex. We call $K_n^{(l)}$ the *complete n-graph with l loops*.

Theorem 4.2. *Let $n \geq 2$ be a positive integer.*

(1) $L(K_n^{(1)}) \cong L(1, n)$.
(2) *Let E be a finite graph such that $L(E)$ is purely infinite simple. If $K_0(L(E)) \cong \mathbb{Z}^n$ and $[L(E)]$ is identity in $K_0(L(E))$, then $L(E) \cong L(K_{n+1}^{(2)})$.*

Proof. Let $w_l : S \to \mathbb{N}$ be the weight function defined by $w_l(0) = l$ and $w_l(i) = 1$ for $1 \leq i \leq n-1$. Then it is direct that $C_n(\mathbb{Z}_n, w_l) \cong K_n^{(l)}$.

(1) We note that

$$\det(I_n - A^t_{K_n^{(1)}}) = \prod_{l=0}^{n-1}(-1)\left(\sum_{j=1}^{n-1}\zeta^{lj}\right) = -(n-1) < 0.$$

Also we have $W - 1 = |S| - 1 = n - 1$. So $\gcd(W-1, n) = 1$ and hence $\sum_{i=0}^{n-1}[v_i]$ is the identity in $M^*_{K_n^{(1)}}$. Also determinant divisors theorem yields that

$$\mathrm{SNF}(I_n - A^t_{K_n^{(1)}}) = \mathrm{diag}(1, 1, \ldots, 1, n-1).$$

Hence, $K_0(L(K_n^{(1)})) \cong \mathbb{Z}_{n-1}$. By Proposition 2.12, the result follows.

(2) We note that $(I_n - A^t_{K_n^{(2)}})$ is the $n \times n$ matrix with every entry -1. Hence $\det(I_n - A^t_{K_n^{(2)}}) = 0$ and $\mathrm{rank}(I_n - A^t_{K_n^{(2)}}) = 1$. Therefore if $n \geq 2$, then

$$K_0(L(K_n^{(2)})) \cong \mathbb{Z}^{n-1}.$$

Also in $K_0(L(K_n^{(2)}))$,

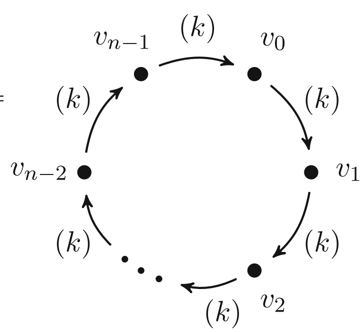
$$\sigma = \sum_{i=0}^{n-1}[v_i] = [v_0] + \sum_{i=1}^{n-1}[v_i] = \left(2[v_0] + \sum_{i=1}^{n-1}[v_i]\right) + \sum_{i=1}^{n-1}[v_i] = 2\sum_{i=0}^{n-1}[v_i] = 2\sigma.$$

Hence $\sum_{i=0}^{n-1}[v_i]$ is the identity in $K_0(L(K_n^{(2)}))$. Applying algebraic KP theorem, we have the result. □

4.2 $|S| = 1$

Let $S = \{i\}$. Since $\langle S\rangle = \mathbb{Z}_n$, $\gcd(i, n) = 1$ and the weight function $w : S \to \mathbb{N}$ is given by $w(i) = W$. Let D_n^k be the graph with n vertices $v_0, v_1, \ldots, v_n$ and kn edges such that every vertex v_i emit k edges to v_{i+1}. We call D_n^k an k-cycle of length n.

$D_n^k =$



It is easy to see that $C_n(S, w) \cong D_n^W$.

The generating relations for $M^*_{D_n^W}$ are given by

$$[v_i] = W[v_{i+1}]$$

for $0 \leq i \leq n$, where the subscripts are interpreted mod n. So for each $0 \leq i \leq n$ we have that

$$[v_i] = W[v_{i+1}] = W^2[v_{i+2}] = \cdots = W^{n-i}[v_{n-1}] = W^{n+1-i}[v_0].$$

In particular, each $[v_i]$ is in the subgroup of $M^*_{D_n^W}$ generated by $[v_0]$. Since the set $\{[v_i] \mid 0 \leq i \leq n-1\}$ generates $M^*_{D_n^W}$, we conclude that $M^*_{D_n^W}$ is cyclic, and $[v_0]$ is a generator.

