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Character on a homogeneous space

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Abstract. In this paper, we look at the notion of cohomological triviality of fibrations of homogeneous spaces of affine algebraic groups defined over $\mathbb{C}$ and use topological methods, primarily the theory of covering spaces. This is made possible because of the structure theory of affine algebraic groups. Further, we generalize our results for arbitrary connected algebraic groups and their homogeneous spaces. As an application of our methods, we give a structure result for quasi- reductive algebraic groups (i.e., algebraic groups whose unipotent radical is trivial), up to isogeny.

Keywords. Algebraic groups; algebraic topology; homogeneous spaces.

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1. Introduction

We will be working over the field of complex numbers throughout this paper. A linear algebraic group G acting linearly on a vector space V, so that there is an open dense orbit U, is known as a prehomogeneous representation and has been extensively studied by Sato and Kimura [21]. Further, $U \approx \frac{G}{H}$ is a homogeneous space, where H is a closed subgroup of G. In [7], Damon considers a special kind of prehomogeneous representations, in which the complement of U is actually a divisor, the defining polynomial f of the divisor is a relative invariant and hence is homogeneous (c.f. [11, Corollary 2.7]) . We note that, when G is reductive the following are equivalent:

(1) U^c is a hypersurface,
(2) H is reductive,
(3) $\frac{G}{H}$ is affine.

The equivalence of 2 and 3 is known as Matsushima's theorem, see [18]. For the proof of the above equivalence, see [11, Theorem 2.28, page 43]. This function f is defined on all of V, and in particular restricting to U we have $f : U \to \mathbb{C}^*$. From the point of view of singularity theory, which Damon is concerned with, the complement of U defines a non-isolated singularity and $f : \frac{G}{H} \to \mathbb{C}^*$ is a global Milnor fibration, with fibres also being homogeneous spaces. For the study of the Milnor fibration, Damon defines (rational) cohomological triviality of fibrations and obtains a very simple numerical criterion for fibrations over the base S^1 to satisfy cohomological triviality.

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In this paper, however, we are not concerned with the topology of the non-isolated singularities. We characterize cohomological triviality of fibrations over $\mathbb{C}^*$ considering the general setting of homogeneous spaces of algebraic groups, instead of the special case of prehomogeneous spaces with the complement a divisor as in [7].

In Section 2, we recall the equivalent notion of cohomological triviality of fibrations as in [7] (the equivalence of both the definitions is also proved in the same article). We will be working throughout with singular cohomology with rational coefficients and will therefore omit the coefficients. We prove some preliminary topological results and give examples of fibrations which are cohomologically trivial and also some examples of fibrations which are not cohomologically trivial.

In Section 3, we come to the main results of the paper. We shall consider homogeneous spaces of connected affine algebraic groups with a nowhere vanishing non- constant function to $\mathbb{C}$. This function defines a character (possibly after dividing by $f(e)$), $\mathcal{X}$, of G which in turn gives rise to an auxiliary fibration.

A character is called primitive if the kernel is connected. Any character is a power of a unique primitive character. Let $\mathcal{X}_0$ be the primitive character of $\mathcal{X}$. We first analyse and show that the fibration, which is also an exact sequence of algebraic groups,

$$S^0 \longrightarrow G \xrightarrow{\mathcal{X}_0} \mathbb{C}^*$$

is cohomologically trivial, where S^0 is the identity component of $\ker(\mathcal{X})$ and the kernel of the primitive character $\mathcal{X}_0$, c.f. Proposition 3.2.

Before starting the analysis of fibrations of affine groups we make a simplifying assumption that the group G is reductive. This assumption is in no loss of generality from the point of topology, the unipotent radical G_u is contractible thus G and $\frac{G}{G_u}$ have the same homotopy type. Also from the point of view of algebraic groups, any character of G restricts to the trivial character on G_u. From this observation we will show that when the kernel of the associated character is connected, then the fibration of homogeneous spaces

$$\frac{S}{H} \longrightarrow \frac{G}{H} \xrightarrow{\mathcal{X}} \mathbb{C}^*$$

is cohomologically trivial, c.f. Theorem 3.7.

From the above, we will deduce that the fibration

$$\frac{S}{H} \longrightarrow \frac{G}{H} \xrightarrow{\mathcal{X}} \mathbb{C}^*$$

is cohomologically trivial if and only if the natural covering map $\frac{G}{S^0 \cap H} \to \frac{G}{H}$ induces an isomorphism $H^*\left(\frac{G}{H}\right) \approx H^*\left(\frac{G}{S^0 \cap H}\right)$, c.f. Corollary 3.9. Our proof, in particular, shows that the monodromy group of the fibration $\mathcal{X}$ is finite and hence is semisimple.

In Section 4, we construct a class of fibrations, with connected fibres, which are not cohomologically trivial and consequently the kernel of the associated character is not connected, c.f. Theorem 4.1. In Section 5, we move to the very general setting of homogeneous spaces of algebraic groups, not necessarily affine. We prove some structure theorems for these groups, viz. the existence of a unipotent radical and maximal central torus. As before, we quotient out the unipotent radical and we will show that our results hold in this general setting for algebraic groups with trivial unipotent radical which we call quasi-reductive groups. As an application we give a structure result for quasi- reductive groups, c.f. Theorem 5.10.

2. Preliminaries

We will use the following equivalent definition of cohomological triviality of fibrations:

DEFINITION 2.1

A fibration $F \hookrightarrow E \to B$ is said to be rationally cohomologically trivial if it satisfies the Künneth formula, i.e., for each k,

$$H^k(E;\mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(F;\mathbb{Q}) \otimes H^q(B;\mathbb{Q}),$$

the isomorphism is of vector spaces. This implies

$$b_k(E) = \sum_{p+q=k} b_p(F) \cdot b_q(B),$$

for all k, where b_k is the k-th Betti number.

Following [7], we can also define (rational) cohomological triviality of a fibration when the base is $\mathbb{C}^*$as follows: Let $F \hookrightarrow E \xrightarrow{\pi} \mathbb{C}^*$ be a fibration, let $\alpha(t) = e^{2\pi i t}$ be a loop at 1, now consider the monodromy map σ_1, the lift of α to a family of homeomorphisms $\sigma_t : F \to F_t = \pi^{-1}\alpha(t)$.

Lemma 2.2. *A fibration $F \hookrightarrow E \xrightarrow{\pi} \mathbb{C}^*$ is* (*rationally*) *cohomologically trivial if and only if $\sigma_1^* : H^*(F;\mathbb{Q}) \to H^*(F;\mathbb{Q})$ is the identity.*

Proof. The proof follows from [7, Proposition 1.8] by observing that in the case $B = \mathbb{C}^*$, the definition of cohomological triviality is equivalent to the assertion of the lemma. □

Observe that if E is connected and the fibre F is not connected, then the fibration $F \hookrightarrow E \to B$ is not cohomologically trivial. This is clear by looking at b_0, the zeroth Betti number. This leads us to make a global assumption that the fibres of any fibration we are considering to be connected.

