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Primes in Beatty sequence

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Abstract. For a polynomial $g(x)$ of deg $k \geq 2$ with integer coefficients and positive integer leading coefficient, we prove an upper bound for the least prime p such that $g(p)$ is in non-homogeneous Beatty sequence $\{\lfloor \alpha n + \beta \rfloor : n = 1, 2, 3, \ldots\}$, where $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$ is irrational and we prove an asymptotic formula for the number of primes p such that $g(p) = \lfloor \alpha n + \beta \rfloor$. Next, we obtain an asymptotic formula for the number of primes p of the form $p = \lfloor \alpha n + \beta \rfloor$ which also satisfies $p \equiv f \pmod{d}$, where f, d are integers with $1 \leq f < d$ and $(f, d) = 1$.

Keywords. Beatty sequence; prime number; estimates on exponential sums.

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1. Introduction

Given a real number $\alpha > 0$ and a non-negative real β, the Beatty sequence associated with α, β is defined by

$$\mathcal{B}(\alpha, \beta) = \{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Beatty sequences appear in a variety of apparently unrelated mathematical settings and because of their versatility, the arithmetic properties of these sequences have been widely studied in the literature; see, for example, [1,5,9–11] and the references contained therein.

If α is rational, then $\mathcal{B}(\alpha, \beta)$ is a union of residue classes, hence *we always assume that α is irrational*. An irrational number θ is said to be of finite type $t \geq 1$, if t is the infimum of $\rho \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $q_0(\varepsilon) > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{q^{\rho+1+\varepsilon}}$$

holds for all integers $q > q_0(\varepsilon)$. In 2016, Jörn Steuding and Marc Technau [13] proved that for every $\varepsilon > 0$, there exists a computable positive integer l such that for every irrational $\alpha > 1$ the least prime p in the Beatty sequence $\mathcal{B}(\alpha, \beta)$ satisfies the inequality

$$p \leq L^{35-16\varepsilon} \alpha^{2(1-\varepsilon)} B p_{m+l}^{1+\varepsilon},$$

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where $B = \max\{1, \beta\}$, $L = \log(2\alpha B)$, p_n denotes the numerator of the n-th convergent to the regular continued fraction expansion of $\alpha = [a_0, a_1, \dots]$ and m is the unique integer such that

$$p_m \leq L^{16}\alpha^2 < p_{m+1}.$$

The first result of this paper is the following.

Theorem 1. *Let $g(x) = a_0 + a_1x + \cdots + a_kx^k$, where $a_0, \dots, a_k \in \mathbb{Z}$ with $a_k \geq 1$ and $k \geq 2$. Put $\gamma = 4^{1-k}$. Then for any positive integer $N \geq 3$, positive real number α with $\left|\frac{a_k}{\alpha} - \frac{a}{q}\right| \leq \frac{1}{q^2}$, $(a, q) = 1$ and any $\varepsilon > 0$, we have*

$$\sum_{\substack{p \leq N \\ g(p) \in \mathcal{B}(\alpha,\beta)}} \log p = \frac{1}{\alpha}\sum_{p \leq N} \log p + O\left(N^{\varepsilon}\left(Nq^{-\gamma} + N^{1-\gamma/2} + q^{\frac{\gamma}{1-\gamma}} N^{\frac{1-(k+1)\gamma}{1-\gamma}}\right)\right).$$

In particular, if a_k/α is an irrational number of finite type $t > 0$, then we have

$$\sum_{\substack{p \leq N \\ g(p) \in \mathcal{B}(\alpha,\beta)}} \log p = \frac{1}{\alpha}\sum_{p \leq N} \log p + O(N^{1-\frac{k\gamma}{t+1}+\varepsilon} + N^{1-\frac{\gamma}{2}+\varepsilon}).$$

Theorem 2. *Let $g(x) = a_0 + a_1x + \cdots + a_kx^k$, where $a_0, \dots, a_k \in \mathbb{Z}$ with $a_k \geq 1$ and $k \geq 2$. Put $\gamma = 4^{1-k}$. For every $\varepsilon > 0$, there exists a computable positive integer l such that for every irrational $\alpha > 1$, the least prime number p such that $g(p)$ is in the Beatty sequence $\mathcal{B}(\alpha, \beta)$ satisfies the inequality*

$$p \leq \alpha^{\frac{2k-1}{k\gamma} - \frac{(\gamma+1)\varepsilon}{\gamma}} B^{\frac{k-1}{k} - \frac{\varepsilon}{\gamma}} p_{m+l}^{\frac{1}{k} + \frac{\varepsilon}{\gamma}},$$

where $B = \max\{1, \beta\}$, p_n denotes the numerator of the n-th convergent to the regular continued fraction expansion of $\frac{\alpha}{a_k}$ and m is the unique integer such that

$$p_m \leq \alpha^{\frac{1+\gamma}{\gamma}} B < p_{m+1}. \tag{1}$$

Remark 1. Instead of using Proposition 1 in (5) if we use Theorem 1 of [15], we may prove Theorem 1 with following error term:

$$\sum_{\substack{p \leq N \\ g(p) \in \mathcal{B}(\alpha,\beta)}} \log p = \frac{1}{\alpha}\sum_{p \leq N} \log p + O\left(N^{1-\frac{1}{25k^2\log(k+2)}} + N^{1-\frac{37\chi}{100k^2\log(\frac{k^2}{\chi}+4)}}\right), \tag{2}$$

where $\chi > 0$ is a real number satisfying the condition $N^{\chi} \ll q$ and if $\frac{a_k}{\alpha}$ of finite type $t > 0$,

$$\sum_{\substack{p\le N\\ g(p)\in\mathcal{B}(\alpha,\beta)}} \log p = \frac{1}{\alpha}\sum_{p\le N}\log p + O(N^{1-\rho}), \tag{3}$$

where

$$\rho = \frac{1}{25k^2\log(k+2)} \quad \text{if } 4k \le t+1,$$
$$\rho = \frac{37}{100k(t+1)\log(k(t+1)+4)} \quad \text{if } 4k > t+1.$$

It is easy to see that error terms obtained in (2) and (3) is better than the error term stated in Theorem 1 for large k.

For irrational α of finite type $\tau = \tau(\alpha)$, Banks and Yeager (Theorem 2, [3]) proved that for any fixed $\varepsilon > 0$, for all integers $1 \le c < d < N^{\frac{1}{4\tau+2}}$ with gcd (c, d) = 1, we have

$$\sum_{\substack{n\le x, n\in\mathcal{B}(\alpha,\beta)\\ n\equiv c \ (\mathrm{mod}\ d)}} \Lambda(n) = \frac{1}{\alpha}\sum_{\substack{n\le x\\ n\equiv c\ (\mathrm{mod}\ d)}} \Lambda(n) + O(x^{1-\frac{1}{4\tau+2}+\varepsilon}).$$

The following theorem improves the error term.

