Proc. Indian Acad. Sci. (Math. Sci.) (2021) 131:15
https://doi.org/10.1007/s12044-021-00613-y

Two results on strong proximinality

T S S R K RAO

Department of Mathematics, Ashoka University, Rajiv Gandhi Education City, Sonipat 131 029, India
E-mail: srin@fulbrightmail.org

MS received 12 June 2020; revised 19 September 2020; accepted 1 October 2020

Abstract. Let $Y \subseteq X$ be a closed subspace. By a simple argument, we show that $Y^{\perp\perp} \subseteq X^{**}$ is strongly proximinal at points of X if and only if Y is a strongly proximinal subspace of X. This substantially improves the main result of Jayanarayanan and Paul (*J. Math. Anal. Appl.* **426** (2015) 1217–1231). As a consequence we get an easy proof of a classical result of Alfsen and Effros (*Ann. Math.* **98** (1972) 98–173), that M-ideals are proximinal subspaces and a result of Dutta and Narayana (Function Spaces, Contemporary Mathematics, vol. 435, American Mathematical Society, Providence (2007) pp. 143–152), that M-ideals are strongly proximinal subspaces.

Keywords. Proximinal subspaces; strongly proximinal subspaces; strong subdifferentiability.

2000 Mathematics Subject Classification. 41A50, 46B28, 46B25.

1. Introduction

Let X be a real Banach space. We recall that a subspace $Y \subseteq X$ is said to be proximinal if for every $x \in X$, there is a $y_0 \in Y$ such that $d(x, Y) = \|x - y_0\|$. This concept is closely related to the geometry of the Banach space X^*. Note that any proximinal subspace is a closed set. It is easy to see that for any $\phi \in X^*$, $\ker(\phi)$ is a proximinal subspace of X if and only if ϕ attains its norm on the closed unit ball of X. Thus the Hahn–Banach theorem ensures plenty of proximinal hyperplanes in a Banach space and the celebrated result of R. C. James states that if every hyperplane is a proximinal subspace, then the Banach space is a reflexive space (see [10]). See [8,9] for examples of Banach spaces, which have no proximinal subspaces of codimension 2.

For a non-reflexive Banach space X, we always consider X as canonically embedded in its bidual X^{**}. Thus for a closed subspace $Y \subseteq X$, $Y^{\perp\perp} \subseteq X^{**}$, using the lower semi-continuity of the norm and compactness of closed balls with respect to the weak*-topology, one has that $Y^{\perp\perp}$ is a proximinal subspace of X^{**}. An interesting question in approximation theory is to put mild additional geometric conditions on $Y^{\perp\perp}$ so that Y becomes a proximinal subspace of X. We do this using a well-studied notion of a strongly proximinal subspace, whose study was initiated by Godefroy and Indumathi in [5]. For a

Published online: 28 April 2021

proximinal subspace $Y \subseteq X$, for $x \in X$, let $P(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$ denote the set of best approximants to x.

DEFINITION 1

A proximinal subspace $Y \subseteq X$ is said to be strongly proximinal, if for any $x \in X$, $\varepsilon > 0$, there exists a $\delta > 0$ such that if $y \in Y$ is such that $\|x - y\| < d(x, Y) + \delta$, there exists $y_0 \in P(x)$ such that $\|y - y_0\| \leq \varepsilon$.

It is easy to see that any finite dimensional subspace is strongly proximinal. In [5], the authors have done a substantial study of subspaces of finite codimension that are strongly proximinal, motivated by the results of Chebyshev and Garkavi on finite codimensional proximinal subspaces. See Chapter 1, Theorem 2.1 of [10] and also the survey article [2]. Garkavi's theorem in particular implies that if a Banach space has a subspace of codimension $n \geq 1$, that is, proximinal, then it also has proximinal finite codimensional subspaces of codimension $1, 2, \ldots, n - 1$. An interesting recent achievement is the main result of [7] (Theorem 3.10), a proximinal subspace $Y \subseteq X$ of finite codimension is strongly proximinal if and only if $Y^{\perp\perp}$ is a strongly proximinal subspace of X^{**}. We note that $Y \subseteq X$ is of finite codimension if and only if $Y^{\perp\perp} \subseteq X^{**}$ is a subspace of the same finite codimension. The substantial proof in [7] relies on the assumptions of proximinality of Y and of its finite codimension. Even after two decades since the appearance of [5], there are very few known classes of Banach spaces in which proximinal subspaces of finite codimension that are of codimension greater than one can be generated, see [9]. Thus there is a need to extend the ambit of [7, Theorem 3.10] beyond finite codimensional spaces.

