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Modules of n-ary differential operators over the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$

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Abstract. We are interested in the study of the space of n-ary differential operators denoted by $\mathfrak{D}_{\underline{\lambda},\mu}$ where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is acting on weighted densities from $\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2} \otimes \cdots \otimes \mathfrak{F}_{\lambda_n}$ to $\mathfrak{F}_\mu$ as a module over the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$. As a consequence, we prove the existence and the uniqueness of a canonical conformally equivariant symbol map from $\mathfrak{D}^k_{\underline{\lambda},\mu}$ to the corresponding space of symbols as well for the explicit expression of the associated quantization map.

Keywords. n-Ary differential operators; densities; orthosymplectic algebra; symbol; quantization maps.

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1. Introduction

The quantization is a concept that comes from physics. The quantization of a classical system whose phase space is a symplectic manifold, consists in the construction of a Hilbert space H and a correspondence between classical and quantum observables. Let M be a smooth manifold, T^*M the cotangent bundle on M and $\mathcal{S}(M)$ the space of all smooth functions that acts as polynomials on the fibers of T^*M. The space $\mathcal{S}(M)$ is usually called the space of symbols of differential operators. The standard quantization procedure consists of constructing a map $\mathcal{Q}$ between the space $\mathrm{Pol}(T^*M)$ of polynomials on T^*M and the space $\mathcal{D}(M)$ of linear differential operators on M called a *quatization map*. The inverse $\sigma = \mathcal{Q}^{-1}$ is thus called a *symbol map*. Generally, there is no quantization and symbol maps equivariant with respect to the action of the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M (or the group $\mathrm{Diff}(M)$ of diffeomorphisms of M) on the two spaces $\mathcal{D}(M)$ and $\mathrm{Pol}(T^*M)$. Thus, we restrict ourselves to equivariant symbols and quantization maps with respect to the action of a given subalgebra of $\mathrm{Vect}(M)$.

More precisely, for every $\lambda \in \mathbb{C}$, let $\mathcal{F}_\lambda(M)$ be the space of tensor densities of degree λ on M (i.e., the space of sections of the line bundle $\Delta_\lambda(M) = |\Lambda^n T^*M|^{\otimes\lambda}$ over M). Clearly, $\mathcal{F}_0(M) \cong C^\infty(M)$ as a $\mathrm{Vect}(M)$-module. Denote $\mathcal{D}_{\lambda,\mu}(M) := \mathrm{Hom}_{\mathrm{diff}}\big(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M)\big)$

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the space of linear differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. This space is an associative (and therefore, a Lie) algebra with the filtration by the order of differentiation:

$$\mathcal{D}^0_{\lambda,\mu}(M) \subset \mathcal{D}^1_{\lambda,\mu}(M) \subset \cdots \subset \mathcal{D}^k_{\lambda,\mu}(M) \subset \cdots .$$

On the other hand, we consider the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on M. As a Vect(M)-module, it is isomorphic to the space Pol(T^*M). The action of $X \in \text{Vect}(M)$ on $\mathcal{S}(M)$ is given by the Hamiltonian vector field

$$\frac{\partial X}{\partial \xi_i}\frac{\partial}{\partial x^i} - \frac{\partial X}{\partial x^i}\frac{\partial}{\partial \xi_i}, \tag{1.1}$$

where (x^i, ξ_i) are local coordinates on T^*M (we identified vector fields on M with the first-order polynomials on T^*M, that is, $X = X^i\xi_i$ in (1.1)). The space of symbols corresponding to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ is there for $\mathcal{S}_\delta(M) = \mathcal{S}(M) \otimes \mathcal{F}_\delta(M)$ where $\delta = \mu - \lambda$. The space $S_\delta(M)$ is also naturally a Vect(M)-module, the Vect(M)-action on $\mathcal{S}_\delta(M)$ is of the form (see [6])

$$L^\delta_X = L_X + \delta D(X), \tag{1.2}$$

where

$$D(X) = \partial_i X^i,\ \partial_i = \frac{\partial}{\partial x^i}.$$

Note that the formula (1.2) does not depend on the choice of local coordinates. Therefore, $\mathcal{S}_\delta(M)$ is a Poisson algebra with a natural gradation given by the decomposition

$$\mathcal{S}_\delta(M) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k_\delta(M),$$

where $\mathcal{S}^k_\delta(M)$ is the space of k-th order tensor fields. The algebra $\mathcal{S}_\delta(M)$ is naturally identified with the associated graded algebra gr$(\mathcal{D}_{\lambda,\mu}(M))$, i.e, the direct sum

$$\text{gr}(\mathcal{D}_{\lambda,\mu}) = \bigoplus_{k=0}^{\infty} \mathcal{D}^k_{\lambda,\mu}/\mathcal{D}^{k-1}_{\lambda,\mu},$$

with the convention that $\mathcal{D}^{-1}_{\lambda,\mu} = \{0\}$. The corresponding projection $\mathcal{D}^k_{\lambda,\mu}/\mathcal{D}^{k-1}_{\lambda,\mu} \to \mathcal{S}^k_\delta(M)$ is called the (principal) symbol.