We will now recall the Leray–Serre spectral sequence. We will refer to [12, Theorem 5.2, Page 135] for the details.

PROPOSITION 2.3 (The Leray–Serre spectral sequence)

Let $F \to E \xrightarrow{p} B$ be a fibration, with B path connected, F connected. Then there is a first quadrant spectral sequence $\{E_r^{\star,\star}, d_r\}$, with $E_2^{p,q} \approx H^p(B;\mathcal{H}^q(F;\mathbb{Q}))$, the cohomology of the space B with the local coefficients in the cohomology of the fibres of p. The spectral sequence gives a filtration on $H^n(E;\mathbb{Q})$ and the associated graded vector space is $H^n(E;\mathbb{Q}) = \oplus_{p+q=n} E_\infty^{p,q}$.

By a simple system of local coefficients, we mean the pullback of any closed path in B induces the identity homomorphism for the cohomology of the fibre F, i.e., let $\omega \subset B$ be any closed path then the homomorphism $h[\omega]_* : H^*(F_{\omega(0)};\mathbb{Q}) \to H^*(F_{\omega(1)};\mathbb{Q})$ is the identity homomorphism. In [23, page 476], a fibration with simple system of local coefficients is referred to as an orientable fibration.

We say that F, B are of finite type if they have the homotopy type of finite CW complexes. When F, B are of finite type and the system of local coefficients is simple, then the E_2 stage of the spectral sequence takes the form $E_2^{p,q} \approx H^p(B;\mathbb{Q}) \otimes H^q(F;\mathbb{Q})$ (c.f. [12, Proposition 5.5, page 139].

A spectral sequence is said to degenerate at the stage r if $d^j = 0, \forall j \geq r$. This gives rise to a filtration on $H^n(E;\mathbb{Q})$ and the associated grading is $H^n(E;\mathbb{Q}) = \oplus_{p+q=n} E_r^{p,q}$.

We will now prove the following lemma when the base of the fibration is $\mathbb{C}^*$.

Lemma 2.4. *Consider the fibration $F \to E \to \mathbb{C}^*$, then Leray–Serre spectral sequence degenerates at E_2 stage. Further, the system of local coefficients being simple is equivalent to cohomological triviality of the fibration.*

Proof. For a fibration with base $\mathbb{C}^*$, the only non-zero terms in the E_2 stage are in $E_2^{0,q}$ and $E_2^{1,q}$ and hence the differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ is zero. By definition, the system of local coefficients is simple if and only if the monodromy is trivial. By the Wang sequence (c.f. [23, Corollary 6, page 456]), monodromy being trivial is equivalent to the variation being zero and thus the fibration is cohomologically trivial. □

Now let B be any path connected space. Assume that F, B are of finite type and let the system of local coefficients defined by F on B be simple. If the Leray–Serre spectral sequence degenerates at the stage E_2, then $H^n(E;\mathbb{Q}) = \oplus_{p+q=n} E_2^{p,q} = \oplus_{p+q=n} H^p(B;\mathbb{Q}) \otimes H^q(F,\mathbb{Q})$. Thus we have showed that the degeneration of the spectral sequence implies cohomological triviality.

Conversely, suppose that $F \to E \to B$ is cohomologically trivial. Then we have that $H^n(E;\mathbb{Q}) = \oplus_{p+q=n} H^p(B;\mathbb{Q}) \otimes H^q(F,\mathbb{Q})$. By the Leray–Serre spectral sequence, we see that $H^n(E;\mathbb{Q}) = \oplus_{p+q=n} E_\infty^{p,q}$. We note that the E_∞ terms are a sub quotient of the E_2 stage and by cohomological triviality we have that $E_\infty^{p,q} = H^n(E;\mathcal{Q}) = \oplus_{p+q=n} H^p(B;\mathbb{Q}) \otimes H^q(F,\mathbb{Q}) = E_2^{p,q}$, i.e., the Leray–Serre spectral sequence degenerates at E_2 stage.

Thus we have shown the following.

Lemma 2.5. *Let $F \to E \xrightarrow{p} B$ be a fibration with F connected and of finite type. Suppose that the system of local coefficients on B is simple, then the degeneration of the Leray–Serre spectral sequence at E_2 stage is equivalent to the cohomological triviality of the fibration.*

Remark 2.6. Note that we prove the equivalence of degeneration of the spectral sequence at E_2 and cohomological triviality only when the system of local coefficients is simple. For a general fibration, we do not know the interplay between the degeneration of the Leray-Serre spectral sequence at E_2 stage, cohomological trivial fibrations and the simple system of local coefficients.

If the base is $\mathbb{C}^*$, we have the following lemma.

Lemma 2.7. *Let $F \to E \xrightarrow{p} \mathbb{C}^*$ be a fibration, such that the fibre F is not connected, and the total space E is connected. Then there exists a connected covering $\mathbb{C}^* \to \mathbb{C}^*$ and a lift $\tilde{p} : E \to \mathbb{C}^*$, such that the fibre bundle $F_0 \to E \xrightarrow{\tilde{p}} \mathbb{C}^*$ has connected fibre F_0.*

Proof. Since E and $\mathbb{C}^*$ are path connected, the long exact homotopy sequence of the fibration $F \to E \xrightarrow{p} \mathbb{C}^*$ reduces to

$$\ldots \pi_1(F, f) \longrightarrow \pi_1(E, e) \xrightarrow[p_*]{} \pi_1(\mathbb{C}^*, 1) \longrightarrow \pi_0(F, f) \longrightarrow 1.$$

This sequence shows, in particular, that p_* is surjective if and only if the fibre F is connected.

Let the number of path components of F be n, then $\mathrm{Im}(\pi_1(E))$ is a subgroup of $\pi_1(\mathbb{C}^*) \approx \mathbb{Z}$, of index n. Hence we have the following commutative diagram,

$$\begin{array}{ccccc} & & & & \mathbb{C}^* \\ & & & \overset{\tilde{p}}{\nearrow} & \big\downarrow z \mapsto z^n \\ F & \longrightarrow & E & \xrightarrow[p]{} & \mathbb{C}^* \end{array},$$

where the dotted arrow is the lift of map p, which exists as the map $\mathbb{C}^* \to \mathbb{C}^*$ is a covering map. Now the fibre bundle $E \xrightarrow{\tilde{p}} \mathbb{C}^*$ has connected fibre. □

Let us now look at some examples of fibrations and check cohomolgical triviality.