We also observe that

$$\det(I_n - A^t_{D_n^W}) = \prod_{l=0}^{n-1}(1 - W\zeta^l) = 1 - W^n < 0.$$

We conclude that $|K_0(L(D_n^W))| = W^n - 1$. Thus we have

$$K_0(L(C_n(S, w)) \cong M^*_{D_n^W} \cong \mathbb{Z}_{W^n-1}.$$

PROPOSITION 4.3

Let $S = \{i\}$, $\gcd(i, n) = 1$, *and* $\gcd(W - 1, n) = 1$, *then*

$$L(C_n(S, w)) \cong M_{W^n-1}(L(1, W^n)).$$

Proof. $\sum_{i=0}^{n-1}[v_i]$ is the identity in the group $M^*_{C_n(S,w)}$. Hence by Proposition 2.12 the result follows. □

COROLLARY 4.4 [5, Proposition 3.4]

Assume the hypothesis of Proposition 4.3 *and* $W = 2$, *then* $L(C_n(S, w)) \cong M_{2^n-1}(L(1, 2^n))$.

4.3 $|S| = 2$

Let $S = \{s_1, s_2\}$ with $s_1 < s_2$. Let $a, b \in \mathbb{N}$. We define $w(s_1) := a$ and $w(s_2) := b$. Thus $W = a + b \geq 2$. Since $\langle S\rangle = \mathbb{Z}_n$, it is sufficient to consider only the following subcases:

(1) $s_1 = 0$ and $s_2 = 1$.
(2) $s_1 = 1$.
(3) s_1 and s_2 divide n with $1 < s_1 < s_2$, and $\gcd(s_1, s_2) = 1$.

In what follows we consider these subcases separately.

Lemma 4.5. *In each of the above subcases if* $a = b = 1$, *then* $\sum_{i=0}^{n-1}[v_i]$ *is the identity in* $M^*_{C_n(S,w)}$.

Proof. Since $W - 1 = |S| - 1 = 1$ in these subcases we have $\gcd(W - 1, n) = 1$ and the result follows from Theorem 3.4. □

PROPOSITION 4.6

Let $n, a, b \in \mathbb{N}$ *be fixed. Let* $0 \leq s_1 < s_2 \leq n - 1$. *Consider the w-Cayley graph* $C_n(S, w)$ *where* $S = \{s_1, s_2\}$ *and* $w(s_1) = a$, $w(s_2) = b$. *Then* $\det(I_n - A^t_{C_n(S,w)}) = 0$ *if and only if exactly one of the following occurs*:

(1) $a = b = 1$, $n \equiv 0 \pmod 6$, $s_2 \equiv 5s_1 \pmod 6$.
(2) $a = b + 1$, *n is even*, *s_1 is even*, *s_2 is odd.*
(3) $b = a + 1$, *n is even*, *s_1 is odd*, *s_2 is even.*

Proof. Let $\Delta = \det(I_n - A^t_{C_n(S,w)})$ and $z_l = a\zeta^{ls_1} + b\zeta^{ls_2}$. Since

$$\Delta = \prod_{l=0}^{n-1}\left(1 - a\zeta^{ls_1} - b\zeta^{ls_2}\right)$$

Then $\Delta = 0$ if and only if $z_l = 1$ for some l. We observe that $z_0 = a + b > 1$ and $z_{n-l} = \overline{z_l}$. So we can write

$$\Delta = \begin{cases} (1 - z_0) \prod_{l=1}^{\frac{n-1}{2}} (1 - z_l)(\overline{1 - z_l}) & \text{if } n \text{ is odd} \\ (1 - z_0)(1 - a(-1)^{s_1} - b(-1)^{s_2}) \prod_{l=1}^{\frac{n}{2}-1} (1 - z_l)(\overline{1 - z_l}) & \text{if } n \text{ is even} \end{cases}$$

Hence we can assume $1 \leq l \leq [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Further, $z_l = 1$ implies $\left|a\zeta^{ls_1} + b\zeta^{ls_2}\right| = 1$. So $1 = \left|a\zeta^{ls_1} + b\zeta^{ls_2}\right| \geq ||a| - |b|| = |a - b| \geq 0$. Since $a, b \in \mathbb{N}$, only possiblities are $a = b$, $a = b + 1$, or $b = a + 1$.