The problem of equivariant quantization is the quest for a quantization map

$$Q_{\lambda,\mu} : \mathcal{S}_\delta(M) \longrightarrow \mathcal{D}_{\lambda,\mu}(M)$$

that commutes with the action of a given Lie subalgebra of Vect(M). In other words, it amounts to an identification of these two spaces which is canonical with respect to the geometry on M. The inverse of the quantization map

$$\sigma_{\lambda,\mu} := (Q_{\lambda,\mu})^{-1}$$

is called symbol map.

The concept of equivariant quantization over $\mathbb{R}^n$ was introduced by Lecomte and Ovsienko in [11]. In their seminal work, they considered spaces of differential operators acting between densities and the Lie algebra of projective vector fields over $\mathbb{R}^n$, $\mathrm{sl}(n+1)$. In this situation, they showed the existence and uniqueness of an equivariant quantization. These results were generalized in many references (see, for instance, [7,9]). In [10], Lecomte globalized the problem of equivariant quantization by defining the problem of natural invariant quantization on arbitrary manifolds. Finally in [3,5,14], the authors proved the existence of such quantizations by using different methods in more and more general contexts.

Recently, several papers dealt with the problem of equivariant quantizations in the context of supergeometry. The papers [12,15] exposed and solved respectively the problems of the $\mathfrak{pgl}(p+1|q)$-equivariant quantization over the superspace $\mathbb{R}^{p|q}$ and of the $\mathfrak{osp}(p+1;q+1|2r)$-equivariant quantization over $\mathbb{R}^{p+q|2r}$, whereas in [13], the authors defined the problem of the natural and projectively invariant quantization on arbitrary supermanifolds and showed the existence of such a map. In [8,16], the problem of equivariant quantizations over the supercircles $S^{1|1}$ and $S^{1|2}$ endowed with canonical contact structures was considered. These quantizations are equivariant with respect to Lie superalgebras $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ of contact projective vector fields respectively.

In [2], for the $S^{1|1}$-case, we were interested in the study of the space $\mathfrak{D}_{\lambda_1,\lambda_2,\mu}$ of bilinear differential operators from $\mathfrak{F}_{\lambda_1}\otimes\mathfrak{F}_{\lambda_2}$ to $\mathfrak{F}_\mu$. For almost all values $(\lambda_1,\lambda_2,\mu)$, we proved the existence and uniqueness (up to normalization) of a projectively, i.e., $\mathfrak{osp}(1|2)$-equivariant symbol map between $\mathfrak{D}_{\lambda_1,\lambda_2,\mu}$ and the corresponding space of symbols $\mathfrak{S}_{\lambda_1,\lambda_2,\mu}$ and calculated the explicit expressions of the symbol and the associated quantization maps.

Our motivation in this work is the generalization of the results proved in [2]. Namely, we consider the superspace $\mathfrak{D}_{\underline{\lambda},\mu}$, $\underline{\lambda}=(\lambda_1,\ldots,\lambda_n)\in\mathbb{C}^n$, of n-ary differential operators $A:\mathfrak{F}_{\lambda_1}\otimes\mathfrak{F}_{\lambda_2}\otimes\cdots\otimes\mathfrak{F}_{\lambda_n}\to\mathfrak{F}_\mu$, where $\mathfrak{F}_\lambda$, $\lambda\in\mathbb{C}$ is the space of tensor densities on the supercircle $S^{1|1}$ of degree λ. The analogue in the super setting of the projective algebra $\mathrm{sl}(2)$ is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$, which is the smallest simple Lie superalgebra, that can be realized as a subalgebra of $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$. Naturally, the Lie superalgebra $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$, and therefore, $\mathfrak{osp}(1|2)$ act on $\mathfrak{D}_{\underline{\lambda},\mu}$, the $\mathfrak{osp}(1|2)$-module $\mathfrak{D}_{\underline{\lambda},\mu}$ is filtered as in the unary case (see [8]):

$$\mathfrak{D}^0_{\underline{\lambda},\mu}\subset\mathfrak{D}^{\frac12}_{\underline{\lambda},\mu}\subset\mathfrak{D}^1_{\underline{\lambda},\mu}\subset\mathfrak{D}^{\frac32}_{\underline{\lambda},\mu}\subset\cdots\subset\mathfrak{D}^{k-\frac12}_{\underline{\lambda},\mu}\subset\mathfrak{D}^k_{\underline{\lambda},\mu}\subset\cdots.$$

The graded module $\mathrm{gr}(\mathfrak{D}_{\underline{\lambda},\mu})$, also called the space of symbols and denoted by $\mathcal{S}_{\underline{\lambda},\mu}$, depends only on the shift, $\delta=\mu-|\underline{\lambda}|$, $|\underline{\lambda}|=\lambda_1+\cdots+\lambda_n$ of the weights. Moreover, as a $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$-module, $\mathcal{S}_{\underline{\lambda},\mu}$ is decomposed as $\bigoplus_{k\in\frac12\mathbb{N}}\mathcal{S}^k_{\underline{\lambda},\mu}$, where

$$\mathcal{S}^k_{\underline{\lambda},\mu}=\bigoplus_{\ell=0}^{2k}\mathfrak{D}^\ell_{\underline{\lambda},\mu}/\mathfrak{D}^{\ell-\frac12}_{\underline{\lambda},\mu}\simeq\bigoplus_{\ell=0}^{2k}\mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}},$$

$\mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}$ stands for the sum $\bigoplus\mathfrak{F}_{\delta-\frac{\ell}{2}}$, where $\mathfrak{F}_{\delta-\frac{\ell}{2}}$ is counted $\begin{pmatrix}\ell+n-1\\ n-1\end{pmatrix}$ times.