Example 2.8. The Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

is not cohomologically trivial, for the term $H^2(S^3)$ is zero, whereas $H^2(S^2) \otimes H^0(S^1) = \mathbb{Z}$.

We now give an example of a fibration which is cohomologically trivial

Example 2.9. For any n, the exact sequence of algebraic groups,

$$1 \to SL_n \to GL_n \xrightarrow{\det} \mathbb{C}^* \to 1$$

is cohomologically trivial, where 'det' is the determinant character. For this, the cohomologies of GL_n and SL_n are exterior algebra on odd degree generators with the cohomology of SL_n beginning at degree 3 and GL_n beginning at degree 1. For details, we refer to [7,16]. Furthermore, note that GL_n is a product of SL_n and $\mathbb{C}^*$ as manifolds but not as groups.

Example 2.10. Similarly by looking at the cohomology algebra, the fibration

$$\frac{SL_n}{SO_n} \to \frac{GL_n}{O_n} \to \mathbb{C}^*$$

is not cohomologically trivial for $n = 2m$ and is cohomologically trivial for $n = 2m + 1$. We refer [7,16] for the details.

Example 2.11. The fibration $\frac{SL_n}{SO_n} \to \frac{GL_n}{SO_n} \to \mathbb{C}^*$ is cohomologically trivial for all n. We refer to [16] for the details.

We shall now prove some preliminary results.

Lemma 2.12. *Consider the following*: $H \hookrightarrow G \to \frac{G}{H}$, *where* G *is an affine algebraic group and* H *is a closed subgroup of* G. *Then* $H \to G \to \frac{G}{H}$ *is a fibre bundle.*

Proof. See for a proof in [22, Example 2.1.1.4(ii), page 105]. □

We note that in the above lemma the group G being affine is not necessary.

Lemma 2.13. *Let* G *be an algebraic group and* $\mathcal{X} : G \longrightarrow \mathbb{C}^*$ *be a non constant morphism of algebraic varieties*, *taking identity to identity. Then* $\mathcal{X}$ *is a character.*

Proof. This is a particular case of [19, Theorem 3]. For a modern proof, we refer the unpublished notes of Conrad [6, Corollary 1.2], where this result is referred to as Rosenlicht unit theorem. □

3. Cohomological triviality of characters

Let $\mathcal{X} : G \to \mathbb{C}^*$ be a non trivial map. By Lemma 2.13, $\mathcal{X}$ is a character. The unipotent radical, G_u being contractible and $\mathcal{X}|_{G_u}$ is trivial, we have the following lemma.

Lemma 3.1. *Let* G *be an arbitrary linear algebraic group and* $\mathcal{X} : G \to \mathbb{C}^*$ *be a non-trivial character with connected kernel* S. *Then the fibration*

$$S \to G \xrightarrow{\mathcal{X}} \mathbb{C}^*$$

is homotopic to the fibration

$$\frac{S}{G_u} \longrightarrow \frac{G}{G_u} \xrightarrow{\mathcal{X}} \mathbb{C}^*.$$

□

Throughout the section, we will assume that G is a connected reductive group by the reasoning of Lemma 3.1.

PROPOSITION 3.2

Let G *be a reductive group and* $\mathcal{X} : G \longrightarrow \mathbb{C}^*$ *be a non trivial character with connected kernel* S. *Then the fibration*

$$1 \to S \to G \xrightarrow{\mathcal{X}} \mathbb{C}^* \to 1$$

is cohomologically trivial.

Proof. By Lemma 2.12, the sequence $1 \to S \to G \xrightarrow{\mathcal{X}} \mathbb{C}^* \to 1$ is a fibre bundle. Since G is reductive there is a central $\mathbb{C}^*$ which surjects onto $\mathbb{C}^*$ under $\mathcal{X}$. This follows from the fact that a character of a reductive group is identically zero on the derived group and hence factors through $\frac{G}{D(G)}$, where $D(G)$ is the derived group of G and $\frac{G}{D(G)}$ is isogenous with the identity component of the center $Z(G)$. Thus if the character is non trivial we can find a central $\mathbb{C}^*$ in G that surjects onto $\mathbb{C}^*$. This can be seen to be as follows: Consider the restriction map $\mathcal{X}|_{Z(G)^0} : Z(G)^0 \to \frac{Z(G)^0}{\ker(\mathcal{X})^0} \to \frac{Z(G)^0}{\ker(\mathcal{X})}$, the last two terms are isomorphic to $\mathbb{C}^*$ and hence the map is given by $z \mapsto z^d$. Now choose any splitting of $\mathcal{X}|_{Z(G)^0} : Z(G)^0 \to \frac{Z(G)^0}{\ker(\mathcal{X})}$, i.e.,

$$\begin{array}{ccccc} S & \longrightarrow & G & \xrightarrow{\mathcal{X}} & \mathbb{C}^* \\ & & \uparrow & \nearrow_{\mathcal{X}|_{\mathbb{C}^*}} & \\ & & \mathbb{C}^* & & \end{array}.$$

The kernel of the character $\mathcal{X}|_{\mathbb{C}^*} : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is a finite cyclic group, say Γ, which induces a fibre product $\tilde{G}$ in $G \times \mathbb{C}^*$,

$$\begin{array}{ccccc} S & \longrightarrow & \tilde{G} & \longrightarrow & \mathbb{C}^* \\ \downarrow & & \pi_1\downarrow & \square & \downarrow \mathcal{X}|_{\mathbb{C}^*} \\ S & \longrightarrow & G & \underset{\mathcal{X}}{\longrightarrow} & \mathbb{C}^* \end{array}.$$

The cover $\tilde{G}$, contained in $G \times \mathbb{C}^*$, is connected (follows by looking at the long exact sequence of homotopy groups associated to the fibration and since the fibre S and the base space are connected) and abstractly isomorphic to $S \times \mathbb{C}^*$ by the map $f : S \times \mathbb{C}^* \longrightarrow \tilde{G}$ given by $(s, t) \longmapsto (st, t)$. This map is also a group homomorphism, which can be seen as follows: $(s_1, t_1)(s_2, t_2) \mapsto (s_1t_1s_2t_2, t_1t_2)$ and since the chosen $\mathbb{C}^*$ is central in G, we have $(s_1t_1s_2t_2, t_1t_2) = (s_1s_2t_1t_2, t_1t_2)$ and the other way, $(s_1, t_1)(s_2, t_2) = (s_1s_2, t_1t_2) \mapsto (s_1s_2t_1t_2, t_1t_2)$.