Theorem 3. *For any positive integers* $N \ge 3$, $1 \le f < d \le \min\{q^{1/2}, N^{1/6}\}$ *such that* $(f, d) = 1$ *and positive real number* α *with* $\left|\frac{1}{\alpha} - \frac{a}{q}\right| \le \frac{1}{q^2}$ *with* $(a, q) = 1$, *for any* $\varepsilon > 0$, *we have*

$$\sum_{\substack{p\le N\\ p\in\mathcal{B}(\alpha,\beta)\\ p\equiv f(d)}} \log p = \frac{1}{\alpha}\sum_{\substack{p\le N\\ p\equiv f(d)}} \log p + O\left(N^{\varepsilon}\left(\frac{N}{q^{1/2}} + N^{1/2}q^{1/2} + N^{3/4}d^{1/2} + \frac{N^{4/5}}{d^{1/5}}\right)\right).$$

Furthermore, if α *is an irrational number of finite type* $t > 0$, *then for all integers* $1 \le f < d \le \min\{N^{\frac{1}{2(t+1)}}, N^{1/6}\}$ *with* $(f, d) = 1$ *and for any* $0 < \varepsilon < \frac{1}{4(t+1)}$, *we have*

$$\sum_{\substack{p\le N\\ p\in\mathcal{B}(\alpha,\beta)\\ p\equiv f(d)}} \log p = \frac{1}{\alpha}\sum_{\substack{p\le N\\ p\equiv f(d)}} \log p + O(N^{1-\frac{1}{2(1+t)}+\varepsilon} + N^{\frac{3}{4}+\varepsilon}d^{1/2} + N^{\frac{4}{5}+\varepsilon}d^{-1/5}).$$

The proof of Theorem 3 depends on the estimation of exponential sum of the type

$$S(\vartheta) = \sum_{|l|\le L}\left|\sum_{\substack{n\le N\\ n\equiv f(d)}} \Lambda(n)e(ln\vartheta)\right|, \tag{4}$$

where ϑ is irrational, $L, N \geq 1$ and $f < d$, $(f, d) = 1$.

We obtain an upper bound for $S(\vartheta)$ in Proposition 2 which is of independent interest.

Remark 2. Let d, f be natural numbers such that $1 \leq f < d \leq 500$ and $(f, d) = 1$. For every $\varepsilon > 0$, there exists a computable positive integer l such that for every irrational $\alpha > 1$, the least prime number $p \in \mathcal{B}(\alpha, \beta)$ such that $p \equiv f \pmod d$ satisfies the inequality

$$p \leq \alpha^{3-7\varepsilon} B^{\frac{1}{2}(1-3\varepsilon)} d^{3-10\varepsilon} p_{m+l}^{1+3\varepsilon},$$

where $B = \max\{1, \beta\}$ and p_n denotes the numerator of the n-th convergent to the regular continued fraction expansion of α and m is the unique integer such that

$$p_m \leq \alpha^{7/3} B^{1/2} d^{10/3} < p_{m+1}.$$

This fact can be proved in a similar way as Theorem 2 using Theorem 3 and Corollary 1.6 of [4].

2. Notation

Throughout this paper, the implied constants in the symbols O and $\ll$ are either absolute or depend only on α and ε. We recall that the notation $f = O(g)$ and $f \ll g$ are equivalent to the assertion that the inequality $|f| \leq cg$ holds for some constant $c > 0$. The notation $f \approx g$ means that $f \ll g$ and $f \gg g$. It is important to note that our bounds are uniform with respect to all of the involved parameters other than α, ε and degree of the polynomial k; in particular, our bounds are uniform with respect to β.

The letters a, d, f, q always denote non-negative integers and m, n, l, u, v and t denote integers. We use $\lfloor x \rfloor$ and $\{x\}$ to denote the greatest integer less than or equal to x and the fractional part of x respectively. Finally, recall that the discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, a_2 \ldots, a_M \in [0, 1)$ is defined by

$$D(M) = \sup_{I \subset [0,1)} \left| \frac{V(I, M)}{M} - |I| \right|,$$

where the supremum is taken over all sub-intervals I of $[0, 1)$, $V(I, M)$ is the number of positive integers $m \leq M$ such that $a_m \in I$ and $|I|$ is the length of I.

3. Preliminaries

3.1 *Case of polynomial values of prime*

Note that an integer $m \in \mathcal{B}(\alpha, \beta)$ if and only if $\frac{m}{\alpha} \in \left(\frac{\beta-1}{\alpha}, \frac{\beta}{\alpha} \right] \pmod 1$ and $m > \alpha + \beta - 1$. This is equivalent to

$$\left\| \frac{m}{\alpha} + \frac{1-2\beta}{2\alpha} \right\| < \frac{1}{2\alpha}.$$

Hence

$$\#\{m \leq N : m \in \mathcal{B}(\alpha, \beta)\} = \sum_{m \leq N} \chi_{\frac{1}{2\alpha}} \left(\frac{m}{\alpha} + \frac{1-2\beta}{2\alpha} \right),$$

where χ_δ for $\delta > 0$ is defined by

$$\chi_\delta(\theta) = \begin{cases} 1 & \text{if } \|\theta\| < \delta, \\ 0 & \text{otherwise,} \end{cases}$$

for $\theta \in \mathbb{R}$. Let $g(x) = a_0 + a_1 x + \cdots + a_k x^k$, where $a_0, \ldots, a_k \in \mathbb{Z}$ with $a_k \geq 1$, $k \geq 2$. Therefore,

$$\#\{p \leq N : g(p) \in \mathcal{B}(\alpha, \beta)\} = \sum_{p \leq N} \chi_{\frac{1}{2\alpha}}\left(\frac{g(p)}{\alpha} + \frac{1-2\beta}{2\alpha}\right).$$

Lemma 1 ([8], *Lemma* 2.1). *For any $L \in \mathbb{N}$, there are coefficients $C_l^{\pm}$ such that*

$$2\delta - \frac{1}{L+1} + \sum_{1 \leq l \leq L} C_l^- e(l\theta) \leq \chi_\delta(\theta) \leq 2\delta + \frac{1}{L+1} + \sum_{1 \leq l \leq L} C_l^+ e(l\theta),$$

with $|C_l^{\pm}| \leq \min\left(2\delta + \frac{1}{L+1}, \frac{3}{2l}\right)$.