Moreover, we prove that, if $\delta=\mu-|\underline{\lambda}|\neq\frac12,1,\frac32,2,\frac52,\ldots,k$, then $\mathfrak{D}^k_{\underline{\lambda},\mu}$ is isomorphic to $\mathcal{S}^k_{\underline{\lambda},\mu}$ as an $\mathrm{osp}(1|2)$-module. This isomorphism, called *a conformally equivariant symbol map*, is unique (once we fix a principal symbol). Explicit expressions of the normalized symbol and its inverse, the *conformally equivariant quantization map*, are given. Finally, we use these results to give a classification of the modules $\mathfrak{D}^k_{\underline{\lambda},\mu}$, $k=\frac12,1,\frac32,2$ for non-resonant values of δ.

2. The main definitions

In this section, we recall the main definition and facts related to the geometry of the supercircle $S^{1|1}$ (see for instance [1,8]).

2.1 *Geometry of the supercircle $S^{1|1}$*

The supercircle $S^{1|1}$ is the simplest supermanifold of dimension $1|1$ generalizing S^1. In order to fix the notation, let us give here the basic definitions of geometric objects on $S^{1|1}$. We define the supercircle $S^{1|1}$ by describing its graded commutative algebra of functions which we denote by $C^\infty_{\mathbb{C}}(S^{1|1})$ and which is constituted by the elements

$$F = f_0(x) + \theta f_1(x), \tag{2.1}$$

where x is an arbitrary parameter on S^1 (the even variable), θ is the odd variable ($\theta^2 = 0$) and f_0, f_1 are C^∞ complex-valued functions. We denote by F' the derivative of F with respect to x, i.e., $F'(x,\theta) = f_0'(x) + \theta f_1'(x)$.

2.2 *Vector fields and differential forms*

Let $\mathrm{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

$$\mathrm{Vect}_{\mathbb{C}}(S^{1|1}) = \{F_0\partial_x + F_1\partial_\theta \mid \ F_i \in C^\infty_{\mathbb{C}}(S^{1|1})\}, \tag{2.2}$$

where ∂_θ (resp ∂_x) means the partial derivative $\frac{\partial}{\partial\theta}$ (resp $\frac{\partial}{\partial x}$).

Let $\Omega^1(S^{1|1})$ be the rank $1|1$ right $C^\infty_{\mathbb{C}}(S^{1|1})$-module with basis $\mathrm{d}x$ and $\mathrm{d}\theta$, we interpret it as the right dual over $C^\infty_{\mathbb{C}}(S^{1|1})$ to the left $C^\infty_{\mathbb{C}}(S^{1|1})$-module $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$, by setting $\langle\partial_{y_i}, \mathrm{d}y_j\rangle = \delta_{ij}$ for $y = (x,\theta)$. The space $\Omega^1(S^{1|1})$ is a left module over $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$, the action being given by the Lie derivative

$$\langle X, L_Y(\alpha)\rangle := \langle [X,Y], \alpha\rangle$$

2.3 *Lie superalgebra of contact vector fields*

The standard contact structure on $S^{1|1}$ is defined as a codimension 1 non-integrable distribution $\langle\bar{D}\rangle$ on $S^{1|1}$, i.e., a subbundle in $TS^{1|1}$ generated by the odd vector field

$$\bar{D} = \partial_\theta - \theta\partial_x. \tag{2.3}$$

This contact structure can be equivalently defined as the kernel of the differential 1-form

$$\alpha = \mathrm{d}x + \theta\mathrm{d}\theta. \tag{2.4}$$

These vector fields satisfy the condition

$$\bar{D}^{2j} = (-1)^j D^{2j} = (-1)^j \partial_x^j, \quad \forall j \in \mathbb{N}, \tag{2.5}$$

where $D = \partial_\theta + \theta\partial_x$.

One can easily check the *super Leibniz formula*:

$$\bar{D}^j \circ F = \sum_{i=0}^{j} \binom{j}{i}_s (-1)^{|F|(j-i)} \bar{D}^i(F)\bar{D}^{j-i}, \tag{2.6}$$

where the notations $\binom{j}{i}_s$ and $|\ |$ stand respectively for the *super combination* defined by

$$\binom{j}{i}_s = \begin{cases} \binom{[\frac{j}{2}]}{[\frac{i}{2}]} & \text{if } i \text{ is even or } j \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

and for the parity function ($[x]$ denotes the integer part of a real number x).