Thus the first projection $\pi_1 : \tilde{G} \longrightarrow G$ can be thought of as twisted by the isomorphism f, and hence by the identification by f, we can see that the upper fibration $S \to \bar{G} \to \mathbb{C}^*$ is not only cohomologically trivial but also a trivial fibration. Now consider any loop in $\mathbb{C}^*$, then its lift to $\mathbb{C}^*$ is a path from 0 to ζ where ζ is a root of unity (corresponding to an element of the finite cyclic group Γ). Since $\zeta \in S$, we have a self-map of S given by translation by ζ and since S is path connected we have that the translations are null-homotopic, therefore, the monodromy of the fibration $\tilde{G} \to \mathbb{C}^*$ is trivial. Thus by commutativity of the second diagram, the fibration $G \to \mathbb{C}^*$ is also cohomologically trivial, by Lemma 2.2. □

This gives another proof for the cohomological triviality of the fibration given in Example 2.5 of Section 2.

We can also see the cohomological triviality of the fibration $S \to G \to \mathbb{C}^*$ by a Wang sequence argument as given in [7], considering a fibration of maximal compact subgroups and exploiting the Hopf algebra structure of the cohomology.

DEFINITION 3.3

A character $\mathcal{X}$ is said to be split if $G = S \times \mathbb{C}^*$ and $\mathcal{X}$ is the second projection, where S is the kernel of $\mathcal{X}$. A character $\mathcal{X}$ is said to be *quasi-split* if $\mathcal{X}$ is split after a finite covering.

As a corollary to Proposition 3.2, we have the following.

COROLLARY 3.4

A character $\mathcal{X} : G \to \mathbb{C}^*$ is split if and only if $\mathcal{X}|_{Z(G)^0}$ is primitive.

Proof. Note that $Z(G)^0 = Z(S)^0 \times \mathbb{C}^*$. Now suppose that the character $\mathcal{X}$ is split, then we have an isomorphic image of $\mathbb{C}^*$, say G_m, in $Z(G)^0$, i.e. there is a section of $\mathcal{X}$ restricted to $Z(G)^0$. Thus $\frac{Z(G)^0}{\ker(\mathcal{X}|_{Z(G)^0})} \approx \mathbb{C}^*$, i.e., $\mathcal{X}|_{Z(G)^0}$ is primitive.

Suppose that the restriction of $\mathcal{X}$ to $Z(G)^0$ is primitive, i.e., the kernel S_Z is connected and hence, by [1, Corollary, page 118], S_Z is a torus and a direct factor in $Z(G)^0$. Thus $Z(G)^0 = S_Z \times G_m$ and since $\ker(\mathcal{X})$ is connected, this G_m maps isomorphically onto $\mathbb{C}^*$, and by the proof of Proposition 3.2, we have that the fibre product $\tilde{G} \approx S \times \mathbb{C}^*$ is isomorphic to G. By the proposition, we need to just prove the injectivity of the composite of f with the projection map π_1. Let (s_1, t_1) and (s_2, t_2) be two arbitrary elements of $S \times G_m$ which map to the same element in G, i.e., $s_1 t_1 = s_2 t_2$. Now, $\mathcal{X}(s_1 t_1) = \mathcal{X}(s_2 t_2)$ and since $\mathcal{X}$ is a homomorphism and s_1, s_2 are in the kernel of $\mathcal{X}$ we have that $\mathcal{X}(t_1) = \mathcal{X}(t_2)$, and since $\mathcal{X}$ restricted to G_m is an isomorphism, we have that $t_1 = t_2$. From this, it follows that $s_1 = s_2$ and thus the composite is injective and hence the character is split. □

This corollary, in particular, shows that GL_n with the determinant character is not split.

COROLLARY 3.5

Consider a fibration of homogeneous spaces of a reductive group G,

$$\frac{S}{H} \to \frac{G}{H} \to \mathbb{C}^*.$$

Suppose that the homogeneous space $\frac{S}{H}$ is connected and that the subgroup H is normal in G, then the fibration is cohomologically trivial.

Proof. Since H is normal, $\frac{G}{H}$ and $\frac{S}{H}$ are groups and by the Proposition 3.2, we are done. □

Remark 3.6. We note that we are not considering all homogeneous spaces of a reductive group G, for example, we will not consider H to be a Borel or a parabolic subgroup of G, for the quotient $\frac{G}{H}$ would then be projective and thus has no non-trivial morphisms to any affine variety. We assume that the subgroup H is closed so that there is a character from $\frac{G}{H} \to \mathbb{C}^*$. For a such character map to exist, it is a necessary condition that the center $Z(G)^0 \not\subset H$. However, this condition is not sufficient: Consider the standard action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$. The stabilizer of the point $(1, 0)$ is the subgroup H consisting of the matrices $\begin{bmatrix} 1 & b \\ 0 & \lambda \end{bmatrix}$, where $b \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$. The quotient variety $\frac{G}{H} \approx \mathbb{C}^2 \setminus \{0\}$. Now if $\frac{G}{H}$ has a character, by Hartog's extension lemma, it extends to all of $\mathbb{C}^2$. But the inverse image of zero must be a divisor in $\mathbb{C}^2$, which is not true. Thus $Z(G)^0 \not\subset H$ but $\frac{G}{H}$ has no character.

Let $\mathcal{X} : \frac{G}{H} \to \mathbb{C}^*$ be a nowhere vanishing function on a homogeneous space of an algebraic group G, with H mapping to 1. Then we have a natural map $\mathcal{X} : G \to \mathbb{C}^*$ by composing with the quotient map. By abuse of notation we will call this composite map also as $\mathcal{X}$. By abuse of terminology, we will refer the map $\mathcal{X}$ on $\frac{G}{H}$ as a character. We will call the the fibration $S \to G \xrightarrow{\mathcal{X}} \mathbb{C}^*$ to be the associated fibration of algebraic groups, where S is the kernel of $\mathcal{X}$. We will first show that if the kernel of the associated fibration is connected then the associated fibration as well as the fibration of homogeneous spaces of affine algebraic groups is cohomologically trivial.

We shall now prove cohomological triviality of fibrations of homogeneous spaces when the kernel S is connected.

Theorem 3.7. *Consider a fibration of homogeneous spaces of a connected reductive group G:*

$$\frac{S}{H} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^*.$$

Suppose that the kernel S of the associated character is connected, then the fibration is cohomologically trivial.

Proof. Consider the fibration $1 \longrightarrow \frac{S}{H} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^* \longrightarrow 1$. By composing with the natural projection map from G, we get a map from G to $\mathbb{C}^*$ as

$$\begin{array}{ccc} \frac{G}{H} & \xrightarrow{\mathcal{X}} & \mathbb{C}^* \\ {\scriptstyle\pi}\uparrow & \nearrow_{\mathcal{X}} & \\ G, & & \end{array}$$

which we will call $\mathcal{X}$. This is a nowhere vanishing map from G onto $\mathbb{C}^*$ and thus is a non trivial character of G, by Lemma 2.13 (note that the identity element e of G is mapped to 1, as the subgroup H is mapped identically to 1). We will show that the lift of a loop in $\mathbb{C}^*$ is homotopic to identity and hence, the fibration is cohomologically trivial. Now consider the fibration of homogeneous space of a closed subgroup H, which is not necessarily connected, contained in S.