Using Lemma 1, we get

$$\sum_{n \leq N} \Lambda(n) \chi_{\frac{1}{2\alpha}}\left(\frac{g(n)}{\alpha} + \frac{1-2\beta}{2\alpha}\right) = \frac{1}{\alpha} \sum_{n \leq N} \Lambda(n) + O\left(\frac{N}{L+1}\right) + O\left(\sum_{1 \leq |l| \leq L} |C_l| \left| \sum_{n \leq N} \Lambda(n) e\left(\frac{lg(n)}{\alpha}\right)\right|\right), \tag{5}$$

where $|C_l| \leq \min\left(\frac{1}{\alpha} + \frac{1}{L+1}, \frac{3}{2l}\right)$. To estimate the exponential sum, we use the following proposition.

PROPOSITION 1 (Equation (22) of [7])

Suppose $\varepsilon > 0$ is given. Let $f(x)$ be a real-valued polynomial in x of degree $k \geq 2$. Put $\gamma = 4^{1-k}$. Suppose α is the leading coefficient of f and there are integers a, q with $(a, q) = 1$ such that

$$|q\alpha - a| < q^{-1}.$$

Then we have

$$\sum_{l \leq L} \left| \sum_{n \leq N} \Lambda(n) e(lf(n)) \right| \ll (NL)^{1+\varepsilon} (q^{-1} + N^{-1/2} + qN^{-k}L^{-1})^\gamma.$$

3.2 *Case of primes in arithmetic progression*

Now we are interested in prime numbers p of the form $p \equiv f \mod d$, which is in $\mathcal{B}(\alpha, \beta)$, where $(f, d) = 1$ and $f < d$. As we discussed above, in order to find a prime number $p \in \mathcal{B}(\alpha, \beta)$, we need to show that

$$\left\| \frac{p}{\alpha} + \frac{1-2\beta}{2\alpha} \right\| < \frac{1}{2\alpha}.$$

By Lemma 1, we have

$$\sum_{\substack{n \le N \\ n \equiv f(d)}} \Lambda(n) \chi_{\frac{1}{2\alpha}}\left(\frac{n}{\alpha} + \frac{1-2\beta}{2\alpha}\right) = \frac{1}{\alpha} \sum_{\substack{n \le N \\ n \equiv f(d)}} \Lambda(n) + O\left(\frac{N}{L\varphi(d)}\right) + O\left(\sum_{1 \le |l| \le L} |C_l| \left| \sum_{\substack{n \le N \\ n \equiv f(d)}} \Lambda(n) e(ln/\alpha) \right|\right), \tag{6}$$

where $|C_l| \le \min\left(\frac{1}{\alpha} + \frac{1}{L+1}, \frac{3}{2l}\right)$. Now we want to estimate the exponential sum of the form (4). To estimate the exponential sum, we use the following proposition.

PROPOSITION 2

Let $S(\vartheta)$ be defined by 4 with $\left|\vartheta - \frac{a}{q}\right| \le q^{-2}$, where a and q are positive integers satisfying $(a, q) = 1$. Then for any real number $\varepsilon > 0$, we have

$$S(\vartheta) \ll_\varepsilon (NL)^\varepsilon \left(\frac{NL}{q^{1/2}} + L^{1/2} N^{1/2} q^{1/2} + L N^{3/4} d^{1/2} + \frac{L N^{4/5}}{d^{1/5}} \right).$$

We will give the proof of Proposition 2 in Section 6. In the end of this section, we give some lemmas require for the proof. The following lemma gives explicit bound for the average of von Mangoldt function.

Lemma 2 [12]. *For any $N \in \mathbb{N}$,*

$$\sum_{n \le N} \Lambda(n) \le c_0 N,$$

for some constant c_0, where one may take $c_0 = 1.04$.

Lemma 3. *Let α be of finite type t if and only if $\frac{1}{\alpha}$ is of finite type t.*

Proof. Let α be an irrational number of finite type t. Let $\frac{p_m}{q_m}$ be the m-th convergent to the continued fraction expansion of α. We denote

$$\mu(\alpha) = \inf\{\rho \in \mathbb{R} : q_{m+1} \le q_m^\rho \text{ holds for all but finitely many } m \in \mathbb{N}\}.$$

We will first show that $\mu(\alpha) = t$. Since $\mu(\alpha)$ is the infimum, for every $\varepsilon > 0$, $q_{m+1} \ge q_m^{\mu(\alpha)-\varepsilon}$ holds for infinitely many $m \in \mathbb{N}$. By using Theorem 171 of [6], we have

$$\left|\alpha - \frac{p_m}{q_m}\right| < \frac{1}{q_m q_{m+1}} \le \frac{1}{q_m^{\mu(\alpha)+1-\varepsilon}}.$$

Therefore $\mu(\alpha) - \varepsilon < t$, for every $\varepsilon > 0$ so that $\mu(\alpha) \le t$. Since the convergents are alternatively less and greater than α, we have

$$\frac{1}{q_m q_{m+1}} = \left|\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m}\right| = \left|\alpha - \frac{p_m}{q_m}\right| + \left|\alpha - \frac{p_{m+1}}{q_{m+1}}\right| \le 2\left|\alpha - \frac{p_m}{q_m}\right|.$$

To prove reverse inequality, we use the definition of finite type to obtain

$$\frac{1}{2q_m q_{m+1}} \le \left|\alpha - \frac{p_m}{q_m}\right| \le \frac{1}{q_m^{t+1-\varepsilon}}$$

which holds for every $\varepsilon > 0$ and for all but finitely many $m \in \mathbb{N}$. Hence, we have $q_m^{t-\varepsilon} \le 2q_{m+1}$, which implies $q_m^{t-\frac{\varepsilon}{2}} \le \frac{2}{q_m^{\varepsilon/2}} q_{m+1} \le q_{m+1}$ holds for all but finitely many $m \in \mathbb{N}$. Therefore, $t - \frac{\varepsilon}{2} < \mu(\alpha)$ for every $\varepsilon > 0$ from which it follows that $t \le \mu(\alpha)$. Thus we have $\mu(\alpha) = t$.

Since the convergents of $\frac{1}{\alpha}$ are reciprocals of the convergents $\frac{p_n}{q_n}$ to the continued fraction expansion of α, and p_n satisfies same recurrence relation that is satisfied by q_n, we have

$$\mu(\frac{1}{\alpha}) = \inf\{\rho \in \mathbb{R} : p_{m+1} \ll p_m^{\rho} \text{ holds for all but finitely many } m \in \mathbb{N}\} = t.$$

This completes the proof of Lemma 3. □

4. Proof of Theorem 1 and Theorem 2

In the previous section, we stated essential results to prove Theorem 1 and Theorem 2. In this section, we give proof of these theorems.