A vector field X is said to be contact if it preserves the contact distribution, i.e.,

$$[X, \bar{D}] = F_X \bar{D}, \tag{2.8}$$

where $F_X \in C^\infty_{\mathbb{C}}(S^{1|1})$ is a function depending on X.

We denote by $\mathcal{K}(1)$ the *Lie superalgebra of contact vector fields* on $S^{1|1}$. It is well-known that every contact vector field can be expressed, for some function $f \in C^\infty_{\mathbb{C}}(S^{1|1})$, by (see [8])

$$X_f = -f\bar{D}^2 + \frac{1}{2}D(f)\bar{D}. \tag{2.9}$$

The vector field (2.9) is said to be the contact vector field with contact Hamiltonian f. One checks that

$$L_{X_f}\alpha = f'\alpha \quad , \quad [X_f, \bar{D}] = -\frac{1}{2}f'\bar{D}$$

The contact bracket is defined by $[X_f, X_g] = X_{\{f,g\}}$. The space $C^\infty_{\mathbb{C}}(S^{1|1})$ is thus equipped with a Lie superalgebra structure isomorphic to $\mathcal{K}(1)$. The explicit formula can be easily calculated as follows:

$$\{f, g\} = fg' - f'g + \frac{1}{2}(-1)^{|f|(|g|+1)}D(f)D(g). \tag{2.10}$$

The action of $\mathcal{K}(1)$ on $C^\infty_{\mathbb{C}}(S^{1|1})$ is defined by

$$\mathfrak{L}_{X_f}(g) = fg' + \frac{1}{2}D(f)\bar{D}(g) = fg' + \frac{1}{2}(-1)^{|f|+1}\bar{D}(f)\bar{D}(g). \tag{2.11}$$

2.4 *The orthosymplectic Lie superalgebra* $\mathfrak{osp}(1|2)$

If we identify S^1 with $\mathbb{RP}^1$ with homogeneous coordinates $(x_1 : x_2)$ and choose the affine coordinate $x = x_1/x_2$, the vector fields

$$\frac{\mathrm{d}}{\mathrm{d}x}, \ x\frac{\mathrm{d}}{\mathrm{d}x}, \ x^2\frac{\mathrm{d}}{\mathrm{d}x}$$

are globally defined and correspond to the standard projective structure on $\mathbb{RP}^1$. In this adapted coordinate, the action of the subalgebra $\mathfrak{sl}(2)$ of the Lie algebra Vect(S^1) is

$$\mathfrak{sl}(2) = \text{Span}\left(\frac{\mathrm{d}}{\mathrm{d}x},\ x\frac{\mathrm{d}}{\mathrm{d}x},\ x^2\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

which is well defined.

Similarly, we consider the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ as a subalgebra of $\mathcal{K}(1)$ as

$$\begin{aligned}\mathfrak{osp}(1|2) = \text{Span}(&X_1 = \partial_x,\ X_x = x\partial_x + \frac{1}{2}\theta\partial_\theta,\ X_{x^2} = x^2\partial_x + x\theta\partial_\theta,\\ &X_\theta = \frac{1}{2}D,\ X_{x\theta} = \frac{1}{2}xD).\end{aligned} \tag{2.12}$$

The space of even elements

$$(\mathfrak{osp}(1|2))_0 = \text{Span}(X_1,\ X_x,\ X_{x^2}) \tag{2.13}$$

is isomorphic to $\mathfrak{sl}(2)$, and the space of odd elements is two dimensional:

$$(\mathfrak{osp}(1|2))_1 = \text{Span}(X_\theta = D,\ X_{x\theta} = xD). \tag{2.14}$$

The new commutation relations are

$$\begin{array}{lll}[X_1, X_{x^2}] = 2X_x, & [X_\theta, X_\theta] = \frac{1}{2}X_1, & [X_x, X_1] = -X_1,\\ [X_x, X_{x^2}] = X_{x^2}, & [X_{x\theta}, X_{x\theta}] = \frac{1}{2}X_{x^2}, & [X_{x^2}, X_\theta] = -X_{x\theta},\\ [X_x, X_\theta] = -\frac{1}{2}X_\theta, & [X_1, X_{x\theta}] = X_\theta, & [X_1, X_\theta] = 0,\\ [X_x, X_{x\theta}] = \frac{1}{2}X_{x\theta}, & [X_{x^2}, X_{x\theta}] = 0, & [X_{x\theta}, X_\theta] = \frac{1}{2}X_x.\end{array}$$

As in the S^1 case, there exist adapted coordinates (x, θ) for which the $\mathfrak{osp}(1|2)$-action is well defined (see [8] for more details).

2.5 *The space of weighted densities on* $S^{1|1}$

In the super setting, by replacing dx by the 1-form α, we get analogous definition for weighted densities, i.e., we define the space of λ-densities as

$$\mathfrak{F}_\lambda = \{F\alpha^\lambda \mid F \in C^\infty_{\mathbb{C}}(S^{1|1})\}. \tag{2.15}$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty_{\mathbb{C}}(S^{1|1})$.