This gives rise to a commutative diagram as follows:

$$\begin{array}{ccccc} \frac{S}{H} & \longrightarrow & \frac{\tilde{G}}{H} & \longrightarrow & \mathbb{C}^* \\ \downarrow & & {\scriptstyle\pi_1}\downarrow & \square & \downarrow{\scriptstyle\mathcal{X}} \\ \frac{S}{H} & \longrightarrow & \frac{G}{H} & \xrightarrow[\mathcal{X}]{} & \mathbb{C}^* \end{array}.$$

As in Proposition 3.2, $\frac{\tilde{G}}{H}$ is isomorphic to $\frac{S}{H} \times \mathbb{C}^*$ induced by the map from $S \times \mathbb{C}^* \longrightarrow \tilde{G}$, i.e., $(\bar{x}, t) \mapsto (\bar{xt}, t)$ to see that the map defined as above is well defined. Note that $\bar{x} = xh$ for some h in H, thus $(\bar{x}, t) = (xh, t) \mapsto (xht, t) = (xth, t)$ (as $\mathbb{C}^*$ is central) which is same as $(\bar{xt}, t)$. Thus the upper fibration $\frac{S}{H} \to \frac{\tilde{G}}{H} \to \mathbb{C}^*$ is trivial. To show that the lift of a loop is homotopic to identity, for the homogeneous spaces, we follow a similar argument as above for the case of the group and we note that a loop in the lower $\mathbb{C}^*$ ends at ζ which is in S and hence in $\frac{S}{H}$ which gives rise to a self-map of $\frac{S}{H}$ given by translation by ζ. But translations in an homogeneous space of a connected algebraic group are isotopic to identity (since $\zeta \in S$, there is a path from ζ to 1 in S and this gives rise to automorphisms of $\frac{S}{H}$ which are homotopic to identity). And thus in the upper level the loop is homotopic to identity. Thus, by the commutativity of the diagram the loop is homotopic to identity in the lower level. Therefore, by Lemma 2.2, the fibration $\frac{S}{H} \to \frac{G}{H} \to \mathbb{C}^*$ is cohomologically trivial. □

Remark 3.8. Note that we have shown in Proposition 3.2 (and for homogeneous spaces in Theorem 3.7) that for a reductive group G and a non trivial, non constant map $\mathcal{X}$ to $\mathbb{C}^*$ with a connected kernel S is a product $S \times \mathbb{C}^*$ (a product $\frac{S}{H} \times \mathbb{C}^*$ for homogeneous spaces) after a finite cover! This, in particular, shows that the monodromy is semisimple.

We will now give a criterion for cohomological triviality when the kernel S of the associated character is not connected.

COROLLARY 3.9

Consider a fibration of homogeneous spaces of a connected reductive group G. Then

$$\frac{S}{H} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^*$$

and the fibration is cohomologically trivial if and only if $H^*\left(\frac{G}{H}\right) \approx H^*\left(\frac{G}{S^0 \cap H}\right)$, *where* S^0 *is the identity component of the kernel of the associated character.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccc} \frac{S}{H} & \longrightarrow & \frac{G}{H} & \xrightarrow{\mathcal{X}} & \mathbb{C}^* \\ \uparrow \approx & & \uparrow & & \uparrow \\ \frac{S^0}{S^0 \cap H} & \longrightarrow & \frac{G}{S^0 \cap H} & \xrightarrow{\tilde{\mathcal{X}}} & \mathbb{C}^* \end{array} .$$

Since S^0 is connected, by Theorem 3.7 the lower fibration is cohomologically trivial. Thus $\frac{S}{H} \to \frac{G}{H} \to \mathbb{C}^*$ is cohomologically trivial if and only if $H^*\left(\frac{G}{H}\right) \approx H^*\left(\frac{G}{S^0 \cap H}\right)$. □

Going back to Example 2.10, by computation of the cohomologies of the homogeneous spaces, $\frac{GL_n}{SO_n}$ and $\frac{GL_n}{O_n}$ show that the above criterion in Corollary 3.9 is satisfied by the fibration $\frac{SL_n}{SO_n} \to \frac{GL_n}{O_n} \to \mathbb{C}^*$ when $n = 2m + 1$ and the fibration does not satisfy the criterion when $n = 2m$.

We can deduce the general case for an arbitrary, but connected linear algebraic group. From this, we obtain as follows.

COROLLARY 3.10

Let $\frac{G}{H}$ be a homogeneous space of an arbitrary connected affine algebraic group with a nowhere vanishing function $\mathcal{X}$ to $\mathbb{C}$. If the kernel of the associated character is connected, then the fibration is cohomologically trivial.

Proof. Since we are looking at the singular cohomology of the homogeneous spaces, G and $\frac{G}{G_u}$, where G_u is the unipotent radical, are of the same homotopy type. This is because G_u is isomorphic to an affine n-space $\mathbb{A}^n$ (see [10], this is in fact true for any unipotent group defined over a perfect field) which is contractible. Thus it suffices to look at the reductive quotient $\frac{G}{G_u}$. Since a subgroup of a unipotent group is unipotent, the fibration

$$\frac{S}{H} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^*$$

reduces to

$$\frac{S/S\cap G_u}{H/H\cap G_u} \longrightarrow \frac{G/G_u}{H/H\cap G_u} \longrightarrow \mathbb{C}^*.$$

□

Remark 3.11. Note that in our analysis quotienting out the unipotent radical is essential. For example, not all characters of G are central when G is solvable, i.e. given a character $\mathcal{X} : G \to \mathbb{C}^*$, the central torus of $Z(G)^0$ may not surject onto $\mathbb{C}^*$. If G is a connected solvable group then the unipotent radical U consists of all unipotent elements of G and the quotient $\frac{G}{U}$ is a torus and G is homotopic to $\frac{G}{U}$, which is homotopic to a maximal torus T.

Thus, consider an exact sequence of a solvable group $S \to G \xrightarrow{\mathcal{X}} \mathbb{C}^*$. Note that $S \subset G$ is also solvable and G is homotopic to T and similarly S is homotopic to T_1, where T and T_1 are maximal tori in G and S respectively. Then the above fibration reduces to $T_1 \to T \xrightarrow{\mathcal{X}'} \mathbb{C}^*$, a fibration of tori where cohomological triviality is equivalent to checking the primitivity of $\mathcal{X}'$.