Case 1. $a = b$. Let $\theta = \frac{2\pi l}{n}$. Then $z_l = 1$ if and only if

$$a\,(\cos s_1\theta + \cos s_2\theta) = 1 \quad \text{and} \quad a\,(\sin s_1\theta + \sin s_2\theta) = 0.$$

The second equation implies that $s_1\theta \equiv -s_2\theta \pmod{2}\pi$. Substituting back in first equation we get,

$$\begin{aligned} 1 &= a\,(\cos(-s_2\theta) + \cos s_2\theta) = 2a \cos s_2\theta \\ \Rightarrow \cos s_2\theta &= \frac{1}{2a} \Rightarrow \frac{2\pi l}{n} s_2 = \arccos\left(\frac{1}{2a}\right). \end{aligned}$$

Thus $n \in \mathbb{N}$ only if $a = 1$. Assuming $a = 1$, we have $\arccos\frac{1}{2} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Substituting back, we see that

$$\frac{2\pi l s_2}{n} = \frac{\pi}{3} \Rightarrow n = 6s_2 l, \quad \text{or} \quad \frac{2\pi l s_2}{n} = \frac{5\pi}{3} \Rightarrow 5n = 6s_2 l.$$

In either case, $n \equiv 0 \pmod{6}$. Also, $s_2\theta \equiv -s_1\theta \pmod{2}\pi$ implies that for some integer m,

$$(s_2 + s_1)\frac{\pi}{3} = 2\pi m \Rightarrow s_2 + s_1 = 6m \quad \text{or} \quad (s_2 + s_1)\frac{5\pi}{3} = 2\pi m \Rightarrow 5(s_2 + s_1) = 6m.$$

In either case, $s_2 + s_1 \equiv 0 \pmod{6}$, or $s_2 \equiv 5s_1 \pmod{6}$.

Conversely, when $a = 1, n \equiv 0 \pmod{6}$ and $s_2 \equiv 5s_1 \pmod{6}$, then letting $l = 6$ implies that

$$z_l = \omega^{ls_1} + \omega^{ls_2} = \left(e^{\frac{2\pi i}{6}}\right)^{s_1} + \left(e^{\frac{2\pi i}{6}}\right)^{-s_1} = 1$$

Case 2. $a = b + 1$. As in Case 1, let $\theta = \frac{2\pi l}{n}$. Then $z_l = 1$ if and only if

$$(b + 1)\cos s_1\theta + b\cos s_2\theta = 1 \quad \text{and} \quad (b + 1)\sin s_1\theta + b\sin s_2\theta = 0.$$

The second equation implies that $s_1\theta = \arcsin\left(\frac{-b}{b+1}\sin s_2\theta\right)$. Substituting back in the first equation, we get

$$(b + 1)\cos\left(\arcsin\left(\frac{-b}{b + 1}\sin s_2\theta\right)\right) + b\cos s_2\theta = 1.$$

Since $\cos(\arcsin x) = \sqrt{1-x^2}$, we have

$$(b+1)\sqrt{1-\left(\frac{b}{b+1}\sin s_2\theta\right)^2}+b\cos s_2\theta = 1.$$

Hence,

$$\sqrt{b^2+2b+1-b^2\sin^2 s_2\theta} = 1-b\cos s_2\theta.$$

Squaring both sides,

$$b^2\cos^2 s_2\theta+2b+1 = b^2\cos^2 s_2\theta - 2b\cos s_2\theta+1 \Rightarrow \cos s_2\theta = -1.$$