For contact vector field X_F, define a one-parameter family of first-order differential operator on $C^\infty_{\mathbb{C}}(S^{1|1})$ as

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F',\lambda \in \mathbb{C}. \tag{2.16}$$

One easily checks that the map $X_F \mapsto \mathfrak{L}^\lambda_{X_F}$ is a homomorphism of Lie superalgebra, i.e., $[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}] = \mathfrak{L}^\lambda_{[X_F, X_G]}$, for every λ. Thus $\mathfrak{F}_\lambda$ becomes a $\mathcal{K}(1)$-module on $C^\infty_{\mathbb{C}}(S^{1|1})$.

Evidently, the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is given by

$$\mathfrak{L}^{\lambda}_{X_F}(G\alpha^\lambda) = (X_F(G) + \lambda F'G)\alpha^\lambda. \tag{2.17}$$

Explicitly, if we put $F = f_0(x) + f_1(x)\theta$, $G = g_0(x) + g_1(x)\theta$,

$$\mathfrak{L}^{\lambda}_{X_F}(G) = L^{\lambda}_{f_0\partial_x}(g_0) + \frac{1}{2}\, f_1 g_1 + \Big(L^{\lambda+\frac{1}{2}}_{f_0\partial_x}(g_1) + \lambda g_0 f_1' + \frac{1}{2} g_0' f_1\Big)\theta. \tag{2.18}$$

2.6 *Multilinear differential operators on weighted densities*

We fix a natural number n. In order to avoid clutter, we have found that it is convenient to use the notations of [5]:

- Denote by i either the n-tuple $(i_1, \ldots, i_n)$ or the indices $i_1, \ldots, i_n$, as, for instance, $a_{\underline{i}} = a_{i_1,\ldots,i_n}$. The difference should be discernable from the context.
- Denote by $|\underline{i}|$ the sum $\sum_{j=1}^{n} i_j$.
- Denote $\mathbf{1}_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the i-th position.
- Denote by $\mathfrak{S}^{(i)}_{\lambda} = \oplus \mathfrak{F}_\lambda$, where $\mathfrak{F}_\lambda$ is counted $\binom{i+n-1}{n-1}$ times.
- $\bigotimes_{t=1}^{n} \bar{D}^{i_t} := \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n}$.
- Throughout the text, we use the classical convention $\sum_{i=1}^{0} c_i = 0$.

Obviously, for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, $\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2} \otimes \cdots \otimes \mathfrak{F}_{\lambda_n}$ is also a $\mathcal{K}(1)$-module with the action

$$\mathfrak{L}^{\underline{\lambda}}_{X_F}(\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_n) = \sum_{p=1}^{n} (-1)^{|F|\left(\sum_{i=1}^{p-1} |\Phi_i|\right)} \Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \mathfrak{L}^{\lambda_p}_{X_F}(\Phi_p) \otimes \cdots \otimes \Phi_n. \tag{2.19}$$

Since $\bar{D}^2 = -D^2 = -\partial_x$, every differential operator $A \in \mathfrak{D}_{\underline{\lambda},\mu}$ can be expressed in the form (see [8]):

$$A = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a_{\underline{i}}(x,\theta) \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n}, \tag{2.20}$$

where the coefficients $a_{\underline{i}}$ are smooth functions on $S^{1|1}$ and $\ell \in \mathbb{N}$. That is, for all $F_1 = f_1\alpha^{\lambda_1} \in \mathfrak{F}_{\lambda_1}$, $F_2 = f_2\alpha^{\lambda_2} \in \mathfrak{F}_{\lambda_2}, \ldots, F_n = f_n\alpha^{\lambda_n} \in \mathfrak{F}_{\lambda_n}$,

$$\begin{aligned} &A(F_1 \otimes F_2 \otimes \cdots \otimes F_n) \\ &= \left(\sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a_{\underline{i}}(x,\theta) (-1)^{\left(\sum_{p=1}^{n-1} |f_p| \sum_{s=p+1}^{n} i_s\right)} \bar{D}^{i_1}(f_1) \bar{D}^{i_2}(f_2) \cdots \bar{D}^{i_n}(f_n) \right) \alpha^\mu. \end{aligned} \tag{2.21}$$

Moreover, if $A \in \mathfrak{D}^k_{\underline{\lambda},\mu}$, then $\ell = 2k$. For short, we will write the operator A as

$$A = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a_{\underline{i}} \bar{D}^{\underline{i}}, \tag{2.22}$$

where $\bar{D}^{\underline{i}} = \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n}$. Thus, we consider a family of $\mathcal{K}(1)$-actions on the superspace of multilinear differential operators $\mathfrak{D}_{\underline{\lambda},\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathfrak{F}_{\lambda_1} \otimes \mathfrak{F}_{\lambda_2} \otimes \cdots \otimes \mathfrak{F}_{\lambda_n}, \mathfrak{F}_\mu)$:

$$\mathfrak{L}^{\underline{\lambda},\mu}_{X_F}(A) = \mathfrak{L}^{\mu}_{X_F} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}^{\underline{\lambda}}_{X_F}. \tag{2.23}$$

2.7 *Explicit formulas for the action of $\mathcal{K}(1)$ on $\mathfrak{D}^k_{\underline{\lambda},\mu}$*

Let us calculate explicitly the action $\mathcal{K}(1)$ on the superspace $\mathfrak{D}^k_{\underline{\lambda},\mu}$. Given a differential operator $A = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a_{\underline{i}} \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n} \in \mathfrak{D}^k_{\underline{\lambda},\mu}$, $F \in C^\infty(S^{1|1})$ and X_F is an arbitrary contact vector field.