Proof of Theorem 1. It follows from Proposition 1 and partial summation formula that

$$\begin{aligned}\sum_{1\le |l|\le L} C_l \sum_{n\le N} \Lambda(n) e\left(l\left(\frac{g(n)}{\alpha} + \frac{1-2\beta}{2\alpha}\right)\right)\\ \ll_\varepsilon N^{1+\varepsilon} L^{\varepsilon}(q^{\gamma} + N^{-\gamma/2} + q^{\gamma} N^{-k\gamma} L^{-\gamma})\\ + N^{1+\varepsilon} q^{\gamma} N^{-k\gamma}. \end{aligned} \tag{7}$$

By (5) and (7), we have

$$\sum_{n\le N} \Lambda(n)\chi_{\frac{1}{2\alpha}}\left(\frac{g(n)}{\alpha} + \frac{1-2\beta}{2\alpha}\right) = \frac{1}{\alpha}\sum_{n\le N}\Lambda(n) + O\left(\frac{N}{L+1}\right)$$
$$+ O(N^{1+\varepsilon} L^{\varepsilon}(q^{-\gamma} + N^{-\gamma/2}) + N^{1+\varepsilon} q^{\gamma} N^{-k\gamma}).$$

Choosing $L = q^{\gamma}$, we have

$$\sum_{n\le N} \Lambda(n)\chi_{\frac{1}{2\alpha}}\left(\frac{g(n)}{\alpha} + \frac{1-2\beta}{2\alpha}\right) = \frac{1}{\alpha}\sum_{n\le N}\Lambda(n)$$

$$+ O\left(N^{\varepsilon}\left(Nq^{-\gamma} + N^{1-\gamma/2} + q^{\frac{\gamma}{1-\gamma}} N^{\frac{1-(k+1)\gamma}{1-\gamma}}\right)\right).$$

This leads to

$$\sum_{\substack{p^{\nu}\le N\\ g(p^{\nu})\in\mathcal{B}(\alpha,\beta)}} \log p + \sum_{\substack{p^{\nu}\le\alpha+\beta-1\\ g(p^{\nu})\in\mathcal{B}(\alpha,\beta-\lfloor\alpha+\beta\rfloor)}} \log p = \frac{1}{\alpha}\sum_{p^{\nu}\le N}\log p + \xi(N,q), \quad (8)$$

where

$$|\xi(N,q)| \le_{\varepsilon} N^{\varepsilon}\left(Nq^{-\gamma} + N^{1-\gamma/2} + q^{\frac{\gamma}{1-\gamma}} N^{\frac{1-(k+1)\gamma}{1-\gamma}}\right). \quad (9)$$

The number of prime powers $p^{\nu} \le N$ with $\nu \ge 2$ is $O(\pi(N^{1/2}))$, thus we have

$$\sum_{\substack{p\le N\\ g(p)\in\mathcal{B}(\alpha,\beta)}} \log p = \frac{1}{\alpha}\sum_{p\le N}\log p + O\left(\frac{N^{1/2}}{\alpha\log N}\right)$$
$$+ O\left(N^{\varepsilon}\left(Nq^{-\gamma} + N^{1-\gamma/2} + q^{\frac{\gamma}{1-\gamma}} N^{\frac{1-(k+1)\gamma}{1-\gamma}}\right)\right). \quad (10)$$

Suppose we assume $\frac{a_k}{\alpha}$ is an irrational number of finite type t. For any positive ε, there is a positive constant c such that

$$\left|\frac{a_k}{\alpha} - \frac{p}{q}\right| \ge \frac{c}{q^{t+1+\varepsilon}}, \quad (11)$$

holds for all integers p, q with $q > 0$. On the other hand, by using Dirichlet's approximation theorem with $Q = N^{\frac{kt}{t+1}}$, we obtain a rational p/q with $1 \le q \le N^{\frac{kt}{t+1}}$ such that

$$\left|\frac{a_k}{\alpha} - \frac{p}{q}\right| \le \frac{1}{qN^{\frac{kt}{t+1}}}. \quad (12)$$

Suppose we assume that there exists a convergent to the simple continued fraction expansion of $\frac{a_k}{\alpha}$ which satisfies (12). Then by (11), its denominator satisfies

$$N^{\frac{k}{t+1}+\varepsilon} \ll q \le N^{\frac{kt}{t+1}}. \quad (13)$$

Therefore, by (10) and (13), we obtain

$$\sum_{\substack{p\le N\\ g(p)\in\mathcal{B}(\alpha,\beta)}} \log p = \frac{1}{\alpha}\sum_{p\le N}\log p + O(N^{1-\frac{k\gamma}{t+1}+\varepsilon} + N^{1-\frac{\gamma}{2}+\varepsilon}).$$

To complete the proof, we want to show that there exists a convergent to the continued fraction of $\frac{a_k}{\alpha}$ which satisfies (12). Let $\frac{p}{q}$ be a rational number which satisfies (12). By using recurrence relation satisfied by the denominator of convergent to the continued fraction of $\frac{a_k}{\alpha}$, we obtain unique $n \in \mathbb{N}$ such that $q_n \le q \le q_{n+1}$. It follows from Theorem 171 of [6] that $\frac{1}{q_n N^{\frac{kt}{t+1}}} < \left|\frac{a_k}{\alpha} - \frac{p_n}{q_n}\right|$ and $q_{n+1} > N^{\frac{kt}{t+1}}$ both cannot hold simultaneously. Therefore, by Theorem 181 of [6], we get that either $\frac{p_n}{q_n}$ or $\frac{p_{n+1}}{q_{n+1}}$ satisfies (12). This completes the proof of Theorem 1.

Proof of Theorem 2. By (8) and (9), we have

$$\sum_{\substack{p^\nu \le N \\ g(p^\nu) \in \mathcal{B}(\alpha,\beta)}} \log p + \sum_{\substack{p^\nu \le \alpha+\beta-1 \\ g(p^\nu) \in \mathcal{B}(\alpha,\beta-\lfloor\alpha+\beta\rfloor)}} \log p = \frac{1}{\alpha} \sum_{p^\nu \le N} \log p + \xi(N,q), \tag{14}$$

where

$$|\xi(N,q)| \le_\varepsilon N^\varepsilon (N q^{-\gamma} + N^{1-\gamma/2} + q^{\frac{\gamma}{1-\gamma}} N^{\frac{1-(k+1)\gamma}{1-\gamma}}). \tag{15}$$

By Lemma 2, the second sum on the left-hand side of (14) is $< 1.04\,(\alpha + \beta - 1)$.