We first note that strong proximinality is a point-wise concept. Thus a closed subspace $Y \subseteq X$ is strongly proximinal in X, if and only if $Y \subseteq \text{span}\{x, Y\}$ is a strongly proximinal subspace, for all $x \notin Y$ (we then say, Y is strongly proximinal at x). We give a simple proof of the result, if $Y \subseteq X$ is any closed subspace such that $Y^{\perp\perp} \subseteq \text{span}\{x, Y^{\perp\perp}\}$ is strongly proximinal for all $x \notin Y$ (we recall that $Y^{\perp\perp} \cap X = Y$), then Y is a proximinal as well as strongly proximinal subspace of X.

We recall from [1,6] that a closed subspace $Y \subseteq X$ is said to be a M-ideal if there is a linear projection $R : X^* \to X^*$ such that $\ker(R) = Y^{\perp}$ and $\|x^*\| = \|R(x^*)\| + \|x^* - R(x^*)\|$ for all $x^* \in X^*$. An important property of M-ideals proved in [1] is that they are proximinal subspaces. The monograph [6] is a standard reference for examples and properties of these spaces. Assuming the proximinality, the authors in [3] note that any M-ideal is a strongly proximinal subspace. The generic arguments involve ball intersection properties that M-ideals possess, see Chapter I of [6]. As a simple consequence of our result, we get an easy proof of both proximinality and strong proximinality of M-ideals.

2. Main result

We will be using the notions and results on strong subdifferentiability from [4].

DEFINITION 2

A non-zero vector $x \in X$ is said to be a point of strong subdifferentiability (in short, SSD point) if the one-sided limit $\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$ exists uniformly for $h \in B_X$, where B_X denotes the closed unit ball of the Banach space X.

If $x^* \in X^*$ is a SSD point, it was shown in [4, Theorem 3.3] that x^* attains its norm. Thus $\ker(x^*)$ is a proximinal subspace of X. To see an example, let μ be positive measure on a measurable space $(\Omega, \mathcal{A})$. Since any extreme point of the unit ball of $L^1(\mu)$ is given by $\pm\frac{\chi_A}{\mu(A)}$ for a μ-atom A, it is easy to see that it is a SSD point.

We recall an interesting relation to strong proximinality, a geometric statement from [5, Lemma 1.1 and Theorem 2.5] that in a dual space X^*, a non-zero vector $x^* \in X^*$ is a SSD point of X^* if and only if $\ker(x^*)$ is a strongly proximinal subspace of X.

In what follows, we use the fact if $Y = \ker(\phi)$ for $\phi \in X^*$, then $Y^{\perp\perp} = \{\Lambda \in X^{**} : \Lambda(\phi) = 0\}$. Using the canonical embedding of $X^* \subseteq X^{***}$, we continue to write $Y^{\perp\perp} = \ker(\phi)$. Also for $x \in X$, $\text{span}\{x, Y^{\perp\perp}\} = (\text{span}\{x, Y\})^{\perp\perp}$.

Theorem 3. *Let $Y \subseteq X$ be a closed subspace. Suppose for every $x \notin Y$, $Y^{\perp\perp}$ is strongly proximinal in* $\text{span}\{x, Y^{\perp\perp}\}$ (*this is so, in particular, when $Y^{\perp\perp} \subseteq X^{**}$ is a strongly proximinal subspace*). *Then $Y \subseteq X$ is proximinal and also strongly proximinal in X. Conversely, if $Y \subseteq X$ is a strongly proximinal subspace, then for every $x \notin Y$, $Y^{\perp\perp}$ is strongly proximinal in* $\text{span}\{x, Y^{\perp\perp}\}$.

Proof. It is enough to show that for $x \notin Y$, Y is a proximinal and strongly proximinal subspace of $\text{span}\{x, Y\}$. Since $(\text{span}\{x, Y\})^{\perp\perp} = \text{span}\{x, Y^{\perp\perp}\}$, if we take $Z = \text{span}\{x, Y\}$, then $Y = \ker(\phi)$, for $0 \neq \phi \in Z^*$ and we may assume that $\phi(x_0) = 1$, where $x_0 = \alpha x$ for some scalar α. Thus $Z = \text{span}\{x_0\} \bigoplus \ker(\phi)$ and $Z^{**} = \text{span}\{x_0\} \bigoplus \ker(\phi) = \text{span}\{x_0\} \bigoplus Y^{\perp\perp}$. By hypothesis, we get that ϕ is a SSD point of Z^{***}. It is easy to see that ϕ is a SSD point of Z^* as well. Consequently, as ϕ attains its norm, Y is proximinal and also strongly proximinal subspace of $\text{span}\{x, Y\}$ for all $x \notin Y$.

Converse part follows from similar arguments, this time one uses the fact from [4] that for a Banach space W, if $\phi \in W^*$ is a SSD point, then it is also a SSD point of W^{***}. □

We use the following standard example in the proof of the corollary that follows.