Remark 3.12. We now consider the case of fibration of homogeneous space of solvable groups $\frac{S}{H} \to \frac{G}{H} \xrightarrow{\mathcal{X}} \mathbb{C}^*$. We first analyse the exact sequence $H \to G \to \frac{G}{H}$. As before, $H \approx T_2$ and $G \approx T$, where T_2^0 and T are maximal tori of H and G respectively. Thus the exact sequence is equivalent to $T_2 \to T \to \frac{T}{T_2}$. For the same reason, the sequence $H \to S \to \frac{S}{H}$ is equivalent to $T_2 \to T_1 \to \frac{T_1}{T_2}$ where T_1^0 is a maximal torus in S. Thus the fibration $\frac{S}{H} \to \frac{G}{H} \xrightarrow{\mathcal{X}} \mathbb{C}^*$ is equivalent to the fibration

$$\frac{T_1}{T_2} \to \frac{T}{T_2} \to \mathbb{C}^*. \tag{1}$$

Since we have that $\frac{S}{H}$ is connected, the fibre $\frac{T_1}{T_2}$ is also connected and is homeomorphic to $\frac{T_1^0}{T_2^0}$ which is a torus. Thus the fibration (1) is equivalent to a fibration of tori

$$\frac{T_1^0}{T_1^0 \cap T_2} \to \frac{T^0}{T_2^0} \to \mathbb{C}^*,$$

which is cohomologically trivial.

4. Construction of homogeneous spaces with characters that are not cohomologically trivial

Let us consider an example.

Let G be a reductive group and $T \subset N(T)$ be a maximal torus of G in the normalizer of T. It is a well known result that $H^*\big(\frac{G}{T}\big) = \mathbb{Q}[W]$, where $W = \frac{N(T)}{T}$ is the Weyl group of G (the Weyl group is a finite group and the cohomology ring of $\frac{G}{T}$ is a regular representation of the Weyl group). Further, $H^*\big(\frac{G}{N(T)}\big) = \mathbb{Q}$. Moreover, for any subgroup H such that $T \subset H \subset N(T)$, the cohomology of $\frac{G}{H}$ can be computed by the covering $\frac{H}{T} \to \frac{G}{T} \to \frac{G}{H}$. Thus $H^*\big(\frac{G}{H}\big) = H^*\big(\frac{G}{T}\big)^{\frac{H}{T}}$ as graded algebras since $\frac{H}{T}$ is finite. $H^*\big(\frac{G}{T}\big)^{\frac{H}{T}} = \mathbb{Q}[\frac{W}{W_H}]$, where W_H is the image of H in W.

Motivated by this example we will now construct a class of fibrations that are not cohomologically trivial.

Theorem 4.1. *Let S be a reductive group and suppose that $H' \subset H$ be subgroups of S such that $\frac{H}{H'}$ is a finite cyclic group of order d and $H^*\big(\frac{S}{H}\big) \overset{\not\cong}{\hookrightarrow} H^*\big(\frac{S}{H'}\big)$, i.e., $b_j\big(\frac{S}{H}\big) \neq b_j\big(\frac{S}{H'}\big)$ for some j. Then there is an embedding of H in group $S \times \mathbb{C}^* = G$ such that the fibration $\frac{S}{H'} \to \frac{G}{H} \to \mathbb{C}^*$ is not cohomologically trivial.*

Proof. Embed H into $G = S \times \mathbb{C}^*$ as follows: $H \xrightarrow{(id,\eta)} G$ defined as $h \mapsto (h, hH')$, where $\eta : \frac{H}{H'} \to \mathbb{C}^*$ is an embedding. The image of H has the same number of connected components as H, so we will denote the image of H in G as H. We observe that $\frac{\mathbb{Z}}{\langle d\rangle} \times \frac{\mathbb{Z}}{\langle d\rangle}$ acts on $\frac{S}{H'} \times \mathbb{C}^*$ with the quotient $\frac{S}{H} \times \mathbb{C}^*$. The diagonal $\frac{\mathbb{Z}}{\langle d\rangle} =: \Gamma$ in $\frac{\mathbb{Z}}{\langle d\rangle} \times \frac{\mathbb{Z}}{\langle d\rangle}$ acts on $\frac{S}{H'} \times \mathbb{C}^* = \frac{S\times\mathbb{C}^*}{H'}$, and the action is defined as $\forall g \in \frac{H}{H'}$, $g(sh, t) = (shg, t^{\eta(g)}) = (sgh', t^{\eta(g)})$ for some $h, h' \in H', s \in S$. This is the same as the quotienting by the subgroup H embedded in G and hence the quotient is $\frac{G}{H}$. Now consider the commutative diagram:

$$\begin{array}{ccc} & G & \xrightarrow{f\circ\pi} \mathbb{C}^* \\ & \downarrow & \nearrow \\ \frac{S}{H'} \longrightarrow & \frac{G}{H} & \end{array}.$$

where π is the second projection and $f : \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto z^d$. We note that the fibre can be identified with $\frac{S}{H'}$, for $H \cap S \times 1 = H'$. We claim that the fibration

$\frac{S}{H'} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^*$ is not cohomologically trivial. Note that the cohomology of $\frac{G}{H}$ is the cohomology of $\frac{S}{H'} \times \mathbb{C}^*$ invariant under the action of Γ, i.e., $H^n(\frac{G}{H}) = \big(H^n(\frac{S}{H'} \times \mathbb{C}^*)\big)^{\Gamma}$. By the Künneth formula, we have $\big(H^n(\frac{S}{H'} \times \mathbb{C}^*)\big)^{\Gamma} = \Big(\big(H^n(\frac{S}{H'}) \otimes H^0(\mathbb{C}^*)\big) \oplus \big(H^{n-1}(\frac{S}{H'} \otimes H^1(\mathbb{C}^*)\big)^{\Gamma} = \big(H^n(\frac{S}{H'}) \otimes H^0(\mathbb{C}^*)\big)^{\Gamma} \oplus \big(H^{n-1}(\frac{S}{H'} \otimes H^1(\mathbb{C}^*)\big)^{\Gamma}$. Now the action of Γ on each graded piece is as follows: $\sigma(a \otimes b) = \sigma(a) \otimes \sigma(b)$, where $\sigma \in \Gamma, a \in H^i(\frac{S}{H'}), b \in H^j(\mathbb{C}^*)$. Further $\sigma(b) = b$ as we are considering cohomology with coefficients in $\mathbb{Q}$ and thus multiplication by d is an isomorphism for $H^1(\mathbb{C}^*)$. Thus if $a \otimes b$ is an invariant cocycle, then $\sigma(a) \otimes \sigma(b) = a \otimes b$, i.e., if and only if $\sigma(a) = a$. However

$H^i(\frac{S}{H'})^\Gamma \approx H^i\left(\frac{\frac{S}{H'}}{\Gamma}\right) \approx H^i(\frac{S}{H})$. By assumption, the invariant cohomology is not the entire cohomology and hence the fibration $\frac{S}{H'} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^*$ is not cohomologically trivial. □

Going back to the discussion in the beginning of the section, we had shown that for any subgroup H of S such that $T \subset H \subset N(T)$ and $\frac{H}{T}$ is a cyclic group, then there is a change in the cohomology of $\frac{S}{H}$ and $\frac{S}{T}$ and thus by the theorem, we construct G so that $\frac{G}{H} \to \mathbb{C}^*$ is not a cohomologically trivial fibration.