PROPOSITION 2.1

The natural action of $\mathcal{K}(1)$ on $\mathfrak{D}^k_{\underline{\lambda},\mu}$ is given by

$$\mathfrak{L}^{\underline{\lambda},\mu}_{X_F}(A) = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a^X_{\underline{i}} \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n}, \tag{2.24}$$

where

$$\begin{aligned} a^X_{\underline{i}} = {}& \mathfrak{L}^{\delta - \frac{|\underline{i}|}{2}}_{X_F}(a_{\underline{i}}) - \sum_{r=1}^{2k-|\underline{i}|} (-1)^{r(|F|+|a_{\underline{i}+r\mathbf{1}_1}|)} \Bigg[\binom{r+i_1}{r+2}_s - \tfrac{1}{2}(-1)^{i_1} \binom{r+i_1}{r+1}_s \\ & + \lambda_1 \binom{r+i_1}{r}_s \Bigg] \bar{D}^r(F') a_{\underline{i}+r\mathbf{1}_1} \\ & - \sum_{t=2}^{n} \sum_{r=1}^{2k-|\underline{i}|} (-1)^{r(|F|+|a_{\underline{i}+r\mathbf{1}_t}|+i_1+i_2+\cdots+i_{t-1})} \Bigg[\binom{r+i_t}{r+2}_s \\ & - \tfrac{1}{2}(-1)^{i_t} \binom{r+i_t}{r+1}_s + \lambda_t \binom{r+i_t}{r}_s \Bigg] \bar{D}^r(F') a_{\underline{i}+r\mathbf{1}_t}. \end{aligned} \tag{2.25}$$

Proof. Let $\phi_1 = \varphi_1 \alpha^{\lambda_1} \in \mathfrak{F}_{\lambda_1}, \phi_2 = \varphi_2 \alpha^{\lambda_2} \in \mathfrak{F}_{\lambda_2}, \ldots, \phi_n = \varphi_n \alpha^{\lambda_n} \in \mathfrak{F}_{\lambda_n}$. Upon using (2.16), (2.19) and (2.23), we get

$$
\begin{aligned}
&\mathfrak{L}^{\underline{\lambda},\mu}_{X_F}(A)(\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_n)=\mathfrak{L}^{\mu}_{X_F}(A((\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_n)))\\
&\quad-(-1)^{|A||F|}A(\mathfrak{L}^{\lambda_1}_{X_F}(\phi_1)\otimes\phi_2\otimes\cdots\otimes\phi_n)\\
&\quad-(-1)^{|A|(|F|+|\phi_1|)}A(\phi_1\otimes\mathfrak{L}^{\lambda_2}_{X_F}(\phi_2)\otimes\cdots\otimes\phi_n)\\
&\quad-\cdots-(-1)^{|A|(|F|+|\phi_1|+\cdots+|\phi_{n-1}|)}A(\phi_1\otimes\phi_2\otimes\cdots\otimes\mathfrak{L}^{\lambda_n}_{X_F}(\phi_n))\\
&=\Big[\sum_{\ell=0}^{2k}\sum_{i_1+i_2+\cdots+i_n=\ell}F((-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}a_{\underline{i}}\bar{D}^{i_1}(\varphi_1)\bar{D}^{i_2}(\varphi_2)\cdots\bar{D}^{i_n}(\varphi_n))'\\
&\quad+\tfrac{1}{2}D(F)\bar{D}((-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}a_{\underline{i}}\bar{D}^{i_1}(\varphi_1)\bar{D}^{i_2}(\varphi_2)\cdots\bar{D}^{i_n}(\varphi_n))\\
&\quad+\mu F'(-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}a_{\underline{i}}\bar{D}^{i_1}(\varphi_1)\bar{D}^{i_2}(\varphi_2)\cdots\bar{D}^{i_n}(\varphi_n)\\
&\quad-(-1)^{|A||F|}(-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}(-1)^{|F|(i_2+\cdots+i_n)}a_{\underline{i}}\\
&\quad\bar{D}^{i_1}(F\varphi_1'+\tfrac{1}{2}D(F)\bar{D}(\varphi_1)+\lambda_1F'\varphi_1)\bar{D}^{i_2}(\varphi_2)\cdots\bar{D}^{i_n}(\varphi_n)\\
&\quad-(-1)^{|A|(|F|+|\varphi_1|)}(-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}(-1)^{|F|(i_3+\cdots+i_n)}\\
&\quad a_{\underline{i}}\bar{D}^{i_1}(\varphi_1)\bar{D}^{i_2}(F\varphi_2'+\tfrac{1}{2}D(F)\bar{D}(\varphi_2)+\lambda_2F'\varphi_2)\bar{D}^{i_3}(\varphi_3)\cdots\bar{D}^{i_n}(\varphi_n)\\
&\quad-\cdots-(-1)^{|A|(|F|+|\varphi_1|+\cdots+|\varphi_n|)}(-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}\\
&\quad a_{\underline{i}}\bar{D}^{i_1}(\varphi_1)\bar{D}^{i_2}(\varphi_2)\bar{D}^{i_3}(\varphi_3)\cdots\bar{D}^{i_n}(F\varphi_n'+\tfrac{1}{2}D(F)\bar{D}(\varphi_n)+\lambda_nF'\varphi_n)]\alpha^{\mu}.
\end{aligned}
$$