Therefore, $s_2\theta \equiv \pi \pmod{2}\pi$. Substituting $\theta = \frac{2\pi l}{n}$, we see that n is even. Also, $s_2\theta \equiv \pi \pmod{2}\pi$ implies that $(s_2-1)\pi = 2\pi m$ for some integer m. So, $s_2 = 2m+1$ or s_2 is odd. Also since, $s_1\theta = \arcsin\left(\frac{-b}{b+1}\sin s_2\theta\right) = \arcsin(0)$, $s_1\pi = 0$ or π. $(b+1)\cos s_1\pi - b = 1 \Rightarrow s_1\pi \equiv 0 \pmod{2}\pi$. Hence s_1 is even.

Conversely, let n, s_1 be even and s_2 be odd then by taking $l = \frac{n}{2}$, we get

$$z_l = (b+)\omega_l^{s_1}+b\omega_l^{s_2} = (b+1)(-1)^{s_1}+b(-1)^{s_2} = b+1-b = 1.$$

Case 3. $b = a+1$. The proof is similar to that of Case 2. □

COROLLARY 4.7

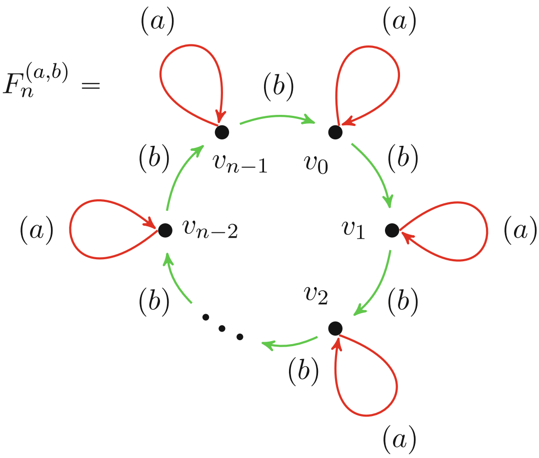
Assume the hypothesis of Proposition 4.6. Further assume that $L(C_n(S,w))$ is unital purely infinite simple. Then $K_0(L(C_n(S,w)))$ is infinite abelian group if and only if one of the following holds:

(1) $a = b = 1, n \equiv 0 \pmod{6}, s_2 \equiv 5s_1 \pmod{6}$.
(2) $a = b+1$, *n is even, s_1 is even, s_2 is odd.*
(3) $b = a+1$, *n is even, s_1 is odd, s_2 is even.*

In which case $\operatorname{rank}(K_0(L(C_n(S,w)))) = n - \operatorname{rank}(I_n - A_{C_n(S,w)})$.

4.3.1. *Subcase* 2.1: $S = \{0, 1\}$. Let $F_n^{(a,b)}$ be the graph with n vertices $v_0, v_1, \ldots, v_{n-1}$ and $ak+bk$ edges such that at every vertex v_l, there are a loops and b edges getting emitted into v_{l+1} (subscripts are mod n).

$F_n^{(a,b)} =$



Then $C_n(S, w) \cong F_n^{(a,b)}$ when $S = \{0, 1\}$. We note that

$$\det(I_n - A^t_{F_n^{(a,b)}}) = \prod_{l=0}^{n-1}(1 - a - b\zeta^l) = (1 - a)^n - b^n.$$

Lemma 4.8. *Let* $n, a, b \in \mathbb{N}$. *Then*

$$\det(I_n - A^t_{F_n^{(a,b)}}) \geq 0 \textit{ if and only if } n \textit{ is even and } a \geq b + 1.$$

Moreover, $\det(I_n - A^t_{F_n^{(a,b)}}) = 0$ *if and only if* n *is even and* $a = b + 1$.