5. On arbitrary algebraic groups

In this section, we analyse cohomological triviality for character maps of homogeneous spaces of arbitrary connected algebraic groups, not necessarily of affine. We will use the notations and conventions of [3], however we continue to work over the field of complex numbers.

To apply our methods, we have to find a central $\mathbb{C}^*$ which surjects onto $\mathbb{C}^*$. This will be possible after we quotient out the unipotent radical, for in a solvable group there are characters which are not central. We will first prove the existence of the unipotent radical in an algebraic group G which is not necessarily affine (for the existence of the unipotent radical, we also refer to [14, Paragraph 8.41], and for commutative algebraic groups, see [4, Theorem 2.9]). After quotienting out the unipotent radical we will show there is a central torus which maps surjectively onto $\mathbb{C}^*$ under a non- trivial character.

Lemma 5.1. *Let G be an arbitrary algebraic group. Then there is maximal unipotent normal affine subgroup U of G called as the unipotent radical of G.*

Proof. Consider the Albanese morphism, $\alpha : G \to \mathrm{Alb}(G)$, where $\mathrm{Alb}(G)$ is an abelian variety called as the Albanese variety. Then the kernel of α is a maximal normal affine algebraic group G_{aff}, G_{aff} being affine has a unipotent radical, say G_u. We claim that G_u is normal in G and is maximal normal unipotent and thus the unipotent radical of G.

Since G_{aff} is normal in G, we have that conjugate of G_u by any element of G is still in G_{aff}, i.e., $gG_ug^{-1} \subset G_{\mathrm{aff}}\ \forall g \in G$. Furthermore G_u is the unipotent radical of G_{aff}, thus gG_ug^{-1} is connected and unipotent subgroup of G_{aff}. Since G_{aff} is normal in G, $g_1g = gg_2$, where $g_1, g_2 \in G_{\mathrm{aff}}$. Therefore $g_1gG_ug^{-1}g_1^{-1} = gg_2G_ug_2^{-1}g^{-1} = gG_ug^{-1}$. Thus gG_ug^{-1} is normal in G_{aff} and hence $gG_ug^{-1} \subset G_u$. And since the dimensions of G_u and gG_ug^{-1} are the same we have that $gG_ug^{-1} = G_u$. Thus G_u is normal in G.

To see that $\mathrm{Alb}(G)$ has no unipotent subgroup, we refer to [6, Lemma 2.3], which says that there are no non constant maps of $\mathbb{C}$-varieties from a linear algebraic group to an abelian variety. □

We then have the following definition:

DEFINITION 5.2

(a) An algebraic group G is called *quasi-reductive* if G_{aff} if reductive.
(b) An algebraic group G is called *anti-affine* if $\mathcal{O}(G) := H^0(G, \mathcal{O}_G) = \mathbb{C}$.

This definition, in particular, says that an anti-affine group has no non-trivial characters, or more generally, no non constant map to an affine variety. We further note that any finite connected cover of an anti-affine group is anti-affine.

DEFINITION 5.3

A *semi-abelian variety* is any group that can be obtained as an extension of an abelian variety A by a torus T, i.e.,

$$1 \longrightarrow T \longrightarrow G \xrightarrow{q} A \longrightarrow 1.$$

Theorem 5.4. *For a quasi-reductive group G, the central torus $Z(G_{\text{aff}})^0$ of G_{aff} is the central torus in G.*

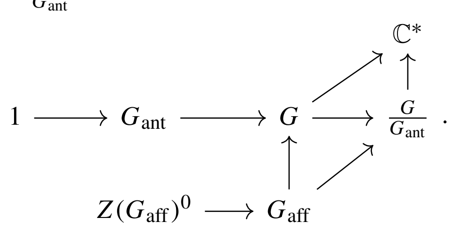
Proof. By [3, Theorem 5.1.1], an algebraic group G is generated by G_{aff} and G_{ant}. Furthermore, $G_{\text{ant}} \subset Z(G)$ the centre of G (c.f. [3, Proposition 3.3.5]), i.e., $Z(G_{\text{aff}})$ commutes with G_{ant} and with G_{aff} and hence with G. Therefore, $Z(G_{\text{aff}})$ is contained in $Z(G)$.

Let T be the central torus in G. G_{aff} contains all the connected affine algebraic subgroups and hence $T \subset Z(G_{\text{aff}})^0$. Thus the central torus $Z(G_{\text{aff}})^0$ of G_{aff} is the central torus in G. □

We will now show that the restriction of any character, defined on a quasi-reductive algebraic group, to the central torus of G_{aff} surjects onto $\mathbb{C}^*$. After this, cohomological triviality of fibration of homogeneous spaces follows by the same methods as in Section 3, for we have never used that G is affine in the section.

Theorem 5.5. *Let G be a quasi-reductive group and $\mathcal{X} : G \to \mathbb{C}^*$ be a character. Then $\mathcal{X}|_{Z(G_{\text{aff}})^0} \to \mathbb{C}^*$ is surjective.*

Proof. Any non-trivial character $\mathcal{X}$ on G factors through $\frac{G}{G_{\text{ant}}}$ as $\mathcal{X}|_{G_{\text{ant}}}$ is trivial and G_{aff} surjects onto $\frac{G}{G_{\text{ant}}}$.



The restriction of $\mathcal{X}$ to G_{aff} gives a character on G_{aff} and since G_{aff} is reductive, $Z(G_{\text{aff}})^0$ surjects onto $\mathbb{C}^*$. □

Now the results of Section 3 go through without any difference and we have the following theorem.

Theorem 5.6. *Let G be an arbitrary connected algebraic group, and $\mathcal{X} : G \longrightarrow \mathbb{C}^*$ be a non trivial character with connected kernel S. Then the fibration*

$$1 \to S \to G \to \mathbb{C}^* \to 1$$

is cohomologically trivial.

Proof. Since $\frac{G}{G_u}$ has the same homotopy type as G, we can assume that the group G is quasi-reductive. Since $Z(G_{\text{aff}})^0$ surjects onto $\mathbb{C}^*$, we can apply the covering space techniques of Proposition 3.2. □

We also have the analogue of Corollary 3.4.