Using the super Leibniz formula (2.6) and by writing (2.1) in the form

$$
\begin{aligned}
&\mathfrak{L}^{\underline{\lambda},\mu}_{X_F}(A)(\phi_1\otimes\phi_2\otimes\cdots\otimes\phi_n)\\
&=\left[\sum_{\ell=0}^{2k}\sum_{i_1+i_2+\cdots+i_n=\ell}(-1)^{\sum_{j=1}^{n-1}|\varphi_j|(i_{j+1}+\cdots+i_n)}a^X_{\underline{i}}\bar{D}^{i_1}(\varphi_1)\bar{D}^{i_2}(\varphi_2)\cdots\bar{D}^{i_n}(\varphi_n)\right]\alpha^{\mu}.
\end{aligned}
$$

By identification, we get easily the formulas (2.25). □

2.8 *Space of symbols of multilinear differential operators*

Consider the graded $\mathcal{K}(1)$-module $\mathrm{gr}(\mathfrak{D}^k_{\underline{\lambda},\mu})$ associated with the filtration

$$\mathfrak{D}^0_{\underline{\lambda},\mu}\subset\mathfrak{D}^{\frac{1}{2}}_{\underline{\lambda},\mu}\subset\mathfrak{D}^1_{\underline{\lambda},\mu}\subset\mathfrak{D}^{\frac{3}{2}}_{\underline{\lambda},\mu}\subset\cdots\mathfrak{D}^{k-\frac{1}{2}}_{\underline{\lambda},\mu}\subset\mathfrak{D}^k_{\underline{\lambda},\mu}\subset\cdots \tag{2.26}$$

i.e., the direct sum

$$\mathrm{gr}(\mathcal{D}_{\underline{\lambda},\mu})=\bigoplus_{k=0}^{\infty}\mathfrak{D}^k_{\underline{\lambda},\mu}/\mathfrak{D}^{k-\frac{1}{2}}_{\underline{\lambda},\mu}. \tag{2.27}$$

We call this $\mathcal{K}(1)$-module the *space of symbols of multilinear differential operators* and denote it $\mathcal{S}_{\underline{\lambda},\mu}$.

The quotient module $\mathfrak{D}^k_{\underline{\lambda},\mu}/\mathfrak{D}^{k-\frac{1}{2}}_{\underline{\lambda},\mu}$, $k\in\frac{1}{2}\mathbb{N}$ can be decomposed into $n_k=\binom{2k+n-1}{n-1}$ components that transform under coordinates change as $\delta-\frac{k}{2}$ densities, where $\delta=\mu-|\underline{\lambda}|$.

Therefore, the multiplication of these components by any non-singular matrix ϖ gives rise to a $\mathcal{K}(1)$-invariant isomorphism called *a principal symbol map*

$$\sigma_{pr}^{\varpi} : \mathfrak{D}^k_{\underline{\lambda},\mu}/\mathfrak{D}^{k-\frac{1}{2}}_{\underline{\lambda},\mu} \xrightarrow{\ \simeq\ } \mathfrak{F}_{\delta-\frac{k}{2}} \oplus \mathfrak{F}_{\delta-\frac{k}{2}} \oplus \cdots \oplus \mathfrak{F}_{\delta-\frac{k}{2}} \ (n_k \text{ copies}). \tag{2.28}$$

The space of symbols of order $\leq k$, $k \in \frac{1}{2}\mathbb{N}$ is

$$\mathcal{S}^k_{\underline{\lambda},\mu} = \bigoplus_{\ell=0}^{2k} \mathfrak{D}^{\ell}_{\underline{\lambda},\mu}/\mathfrak{D}^{\ell-\frac{1}{2}}_{\underline{\lambda},\mu}. \tag{2.29}$$

The $\mathcal{K}(1)$-module $\mathcal{S}_{\underline{\lambda},\mu}$ depends only on the shift, δ, of the weights and not on $\mu, \lambda_1, \lambda_2, \ldots, \lambda_n$ independently. Moreover, for every $k \in \frac{1}{2}\mathbb{N}$, we have

$$\mathcal{S}^k_{\mu-|\underline{\lambda}|} = \mathcal{S}^k_{\delta} = \bigoplus_{\ell=0}^{2k} \mathfrak{D}^{\ell}_{\underline{\lambda},\mu}/\mathfrak{D}^{\ell-\frac{1}{2}}_{\underline{\lambda},\mu} \cong \bigoplus_{\ell=0}^{2k} \mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}}, \tag{2.30}$$

here the notation $\mathfrak{F}^{(i)}_{\lambda}$, $i \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ stands for the sum $\bigoplus \mathfrak{F}_{\lambda}$, where $\mathfrak{F}_{\lambda}$ is counted $\binom{i+n-1}{n-1}$ times.