Therefore, we have

$$\sum_{\substack{p \le N \\ g(p) \in \mathcal{B}(\alpha,\beta)}} \log p \ge \frac{1}{\alpha} \sum_{p \le N} \log p + \xi(N,q) - 1.04(\alpha+\beta-1)$$
$$+ \left(\frac{1}{\alpha} - 1\right) \sum_{\substack{p^\nu \le N \\ \nu \ge 2}} \log p.$$

Notice that the last term is negative, it is obviously bounded by

$$\left(1 - \frac{1}{\alpha}\right) \sum_{\substack{p^\nu \le N \\ \nu \ge 2}} \log p < \left(1 - \frac{1}{\alpha}\right) \pi(N^{1/2}) \log N.$$

We will use inequality (3.2) of Rosser and Schoenfeld [12] for $\pi(x)$. We have

$$\left(1 - \frac{1}{\alpha}\right) \sum_{\substack{p^\nu \le N \\ \nu \ge 2}} \log p < 2\left(1 + \frac{3}{\log N}\right) N^{1/2}.$$

We also use inequality (3.16) of Rosser and Schoenfeld which is

$$\sum_{p \le N} \log p > N - \frac{N}{\log N},$$

for $N \ge 41$. Therefore, we obtain

$$\sum_{\substack{p \le N \\ g(p) \in \mathcal{B}(\alpha,\beta)}} \log p \ge \frac{N}{\alpha}\left(1 - \frac{1}{\log N}\right)$$

$$+\xi(N,q) - 1.04(\alpha+\beta-1) - \left(1+\frac{3}{\log N}\right)N^{1/2}.$$

We thus find a prime $p \leq N$ such that $g(p) \in \mathcal{B}(\alpha, \beta)$ if we show that the following inequality:

$$\frac{N}{\alpha}\left(1-\frac{1}{\log N}\right) > \xi(N,q) + 1.04(\alpha+\beta-1) + \left(1+\frac{3}{\log N}\right)N^{1/2},$$

may also be replaced by

$$0.73\frac{N}{\alpha} > 1.04(\alpha+\beta-1) + 1.81N^{1/2} + \xi(N,q).$$

By (15), we have

$$0.73 > 1.04\frac{\alpha}{N}(\alpha+\beta-1) + 1.81\frac{\alpha}{N^{1/2}} + C(\varepsilon)N^{\varepsilon}\alpha(q^{-\gamma} + N^{-\gamma/2} + q^{\frac{\gamma}{1-\gamma}}N^{\frac{-k\gamma}{1-\gamma}})$$

and appropriate absolute constant $C(\varepsilon)$ depending only on ε but not on α.
Obviously N need to be larger than $\mathrm{Max}\{\alpha^{2/\gamma}, B\}$ and q larger than $\alpha^{1/\gamma}$. We shall take both N and q somewhat larger so that the above inequality holds. Now choose

$$N = \alpha^{\frac{2k-1}{k\gamma}}B^{\frac{k-1}{k}}q^{\frac{1}{k}}\eta^{\frac{\varepsilon}{\gamma}}, \quad q = \alpha^{\frac{\gamma+1}{\gamma}}B\eta$$

with some large parameter η to be specified later and $B = \max\{1, \beta\}$. Then the latter inequality can be rewritten as

$$\begin{aligned}0.73 >&1.04(\alpha+\beta-1)(\alpha^{\frac{(k-1)\gamma-2k}{k\gamma}}B^{-1}\eta^{-\frac{1}{k}-\frac{\varepsilon}{\gamma}})\\ &+1.81(\alpha^{\frac{(2k-1)\gamma-2k}{2k\gamma}}B^{-1/2}\eta^{-\frac{1}{2k}-\frac{\varepsilon}{2\gamma}})\\ &+C\Big(\alpha^{-\gamma+\frac{(2k+\gamma)\varepsilon}{k\gamma}}B^{-\gamma+\varepsilon}\eta^{-\gamma+\frac{\varepsilon}{k}+\frac{\varepsilon^2}{\gamma}}\\ &+\alpha^{-\frac{\gamma}{2k}+\frac{(2k+\gamma)\varepsilon}{k\gamma}}B^{-\frac{\gamma}{2}+\varepsilon}\eta^{-\frac{\gamma}{2k}+\frac{(2-k)\varepsilon}{2k}+\frac{\varepsilon^2}{\gamma}}\\ &+\alpha^{\frac{2(1-k)-\gamma}{1-\gamma}+\frac{(2k+\gamma)\varepsilon}{k\gamma}}B^{\frac{(1-k)\gamma}{1-\gamma}+\varepsilon}\eta^{\frac{(1-k^2-\gamma)\varepsilon}{1-\gamma}+\frac{\varepsilon^2}{\gamma}}\Big).\end{aligned}$$

Since $k \geq 2$ and $\gamma = 4^{1-k}$ assuming $\varepsilon < \frac{\gamma^2}{2(2k+\gamma)}$, as we may, all exponents of α, B and η are negative. Therefore, the above inequality is satisfied for all sufficiently large η, say $\eta \geq \eta_0$. Since η is intertwined with q, a little care needs to be taken. Since α is irrational there are infinitely many rational $\frac{a}{q}$ satisfying

$$\left|\frac{\alpha}{a_k} - \frac{a}{q}\right| < \frac{1}{q^2}$$

Let $\frac{q_n}{p_n}$ be the n-th convergent to the continued fraction expansion of $\frac{\alpha}{a_k}$. We shall choose l such that $\eta_0 \le \frac{p_{m+l}}{p_m}$, where m is defined by (1), for then the choice $q = p_{m+l}$ will yield an $\eta \ge \eta_0$. The choice of η is as in (12) of [13]. Once η is chosen, l is also determined as it depends only on η.

5. Proof of Theorem 3

The present section is devoted to a proof of Theorem 3.