Example 4. Let $X = Y \bigoplus_\infty Z$ (ℓ^∞-direct sum) for Banach spaces X, Y, Z. Let $Q : X \to Y$ be the linear projection onto Y such that $\|x\| = \max\{\|Q(x)\|, \|x - Q(x)\|\}$ for all $x \in X$.

It is easy to see that $Y \subseteq X$ is a proximinal subspace, since $d(x, Y) = \|x - Q(x)\|$ also $P(x) = B(Q(x), \|x - Q(x)\|)$ (closed ball in Y). Now for $x \notin Y$ and $\varepsilon > 0$, let $\delta = \varepsilon$. If $y \in Y$ and $\|x - y\| < \|x - Q(x)\| + \delta$, note that $\|x - y\| = \max\{\|Q(x) - y\|, \|x - Q(x)\|\}$. Thus $\|Q(x) - y\| < \|x - Q(x)\| + \delta$. Since $P(x) = B(Q(x), \|x - Q(x)\|)$, let $y_0 \in B(Q(x), \|x - Q(x)\|)$ such that $\|y - y_0\| \leq \varepsilon$. Therefore, Y (and for the same reason Z) are strongly proximinal subspace of X.

COROLLARY 5

Let $Y \subseteq X$ be a M-ideal. Then Y is a proximinal as well as strongly proximinal subspace of X.

Proof. Since $Y \subseteq X$ is a M-ideal, then $X^{**} = Y^{\perp\perp} \bigoplus_\infty R(X^*)^\perp$, where R is the projection in the definition of a M-ideal. Thus by our example, $Y^{\perp\perp}$ is a strongly proximinal subspace of X^{**} and hence by Theorem 3, Y is a proximinal and strongly proximinal subspace of X. □

Remark 6. We do not know if $Y^{\perp\perp}$ is strongly proximinal in span$\{x, Y^{\perp\perp}\}$ for all, $x \in X$, $x \notin Y$, implies that $Y^{\perp\perp}$ is a strongly proximinal subspace of X^{**}? In particular, we have strengthened the converse part of Theorem 3.10 by two ways, by requiring strong proximinality of $Y^{\perp\perp}$ only at points of $x \in X$ and by removing proximinality assumption on Y. Let $Y \subset X$ be an infinite dimensional reflexive subspace which is strongly proximinal in X . Now $Y^{\perp\perp} = Y$, even here we do not know if Y it is always strongly proximinal in X^{**} ? Another variation is to consider Banach space X such that $X \subset X^{**}$ is strongly proximinal. Here again we do not know if $X^{\perp\perp}$ is always a strongly proximinal subspace of the fourth dual, $X^{(IV)}$.

Remark 7. Since X is a weak*-dense subspace of X^{**}, one view point is to conclude strong proximinality from information on a weak*-dense set. This is particularly useful in application in vector valued optimization theory, where one can often approximate by functions taking finitely many values. We do not know if a proximinal subspace $Y \subseteq X$ is strongly proximinal points from a norm dense set of X, then is it strongly proximinal in X?

Acknowledgements

The author would like to thank the referee for his useful comments.

References

[1] Alfsen E and Effros E G, Structure in real Banach spaces, *Ann. Math.* **98** (1972) 98–173
[2] Bhattacharyya T and Rao T S S R K, Survey of last ten years of work done in India in some selected areas of functional analysis and operator theory, *Indian J. Pure Appl. Math.* **50** (2019) 599–617
[3] Dutta S and Narayana D, Strongly proximinal subspaces in Banach spaces, Function Spaces, Contemp. Math., vol. 435 (2007) (Providence: American Mathematical Society) pp. 143–152
[4] Franchetti C and Paya R, Banach spaces with strongly subdifferentiable norm, *Boll. Un. Mat. Ital. B* **7** (1993) 45–70
[5] Godefroy G and Indumathi V, Strong proximinality and polyhedral spaces, *Rev. Mat. Complut.* **14** (2001) 1139–1148
[6] Harmand P, Werner D and Werner W, *M*-ideals in Banach spaces and Banach algebras, Lecture Notes in Mathematics, 1547 (1993) (Springer)
[7] Jayanarayanan C R and Paul T, Strong proximinality and intersection properties of balls in Banach spaces, *J. Math. Anal. Appl.* **426** (2015) 1217–1231.
[8] Kadets V, López G and Martin M, Some geometric properties of Read's space, *J. Funct. Anal.* **274** (2018) 889–899
[9] Rmoutil M, Norm-attaining functionals need not contain 2-dimensional subspaces, *J. Funct. Anal.* **272** (2017) 918–928
[10] Singer I, Best approximation in normed linear spaces by elements of linear subspaces, Die Grundlehren der mathematischen Wissenschaften Band 171 (1970) (Berlin: Springer)

COMMUNICATING EDITOR: B V Rajarama Bhat