Proof. We refer to the proof of Proposition 4.6. We need to substitute $s_1 = 0$, and $s_2 = 1$. Since,

$$\Delta = \begin{cases} (1 - z_0) \prod_{l=1}^{\frac{n-1}{2}} (1 - z_l)\overline{(1 - z_l)} & \text{if } n \text{ is odd,} \\ (1 - z_0)(1 - a(-1)^j - b(-1)^k) \prod_{l=1}^{\frac{n}{2}-1} (1 - z_l)\overline{(1 - z_l)} & \text{if } n \text{ is even,} \end{cases}$$

$\Delta \geq 0$ if and only if n is even and $a \geq b + 1$, in which case $1 - z_{\frac{n}{2}} = 1 - a + b \leq 0$. Also, it follows that, $\det(I_n - A^t_{F_n^{(a,b)}}) = 0$ if and only if n is even and $a = b + 1$. □

We describe the Smith Normal Form of $I_n - A^t_{F_n^{(a,b)}}$.

Lemma 4.9. *Suppose* $n \in \mathbb{N}$. *Let* T *be the* $n \times n$ *circulant matrix whose first row is* $\vec{t} = ((1 - a), -b, 0, \ldots, 0)$.*Let* $\gcd(1 - a, b) = d$. *Then the Smith Normal Form*

$$\text{SNF}(T) = \text{diag}\left(d, d, \ldots, d, \frac{|(1 - a)^n - b^n|}{d^{n-1}}\right)$$

Proof. In order to compute Smith Normal Form of T, we use at the determinant divisors theorem and look at $i \times i$ minors of T for each $1 \le i \le n$. Let α_i be the gcd of the set of all $i \times i$ minors of T and $\alpha_0 = 1$. Then

$$\mathrm{SNF}(T) = \mathrm{diag}\left(\frac{\alpha_1}{\alpha_0}, \frac{\alpha_2}{\alpha_1}, \dots, \frac{|\det(T)|}{\alpha_{n-1}}\right).$$

By the definition of T, it is easy to observe that $\alpha_i = \gcd\left((a-1)^i, b^i\right) = \gcd(a,b)^i = d^i$ for $1 \le i \le n-1$ and $|\det(T)| = |(1-a)^n - b^n|$. Therefore,

$$\begin{aligned}\mathrm{SNF}(T) &= \mathrm{diag}\left(\frac{d}{1}, \frac{d^2}{d}, \frac{d^3}{d}, \dots, \frac{|(1-a)^n - b^n|}{d^{n-1}}\right)\\ &= \mathrm{diag}\left(d, d, d, \dots, \frac{|(1-a)^n - b^n|}{d^{n-1}}\right)\end{aligned}$$

□

Theorem 4.10. *Let $n, a, b \in \mathbb{N}$ be fixed. Suppose $S = \{0, 1\} \subset \mathbb{Z}_n$, $w : S \to \mathbb{N}$ be defined as $w(0) = a$ and $w(1) = b$. Let $d = \gcd(a-1, b)$. Then*

$$K_0(L(C_n(S,w))) \cong \begin{cases} (\mathbb{Z}_d)^{n-1} \oplus \mathbb{Z} & \text{if } a = b+1 \text{ and } n \text{ is even},\\ (\mathbb{Z}_d)^{n-1} \oplus \mathbb{Z}_{\frac{|(1-a)^n - b^n|}{d^{n-1}}} & \text{otherwise}.\end{cases}$$

Proof. Follows from the above lemmas 4.8 and 4.9. □

Example 4.11. $L(C_n(0,1)) \cong L(1,2)$.

Proof. In Theorem 4.10, we take $a = b = 1$. Then $\gcd(a-1, b) = 1$. Hence $\det(I_n - A^t_{C_n(0,1)}) = -1 < 0$ and $K_0(L(C_n(0,1)))$ is trivial. By Proposition 2.12, we have $L(C_n(0,1)) \cong L(1,2)$. □

The above example was observed in [5], Proposition 3.3.

4.3.2. *Subcase* 2.2: $S = \{1, j\}$ *with* $j > 1$. We note that by Proposition 3.6, $\det(I_n - A^t_{C_n(S,w)}) > 0$ if and only if n, j are even and $b > a+1$. Also by Proposition 4.6, $\det(I_n - A^t_{C_n(S,w)}) = 0$ if and only if o ne of the following occurs:

(1) $a = b = 1, n \equiv 0 \pmod 6, j \equiv 5 \pmod 6$
(2) $b = a+1, n, j$ are even.