Proof of Theorem 3. It follows from Proposition 2 and partial summation formula that

$$\sum_{1\le|l|\le L} C_l \sum_{\substack{n\le N\\ n\equiv f(d)}} \Lambda(n)e(ln/\alpha) \ll_\varepsilon (NL)^\varepsilon\left(\frac{N}{q^{1/2}} + N^{1/2}q^{1/2} + \frac{N^{1/2}q^{1/2}}{L^{1/2}} + N^{3/4}d^{1/2} + \frac{N^{4/5}}{d^{1/5}}\right). \quad (16)$$

By (6) and (16), we obtain

$$\sum_{\substack{n\le N\\ n\equiv f(d)}} \Lambda(n)\chi\left(\frac{n}{\alpha}+\frac{1-2\beta}{2\alpha}\right) = \frac{1}{\alpha}\sum_{\substack{n\le N\\ n\equiv f(d)}} \Lambda(n) + O\left(\frac{N}{L(\varphi(d))}\right) + O\left((NL)^\varepsilon\left(\frac{N}{q^{1/2}} + N^{1/2}q^{1/2} + N^{3/4}d^{1/2} + \frac{N^{4/5}}{d^{1/5}}\right)\right).$$

Choose

$$L = \frac{q^{1/2}}{\varphi(d)}.$$

Therefore, we obtain an estimate

$$\sum_{\substack{n\le N\\ n\equiv f(d)}} \Lambda(n)\chi\left(\frac{n}{\alpha}+\frac{1-2\beta}{2\alpha}\right) = \frac{1}{\alpha}\sum_{\substack{n\le N\\ n\equiv f(d)}} \Lambda(n) + O\left(N^\varepsilon\left(\frac{N}{q^{1/2}} + N^{1/2}q^{1/2} + N^{3/4}d^{1/2} + \frac{N^{4/5}}{d^{1/5}}\right)\right).$$

We rewrite the above equality as

$$\sum_{\substack{p^\nu\le N\\ p^\nu\in\mathcal{B}(\alpha,\beta)\\ p^\nu\equiv f(d)}} \log p + \sum_{\substack{p^\nu\le\alpha+\beta-1\\ p^\nu\in\mathcal{B}(\alpha,\beta-\lfloor\alpha+\beta\rfloor)\\ p^\nu\equiv f(d)}} \log p = \frac{1}{\alpha}\sum_{\substack{p^\nu\le N\\ p^\nu\equiv f(d)}} \log p + O\left(N^\varepsilon\left(\frac{N}{q^{1/2}} + N^{1/2}q^{1/2} + N^{3/4}d^{1/2} + \frac{N^{4/5}}{d^{1/5}}\right)\right).$$

By using prime number theorem, we have

$$\sum_{\substack{p^\nu \le N \\ p^\nu \equiv f(d) \\ \nu \ge 2}} \log p \ll \sqrt{N}.$$

Thus we have

$$\sum_{\substack{p \le N \\ p \in \mathcal{B}(\alpha,\beta) \\ p \equiv f(d)}} \log p = \frac{1}{\alpha} \sum_{\substack{p \le N \\ p \equiv f(d)}} \log p + O\left(\frac{N^{1/2}}{\alpha}\right)$$
$$+ O\left(N^{\varepsilon}\left(\frac{N}{q^{1/2}} + N^{1/2}q^{1/2} + N^{3/4}d^{1/2} + \frac{N^{4/5}}{d^{1/5}}\right)\right). \tag{17}$$

Suppose we assume that α is an irrational number of finite type t. Note that by Lemma 3, $\frac{1}{\alpha}$ and α are of same type. Hence, for any positive ε, there is a positive constant c such that

$$\left|\frac{1}{\alpha} - \frac{p}{q}\right| \ge \frac{c}{q^{t+1+\varepsilon}} \tag{18}$$

holds for all integers p, q with $q > 0$. As discussed in Theorem 1, by using Dirichlet's approximation theorem with $Q = N^{\frac{t}{1+t}}$, we obtain a convergent to the simple continued fraction expansion of $\frac{1}{\alpha}$, say $\frac{p}{q}$, with $1 \le q \le N^{\frac{t}{1+t}}$ such that

$$\left|\frac{1}{\alpha} - \frac{p}{q}\right| \le \frac{1}{qN^{\frac{t}{t+1}}}. \tag{19}$$

Then by (19) and (18), there exists a convergent to the simple continued fraction expansion of $\frac{1}{\alpha}$ whose denominator satisfies

$$N^{\frac{1}{1+t}+\varepsilon} \ll q \le N^{\frac{t}{1+t}}. \tag{20}$$

Therefore, by (17) and (20), we obtain

$$\sum_{\substack{p \le N \\ p \in \mathcal{B}(\alpha,\beta) \\ p \equiv f(d)}} \log p = \frac{1}{\alpha} \sum_{\substack{p \le N \\ p \equiv f(d)}} \log p + O(N^{\varepsilon}(N^{1-\frac{1}{2(1+t)}} + N^{3/4}d^{1/2} + N^{4/5}d^{-1/5})).$$

This completes the proof of Theorem 3.

If N is a prime number, Fermat's little theorem asserts that $a^N \equiv a \pmod N$ for all $a \in \mathbb{Z}$. A Carmichael number is a composite number N with the same property.

Banks and Yeager (Theorem 1, [3]) proved that for $\alpha > 1$ of finite type irrational, there are infinitely many Carmichael numbers composed solely of primes from the Beatty sequence $\mathcal{B}(\alpha, \beta)$. They also proved the quantitative version of the above result (Theorem 3 of [3]). As an application of Theorem 3, we deduce the following corollary which extends the range of B in Theorem 3 of [3] for type $t > 1$. To state the corollary, we require the following notations of [3]:

Let $\pi(x)$ be the number of primes $p \leq x$, and let $\pi(x, y)$ be the number of those primes for which $p - 1$ is free of prime factors exceeding y. We denote by $\mathcal{E}$ the set of numbers E in the range $0 < E < 1$ for which there exists a number $x_0(E), \gamma(E) > 0$ such that

$$\pi(x, x^{1-E}) \geq \gamma(E)\pi(x) \quad \text{for all } x \geq x_0(E).$$

COROLLARY 1

For each $E \in \mathcal{E}$, $B \in (0, \min\{\frac{1}{2t+2}, \frac{1}{6}\})$ and $\varepsilon > 0$, there is a number $x_0 = x_0(\alpha, \beta, E, B, \varepsilon)$ such that for any $x \geq x_0$, there are at least $x^{EB-\varepsilon}$ Carmichael numbers up to x composed solely of primes from $\mathcal{B}(\alpha, \beta)$.

The proof of Corollary 1 follows from Lemma 5, Lemma 6 of [3] by using Theorem 3 in place of Theorem 2 of [3].

6. Proof of Proposition 2

The proof of Proposition 2 is based on the work of Balog and Perelli [2].

6.1 *Some lemmas*

Here we list several lemmas required for the proof.