COROLLARY 5.7

A character $\mathcal{X} : G \to \mathbb{C}^*$ of quasi-reductive groups is split if and only if $\mathcal{X}|_{Z(G_{\text{aff}})^0}$ is primitive. □

Further, Theorem 3.7 is also generalized as follows.

Theorem 5.8. *Consider a fibration of homogeneous spaces of a connected algebraic group G:*

$$\frac{S}{H} \longrightarrow \frac{G}{H} \longrightarrow \mathbb{C}^*.$$

Suppose that the kernel S of the associated character is connected, then the fibration is cohomologically trivial. □

We note that this method of study generalizes to a surjective map of quasi-reductive group to a torus and the same results are valid in this context. In general, the question of cohomological triviality is hard. When the base is a commutative group, our method of analysis requires an isogenous central subgroup that maps onto the base, which naturally leads to the covering space theory to produce a trivial fibre bundle over the group G. This requirement is quite strong: in fact, if the base is an abelian variety then there is no guarantee of an abelian variety in G that maps onto the base. However when the base is the universal semiabelian variety $\frac{G}{D(G_{\text{aff}})}$ of a reductive group (c.f. [3, page 51]), we have the following.

Theorem 5.9. *Let G be a quasi-reductive group and G_{sab} be the quotient semi-abelian variety $\frac{G}{D(G_{\text{aff}})}$, where $D(G_{\text{aff}})$ is the derived group of the affine part of G and also the derived group of G (c.f. [3, Corollary 5.1.5]). Then the fibration*

$$1 \to D(G_{\text{aff}}) \to G \xrightarrow{\pi} G_{\text{sab}} \to 1$$

is cohomologically trivial.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & D(G_{\mathrm{aff}}) & \longrightarrow & G & \xrightarrow{\pi} & G_{\mathrm{sab}} & \longrightarrow & 1 \\ & & & & \uparrow & \nearrow & & & \\ & & & & G_{\mathrm{ant}} \cdot Z(G_{\mathrm{aff}})^0 = G' & & & & \end{array}.$$

We note that G' is central in G, as G_{ant} and $Z(G_{\mathrm{aff}})^0$ are central. Since G is quasi-reductive, G is generated by $D(G_{\mathrm{aff}})$ and G', thus $\pi|_{G'}$ is surjective. Further, $\ker(\pi|_{G'}) = G' \cap D(G_{\mathrm{aff}})$ is a central torus contained in $D(G_{\mathrm{aff}})$, thus $\ker(\pi|_{G'})$ is finite and hence $\pi|_{G'}$ is an isogeny and hence a covering of G_{sab}. Now following the method of proof of Proposition 3.2, we have the result. □

We now give a result for the structure for arbitrary connected quasi-reductive algebraic group G up to isogeny, as an application of our methods. The existence of the maximal abelian sub variety can already be seen in the work of Rosenlicht in [20, Corollary, page 434]. For a modern treatment, we refer to [3, Theorem 4.2.5].

First, we make a general comment which we will use: Let Q be a quotient group of a group G, let H be a central sub-group of G, that is isogenous to Q. Consider the fibre product

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & H \\ \downarrow & & \downarrow \\ G & \xrightarrow{q} & Q \end{array}.$$

By the technique of Proposition 3.2, we have $\tilde{G} \approx H \times \ker(q)$. Thus it is enough to find a central subgroup that is isogenous to a quotient group to get such a decomposition of G up to isogeny.

Consider the morphism $q : G \to \frac{G}{G_{\mathrm{ant}}} \to T'$, where T' is the maximal toral quotient of the group $\frac{G}{G_{\mathrm{ant}}}$ (and hence a maximal toral quotient of G). Now the center $Z(G_{\mathrm{aff}})^0$ surjects onto T' and thus there is a torus $T \subset Z(G_{\mathrm{aff}})^0$ that is isogenous to T'. Hence by the earlier comment in the previous paragraph, we have an isogeny of $G \approx T \times G_1$, where G_1 is the kernel of the composition.

We note that the derived groups of G and G_1 are the same, since the derived groups are affine, as G_{ant} is commutative G_{ant} is trivial in $D(G)$ (also [3, Corollary 5.1.5] and since G_{aff} is reductive $D(G)$ is semisimple) and $G_{\mathrm{ant}} \subset G_1$. Now consider the morphism $f : G_1 \to \frac{G_1}{D(G)} = Q$, G_{ant} maps surjectively onto Q and the kernel $G_{\mathrm{ant}} \cap D(G_1)$ is finite (the affine part of G_{ant} is a torus!). Thus G_{ant} is isogenous to Q and hence $G_1 \approx D(G) \times G_{\mathrm{ant}}$.

Let A be a maximal abelian subvariety of G_{ant} and consider the image of A, say A' under the albanese morphism α. By Poincaré's complete reducibility theorem (c.f. [17, Theorem 1, page 160]), A' has a complementary abelian subvariety in $\alpha(G_{\mathrm{ant}})$, say B, up to isogeny. Taking the composition of α with the quotient map to $\frac{\alpha(G_{\mathrm{ant}})}{B} \approx A'$, we have a map $q : G_{\mathrm{ant}} \to A'$ and the restriction of q to A is an isogeny (the kernel is $((G_{\mathrm{ant}})_{\mathrm{aff}} \cap A$ which is finite). Thus $G_{\mathrm{ant}} \approx A \times G_S$, where G_S is a semi-abelian, anti-affine variety that has no proper abelian subvarieties. However, we note that the albanese map of G_S is non trivial. Further, A is the unique maximal abelian variety in G_{ant}, this follows as A is the unique abelian variety in $\tilde{G}_{\text{ant}}$.

We thus have the following result:

Theorem 5.10. *Up to isogeny, any quasi-reductive algebraic group has the following structure* $G_{\text{ss}} \times T \times A \times G_S$, *where* G_{ss} *is affine semisimple*, T *is a torus*, A *is the unique abelian variety that is split* (*in* G_{ant}) *and* G_S *is a semi-abelian, anti-affine variety with no proper abelian subvariety.*

Remark 5.11. Further, consider a homogeneous space $\frac{G}{H}$ with surjective morphism onto a commutative algebraic group B, so that the composition with the quotient map $q : G \to \frac{G}{H}$ is a homomorphism of algebraic groups, with kernel S connected. Now suppose that, there is a central subgroup $G_1 \subset G$ which is isogenous to B, then the cover of $\frac{G}{H}$ is $\frac{\tilde{G}}{H} \approx \frac{S}{H} \times G_1$, where $\frac{S}{H}$ is the fibre over e and the fibration $\frac{S}{H} \to \frac{G}{H} \to B$ is cohomologically trivial.

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