In order to compute $K_0(L(C_n(S,w)))$, we apply Theorem 3.9 and compute the Smith normal form of $T^n_{C_n(S,w)} - I_n$. This procedure is performed for unweighted Cayley graph in [6]. However, the record an interesting example here.

4.3.3. *Leavitt path algebras of Cayley graphs of dihedral groups.* Let $\tilde{D}_n$ be the dihedral group of order $2n$, i.e. $\tilde{D}_n = \langle r, s \mid r^n = s^2 = e, rsr = s\rangle$. Let D_n denote the Cayley graph of $\tilde{D}_n$ with respect to the generating subset $S = \{r, s\}$.

The following discussion is taken from [7]. A graph transformation is called standard if it is one of the following types: in-splitting, in-amalgamation, out-splitting, out-amalgamation, expansion, or contraction. For definitions the reader to referred to [3]. If E and F are graphs having no sources and no sinks, a flow equivalence from E to F is a sequence $E = E_0 \to E_1 \to \cdots \to E_n = F$ of graphs and standard graph transformations which starts at E and ends at F.

PROPOSITION 4.12. [3, Corollary 6.3.13]

Suppose E and F are finite graphs with no sources whose corresponding Leavitt path algebras are purely infinite simple. Then E is flow equivalent to F if and only if $\det(I_{|E|} - A_E) = \det(I_{|F|} - A_F)$ *and* $\mathrm{Coker}(I_{|E|} - A_E) \cong \mathrm{Coker}(I_{|F|} - A_F)$.

DEFINITION 4.13 (In-splitting)

Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $r^{-1}(v) \neq \phi$, partition the set $r^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_1^v, \ldots, \mathcal{E}_{m(v)}^v$ where $m(v) \geq 1$. If v is a source then set $m(v) = 0$. Let $\mathcal{P}$ denote the resulting partition of E^1. We form the in-split graph $E_r(\mathcal{P})$ from E using the partition $\mathcal{P}$ as follows:

$$E_r(\mathcal{P})^0 = \{v_i \mid v \in E^0, 1 \leq i \leq m(v)\} \cup \{v \mid m(v) = 0\},$$
$$E_r(\mathcal{P})^1 = \{e_j \mid e \in E^1, 1 \leq j \leq m(s(e))\} \cup \{e \mid m(s(e)) = 0\},$$

and define $r_{E_r(\mathcal{P})}, s_{E_r(\mathcal{P})} : E_r(\mathcal{P})^1 \to E_r(\mathcal{P})^0$ by

$$s_{E_r(\mathcal{P})}(e_j) = s(e)_j \quad \text{and} \quad s_{E_r(\mathcal{P})}(e) = s(e)$$
$$r_{E_r(\mathcal{P})}(e_j) = r(e)_i \quad \text{and} \quad s_{E_r(\mathcal{P})}(e) = s(e)_i, \text{ where } e \in \mathcal{E}_i^{r(e)}.$$

We observe that D_n can be obtained from C_n^{n-1} by the standard operation in-splitting with respect to the partition $\mathcal{P}$ of the edge set of C_n^{n-1} that places each edge in its own singleton partition class. In [8] the collection of Leavitt path algebras $\{L(C_n^{n-1}) \mid n \in \mathbb{N}\}$ is completely described and by Proposition 4.12 we have that the same description holds true if we replace C_n^{n-1} with D_n for every $n \in \mathbb{N}$. Hence we have

Theorem 4.14. *For each $n \in \mathbb{N}$, $\det(I_n - A^t_{D_n}) \leq 0$. And*

(1) *If $n \equiv 1$ or $5 \pmod 6$ then $K_0(L(D_n)) \cong \{0\}$ and $L(D_n) \cong L(1, 2)$.*
(2) *If $n \equiv 2$ or $4 \pmod 6$ then $K_0(L(D_n)) \cong \mathbb{Z}/3\mathbb{Z}$ and $L(D_n) \cong M_3(L(1, 4))$.*
(3) *If $n \equiv 3 \pmod 6$ then $K_0(L(D_n)) \cong (\mathbb{Z}/2\mathbb{Z})^2$*
(4) *If $n \equiv 0 \pmod 6$ then $K_0(L(D_n)) \cong \mathbb{Z}^2$ and $L(D_n) \cong L(K_3^{(2)})$*