Lemma 4.

$$\sum_{\substack{x<m\leq x' \\ m\equiv f(d)}} e(m\theta) \ll \min\left(\frac{x'}{d} + 1, \|\theta d\|^{-1}\right).$$

Lemma 5 [16]. *Suppose that $X, Y \geq 1$ are positive integers. Also suppose that $|\alpha - a/q| < q^{-2}$, where α is a real number, a and q integers satisfying $(a, q) = 1$. Then*

$$\sum_{x\leq X} \min(Y, \|\alpha x\|^{-1}) \ll \frac{XY}{q} + (X + q)\log 2q,$$

$$\sum_{x\leq X} \min\left(\frac{XY}{x}, \|\alpha x\|^{-1}\right) \ll \frac{XY}{q} + (X + q)\log(2XYq).$$

Lemma 6 [14]. *For any real number ϑ and natural numbers N, l and $1 \leq f < d$ such that $(f, d) = 1$, we have*

$$\sum_{\substack{n\leq N \\ n\equiv f(d)}} \Lambda(n)e(ln\,\vartheta) = O(N^{1/2}) + S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{m \le U} \sum_{\substack{n \le N/m \\ mn \equiv f(d)}} \mu(m)(\log n) e(lmn\vartheta),$$

$$S_2 = \sum_{m \le U^2} \sum_{\substack{n \le N/m \\ mn \equiv f(d)}} \phi_1(m) e(lmn\vartheta),$$

$$S_3 = \sum_{U < m \le N/U} \sum_{\substack{U < n \le N/m \\ mn \equiv f(d)}} \phi_2(m) \Lambda(n) e(lmn\vartheta),$$

and

$$\phi_1(m) \ll \log m, \quad \phi_2(m) \ll d_2(m).$$

Here U is an arbitrary parameter to be chosen later satisfying $1 \le U \le N^{1/2}$.

Lemma 7. *Suppose that $\varepsilon > 0$ and that $\phi(u)$, $\psi(v)$ are real-valued functions such that $|\phi(u)| \ll T$, $|\psi(v)| \ll F$. Suppose that $|\vartheta - a/q| < q^{-2}$, where ϑ is a real number, a and q are integers satisfying $(a, q) = 1$. For positive integers N, W, X and L, write*

$$S = \sum_{|l| \le L} \Bigg| \sum_{X < v \le 2X} \sum_{\substack{u \le W \\ uv \le N \\ uv \equiv f(d)}} \phi(u)\psi(v) e(luv\vartheta) \Bigg|. \tag{21}$$

Then

$$S \ll TF \Bigg(\frac{LWX^{1/2}}{d^{1/2}} + (LXd)^{\varepsilon} \Bigg(\frac{LXW}{q^{1/2}} + LXW^{1/2}d^{1/2} + L^{1/2}q^{1/2}X^{1/2}W^{1/2} \Bigg) \Bigg).$$

Proof. For the moment, we shall ignore the condition $uv \le N$ in (21). Consider

$$S = \sum_{|l| \le L} \Bigg| \sum_{X < v \le 2X} \sum_{\substack{u \le W \\ uv \equiv f(d)}} \phi(u)\psi(v) e(luv\vartheta) \Bigg|. \tag{22}$$

We observe that

$$S = \sum_{\substack{f_1 f_2 \equiv f(d) \\ (f_1, d) = (f_2, d) = 1}} R_{f_1, f_2} \ll d \max_{\substack{f_1 f_2 \equiv f(d) \\ (f_1, d) = (f_2, d) = 1}} |R_{f_1, f_2}|, \tag{23}$$

where

$$R_{f_1, f_2} = \sum_{|l| \le L} \Bigg| \sum_{\substack{X < v \le 2X \\ v \equiv f_2(d)}} \sum_{\substack{u \le W \\ u \equiv f_1(d)}} \phi(u)\psi(v) e(luv\vartheta) \Bigg|.$$

By using Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
|R_{f_1,f_2}|^2 &\le L \sum_{|l|\le L} \Bigg| \sum_{\substack{u\le W\\ u\equiv f_1(d)}} \sum_{\substack{X<v\le 2X\\ v\equiv f_2(d)}} \phi(u)\psi(v)e(luv\vartheta)\Bigg|^2\\
&\le L \sum_{|l|\le L} \Bigg(\sum_{\substack{u\le W\\ u\equiv f_1(d)}} |\phi(u)|^2\Bigg)\Bigg(\sum_{\substack{u\le W\\ u\equiv f_1(d)}} \Bigg| \sum_{\substack{X<v\le 2X\\ v\equiv f_2(d)}} \psi(v)e(luv\vartheta)\Bigg|^2\Bigg)\\
&\le \frac{T^2LW}{d}\left(\frac{F^2LWX}{d^2}+R_1\right), \qquad (24)
\end{aligned}
$$

where

$$
R_1 = \sum_{|l|\le L} \sum_{\substack{u\le W\\ u\equiv f_1(d)}} \sum_{\substack{X<v_1,v_2\le 2X\\ v_1,v_2\equiv f_2(d)\\ v_1\ne v_2}} \psi(v_1)\psi(v_2)e(lu(v_1-v_2)\vartheta).
$$

We may write R_1 in the form

$$
\begin{aligned}
R_1 &= \sum_{|l|\le L} \sum_{\substack{u\le W\\ u\equiv f_1(d)}} \sum_{\substack{|k|\le X\\ k\equiv 0(d)}} \sum_{\substack{X<v_1,v_2\le 2X\\ v_1,v_2\equiv f_2(d)\\ v_1\ne v_2\\ v_1-v_2=k}} \psi(v_1)\psi(v_2)e(luk\vartheta)\\
&\le \sum_{|l|\le L} \sum_{\substack{|k|\le X\\ k\equiv 0(d)}} \zeta_1(k) \sum_{\substack{u\le W\\ u\equiv f_1(d)}} e(luk\vartheta), \qquad (25)
\end{aligned}
$$

where

$$
\zeta_1(k) = \sum_{\substack{X<v_1,v_2\le 2X\\ v_1,v_2\equiv f_2(d)\\ v_1\ne v_2\\ v_1-v_2=k}} \psi(v_1)\psi(v_2) \ll \frac{F^2X}{d}.
$$

Applying Lemma 4 for the innermost sum of (25), we get

$$
|R_1| \le \frac{F^2X}{d} \sum_{|l|\le L} \sum_{\substack{|k|\le X\\ k\equiv 0(d)}} \min\left(\frac{W}{d}+1, \|lkd\vartheta\|^{-1}\right).
$$

Let $r = lkd$ so that $1 \le |r| \le 2LXd$ and r will run through all the integers in the interval above, also the number of representations of r is not more than $d_2(r)$. Therefore we have

$$
|R_1| \le \frac{F^2X}{d}(LXd)^{\varepsilon} \sum_{|r|\le 2LXd} \min\left(\frac{W}{d}+1, \|r\vartheta\|^{-1}\right).
$$