4.3.4. *Subcase 2.3*: $S = \{s_1, s_2\}$ *where s_1, s_2 divide n, $1 < s_1 < s_2$ and* $\gcd(s_1, s_2) = 1$. By Proposition 4.6 and by Proposition 3.6, we have that $\det(I_n - A^t_{C_n(S,w)}) = 0$ if and only if one of the following occurs:

(1) $a = b = 1, n \equiv 0 \pmod 6, d_2 \equiv 5d_1 \pmod 6$.
(2) $a = b + 1$, n, d_1 are even, d_2 is odd.
(3) $b = a + 1$, n, d_2 are even, d_1 is odd.

and $\det(I_n - A^t_{C_n(S,w)}) > 0$ if and only if one of the following occurs:

(1) $a > b+1$, n, d_1 are even, d_2 is odd.
(2) $b > a+1$, n, d_2 are even, d_1 is odd.

In order to compute $K_0(L(C_n(S,w)))$, we apply Theorem 3.9 and compute the Smith Normal Form of $T^n_{C_n(S,w)} - I_n$.

We illustrate this when $S = \{d_1, d_2\}$, where d_1, d_2 divides n, $\gcd(d_1, d_2) = 1$ and $a = b = 1$. In this special case we have $\det(I_n - A^t_{C_n(d_1,d_2)}) = 0$ if and only if $n \equiv 0 \pmod 6$ and $d_2 \equiv 5d_1 \pmod 6$. In all other cases, we have $\det(I_n - A^t_{C^k_n}) < 0$. Define $H_{(d_1,d_2)}(n) := |\det(I_n - A^t_{C^k_n})|$. In order to compute $K_0(L(C_n(d_1,d_2)))$, we apply Theorem 3.9 and compute the Smith normal form of $T^n_{C_n(d_1,d_2)} - I_n$.

For $1 \le j, k \in \mathbb{N}$ let us define a sequence $F_{(j,k)}$ recursively as follows:

$$F_{(j,k)}(n) = \begin{cases} 0, & \text{if } \; 1 \le n \le k-2, \\ 1, & \text{if } \; n = k-1, \\ 0, & \text{if } \; n = k, \\ F_{(j,k)}(n-j) + F_{(j,k)}(n-k), & \text{if } \; n \ge k+1. \end{cases}$$

In $M^*_{C_n(d_1,d_2)}$, we have

$$\begin{aligned} [v_0] &= [v_j] + [v_k] \\ &= [v_{2j}] + [v_k] + [v_{j+k}] \\ &= [v_{3j}] + [v_k] + [v_{j+k}] + [v_{2j+k}] \\ &= \ldots . \end{aligned}$$

The coefficients appearing in the above equations are terms in the sequence $F_{(d_1,d_2)}$ and corresponding $T_{C_n(d_1,d_2)}$ is given by the following

Lemma 4.15. *For fixed d_1, d_2, let $d_2 - d_1 = k$. Let $T = T_{C_n(d_1,d_2)}$. Suppose $G(n) := F_{(d_1,d_2)}(n)$ be the sequence defined above. Then for each $n \in \mathbb{N}$,*

$$T^n = \begin{pmatrix} G(n-1) & G(n) & \ldots & G(n+d_2-2) \\ G(n-2) & G(n-1) & \ldots & G(n+d_2-3) \\ \vdots & \vdots & \ldots & \vdots \\ G(n+d_1-1) & G(n+d_1) & \ldots & G(n+d_2+d_1-2) \\ \vdots & \vdots & \ldots & \vdots \\ G(n) & G(n+1) & \ldots & G(n+d_2-1) \end{pmatrix},$$

where the highlighted row is $(k+1)$-th row.