Then by using Lemma 5, we obtain

$$R_1 \ll F^2(LXd)^{\varepsilon}\left(\frac{LX^2W}{qd} + LX^2 + \frac{qX}{d}\right). \tag{26}$$

By (24) and (26), we have

$$R_{f_1,f_2} \ll TF\left(\frac{LWX^{1/2}}{d^{3/2}} + (LXd)^{\varepsilon}\left(\frac{LXW}{q^{1/2}d} + \frac{LXW^{1/2}}{d^{1/2}} + \frac{L^{1/2}q^{1/2}X^{1/2}W^{1/2}}{d}\right)\right). \tag{27}$$

Thus lemma follows from (23) and (27) . □

Lemma 8. *Suppose we have the hypotheses and notations of Lemma 7 with either $\phi(x) = 1$ or $\phi(x) = \log x$ for all x. Then*

$$S \ll F(LXd)^{\varepsilon}(LXWq^{-1} + LXd + q). \tag{28}$$

Proof. The $\log x$ factor may easily be removed by partial summation formula, so we presume that $\phi(x) \equiv 1$. Again we may ignore the condition $uv \leq N$. Therefore, we need to estimate

$$\begin{aligned} S &= \sum_{|l|\leq L}\left|\sum_{\substack{X<v\leq 2X \\ uv\equiv f(d)}}\sum_{u\leq W}\psi(v)e(luv\vartheta)\right| \\ &\leq F\sum_{|l|\leq L}\sum_{\substack{X\leq v<2X \\ (v,d)=1}}\left|\sum_{\substack{u\leq W \\ u\equiv f\bar{v}(d)}} e(luv\vartheta)\right|, \end{aligned}$$

where $\bar{v}$ is defined by $v\bar{v} \equiv 1 \pmod d$. Then by using Lemma 4, we have

$$S \leq F\sum_{|l|\leq L}\sum_{\substack{X\leq v<2X \\ (v,d)=1}} \min\left(\frac{W}{d} + 1, \|lvd\vartheta\|^{-1}\right).$$

Let $r = lvd$ so that $1 \leq |r| \leq 2LXd$ and r will run through all the integers in the interval above. Also, the number of representations of r is not more than $d_2(r)$. Therefore, we have

$$S \leq F(LXd)^{\varepsilon}\sum_{|r|\leq 2LXd} \min\left(\frac{W}{d} + 1, \|r\vartheta\|^{-1}\right). \tag{29}$$

Thus (28) follows from (29) and Lemma 5. □

6.2 *Proof of the Proposition* 2

We may assume that

$$N \geq \max(qd^2L^{-1}, d^6), \quad q \geq d^2 \tag{30}$$

otherwise (16) is a consequence of the trivial bound,

$$S(\vartheta) \leq \frac{LN}{d}.$$

Using Lemma 6, we have the following sums to estimate:

$$S_1' = \sum_{|l|\leq L} \left| \sum_{\substack{m\leq U \\ mn\equiv f(d)}} \sum_{n\leq N/m} \mu(m)(\log n)e(lmn\vartheta) \right|,$$

$$S_2' = \sum_{|l|\leq L} \left| \sum_{\substack{m\leq U^2 \\ mn\equiv f(d)}} \sum_{n\leq N/m} \phi_1(m)e(lmn\vartheta) \right|,$$

$$S_3' = \sum_{|l|\leq L} \left| \sum_{\substack{U<m\leq N/U \\ mn\equiv f(d)}} \sum_{U<n\leq N/m} \phi_2(m)\Lambda(n)e(lmn\vartheta) \right|.$$

By dyadic division, we write

$$S_1' \leq \sum_{t=0}^{\left[\frac{\log U}{\log 2}\right]} S_{1t},$$

where

$$S_{1t} = \sum_{l\leq L} \left| \sum_{\substack{2^t<m\leq 2^{t+1} \\ mn\equiv f(d)}} \sum_{n\leq N/m} \mu(m)(\log n)e(lmn\vartheta) \right|.$$

Then using Lemma 8, we get

$$S_1' \ll (NL)^{\varepsilon}(LNq^{-1} + LUd + q).$$

S_2' can be estimated similar to S_1' by partitioning into dyadic sub-sums, say S_{2t}. We estimate S_{2t} using Lemma 8, and we get

$$S_2' \ll (NL)^{2\varepsilon}(LNq^{-1} + LU^2d + q).$$

We write $S_3' = S_{31}' + S_{32}'$, where

$$S_{31}' = \sum_{l\leq L} \left| \sum_{\substack{U<m\leq N^{1/2} \\ mn\equiv f(d)}} \sum_{U<n\leq N/m} \phi_2(m)\Lambda(n)e(lmn\vartheta) \right|,$$

$$S_{32}' = \sum_{l\leq L} \left| \sum_{\substack{U<n\leq N^{1/2} \\ mn\equiv f(d)}} \sum_{N^{1/2}<m\leq N/n} \phi_2(m)\Lambda(n)e(lmn\vartheta) \right|.$$

By dividing S_{31}' dyadically, we obtain

$$S_{31}' = \sum_{t=0}^{R} S_{31t},$$

where

$$S_{31t} = \sum_{l \le L} \Bigg| \sum_{\substack{U2^t < m \le U2^{t+1} \\ mn \equiv f(d)}} \sum_{U < n \le N/m} \mu(m)(\log n)e(lmn\vartheta) \Bigg|$$

and

$$R = \left[\frac{\log(\frac{N^{1/2}}{U})}{\log 2} \right].$$

Then using Lemma 7, we have

$$S'_{31} \ll (NL)^{3\varepsilon} \left(\frac{LN}{q^{1/2}} + LN^{1/2}q^{1/2} + LN^{3/4}d^{1/2} + \frac{LN}{U^{1/2}d^{1/2}} \right).$$

Similarly we can show that S'_{32} has the same upper bound. Therefore,

$$S(\vartheta) \ll_\varepsilon (NL)^{\varepsilon} \left(\frac{NL}{q^{1/2}} + L^{1/2}N^{1/2}q^{1/2} \right.$$
$$\left. + LN^{3/4}d^{1/2} + LU^2 d + q + \frac{LN}{U^{1/2}d^{1/2}} \right). \qquad (31)$$

Then (16) follows from (31) with the choice of

$$U = \frac{N^{2/5}}{d^{3/5}},$$

and the observation $q \le L^{1/2}N^{1/2}q^{1/2}$.

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