Proof. We prove the lemma by induction on n. We extend the definition of G to the negative integers as well. Then

$$T = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & 1 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} G(0) & G(1) & \dots & G(k-1) & \dots & G(d_2-2) & G(d_2-1) \\ G(-1) & G(0) & \dots & G(k-2) & \dots & G(d_2-3) & G(d_2-2) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ G(d_1) & G(d_1+1) & \dots & G(d_2-1) & \dots & G(d_2+d_1-2) & G(d_2+d_1-1) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ G(2) & G(3) & \dots & G(k+1) & \dots & G(d_2) & G(d_2+1) \\ G(1) & G(2) & \dots & G(k) & \dots & G(d_2-1) & G(d_2) \end{pmatrix}$$

where highlighted column is k-th column.

Thus we have the statement true for $n = 1$. Now suppose

$$T^{n-1} = \begin{pmatrix} G(n-2) & G(n-1) & \dots & G(n+d_2-3) \\ G(n-3) & G(n-2) & \dots & G(n+d_2-4) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-2) & G(n+d_1-1) & \dots & G(n+d_2+d_1-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n-1) & G(n) & \dots & G(n+d_2-2) \end{pmatrix}.$$

Then

$$T^n = T^{n-1}T$$

$$= \begin{pmatrix} G(n-2) & \dots & G(n+k-2) & \dots & G(n+d_2-3) \\ G(n-3) & \dots & G(n+k-3) & \dots & G(n+d_2-4) \\ \vdots & & \vdots & & \vdots \\ G(n+d_1-2) & \dots & G(n+d_2-2) & \dots & G(n+d_2+d_1-3) \\ \vdots & & \vdots & & \vdots \\ G(n-1) & \dots & G(n+k-2) & \dots & G(n+d_2-2) \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} G(n-1) & G(n) & \dots & G(n-2)+G(n+k-2) \\ G(n-2) & G(n-1) & \dots & G(n-3)+G(n+k-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-1) & G(n+d_1) & \dots & G(n+d_1-2)+G(n+d_2-2) \\ \vdots & \vdots & \dots & \vdots \\ G(n) & G(n+1) & \dots & G(n-1)+G(n+k-2) \end{pmatrix}$$

$$= \begin{pmatrix} G(n-1) & G(n) & \dots & G(n+d_2-2) \\ G(n-2) & G(n-1) & \dots & G(n+d_2-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-1) & G(n+d_1) & \dots & G(n+d_2+d_1-2) \\ \vdots & \vdots & \dots & \vdots \\ G(n) & G(n+1) & \dots & G(n+d_2-1) \end{pmatrix}.$$

□

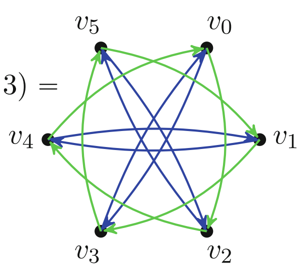
Using the determinant divisors theorem, the Smith normal form of $T^n_{C_n(d_1,d_2)} - I_k$ can be reduced to

$$\mathrm{SNF}(T^n_{C_n(d_1,d_2)} - I_k) = \begin{pmatrix} \alpha_1(n) & & & \\ & \frac{\alpha_2(n)}{\alpha_1(n)} & & \\ & & \ddots & \\ & & & \frac{\alpha_{d_2}(n)}{\alpha_{d_2-1}(n)} \end{pmatrix},$$

where α_i is the greatest common divisor of the set of all $i \times i$ minors of $T^n_{C_n(d_1,d_2)}$.

Example 4.16. Let $n = 6$, $d_1 = 2$, $d_2 = 3$. The corresponding Cayley graph is

$C_6(2,3) =$



Corresponding companion matrix is given by

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$T^6 - I_3 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix},$$

whose Smith Normal Form is given by

$$\mathrm{SNF}(T^6 - I_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

Hence, $K_0(L(C_6(2,3))) \cong \mathbb{Z}_7$ and $L(C_6(2,3)) \cong L(1,8)$.

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