Thanks to the isomorphism (2.28), an element P of $\mathcal{S}^k_{\delta}$ can be written in a unique way in the form

$$P = \alpha^{\delta} \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} \bar{a}_{\underline{i}}(x,\theta)\, \alpha^{-\frac{|\underline{i}|}{2}}, \tag{2.31}$$

where $\bar{a}_{\underline{i}}$ are arbitrary functions in $C^{\infty}(S^{1|1})$.

As the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$ is a subalgebra $\mathcal{K}(1)$, the space of symbols $\mathcal{S}_{\delta}$ can be viewed as an $\mathfrak{osp}(1|2)$-module.

3. $\mathfrak{osp}(1|2)$-equivariant symbol and quantization maps

We restrict the $\mathcal{K}(1)$-module structures to the particular subalgebra $\mathfrak{osp}(1|2)$ and look for $\mathfrak{osp}(1|2)$-isomorphisms between $\mathfrak{D}_{\underline{\lambda},\mu}$ and $\mathcal{S}_{\delta}$. We fix a principal symbol map σ^{ϖ}_{pr} as in (2.28), where ϖ is a non singular matrix.

DEFINITION 3.1

A symbol map is a linear bijection

$$\sigma^{\varpi}_{\underline{\lambda},\mu} : \mathfrak{D}_{\underline{\lambda},\mu} \to \mathcal{S}_{\delta} \tag{3.1}$$

such that the highest-order term of $\sigma^{\varpi}_{\underline{\lambda},\mu}(A)$, where $A \in \mathfrak{D}_{\underline{\lambda},\mu}$ coincides with the principal symbol $\sigma^{\varpi}_{\text{pr}}(A)$. Hence, the inverse map $Q = (\sigma^{\varpi}_{\underline{\lambda},\mu})^{-1}$ will be called a *quantization map*.

The problem of existence and uniqueness of $\mathfrak{osp}(1|2)$-equivariant symbol (and so quantization) map can be tackled once the symbol map $\sigma_{\rm pr}^{\varpi}$ is fixed.

The first main result of this paper is the following:

Theorem 3.2. *If δ is non-resonant, i.e., $\delta = \mu - |\lambda| \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, k$, then $\mathfrak{D}^k_{\underline{\lambda},\mu}$ and $\mathcal{S}^k_\delta$ are $\mathfrak{osp}(1|2)$-isomorphic through the family of $\mathfrak{osp}(1|2)$-equivariant maps $\sigma^{\varpi}_{\underline{\lambda},\mu}$ defined by*

$$\sigma^{\varpi}_{\underline{\lambda},\mu}(A) = \alpha^\delta \sum_{p=0}^{2k} \sum_{|\underline{i}|=p} \sum_{\ell=p}^{2k} \sum_{|\underline{s}|=\ell} \varpi_{\underline{i}}^{\underline{s}} D^{\ell-p}(a_{\underline{s}}) \alpha^{-\frac{|\underline{i}|}{2}}, \tag{3.2}$$

where $A = \sum_{p=0}^{2k} \sum_{|\underline{i}|=p} a_{\underline{i}} \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n} \in \mathfrak{D}^k_{\underline{\lambda},\mu}$ and $\varpi_{\underline{i}}^{\underline{s}}$ are constants given by the induction formula

$$\begin{aligned}(-1)^{\ell-p}\Big(\big[\tfrac{\ell-p}{2}\big] + (1-(-1)^{\ell-p})(\delta - \tfrac{\ell}{2})\Big)\varpi_{\underline{i}}^{\underline{s}} - \Big(\big[\tfrac{s_1}{2}\big] + (1-(-1)^{s_1})\lambda_1\Big)\varpi_{\underline{i}}^{\underline{s}-\mathbf{1}_1} \\ - \textstyle\sum_{j=2}^{n}(-1)^{s_1+s_2+\cdots+s_{j-1}}\Big(\big[\tfrac{s_j}{2}\big] + (1-(-1)^{s_j})\lambda_j\Big)\varpi_{\underline{i}}^{\underline{s}-\mathbf{1}_j} = 0.\end{aligned} \tag{3.3}$$

If ϖ is the identity map, we obtain the "normalized" symbol map $\sigma^{\rm Id}_{\underline{\lambda},\mu}$ given by the rule

$$\sigma^{\rm Id}_{\underline{\lambda},\mu}(A) = \alpha^\delta \sum_{p=0}^{2k} \sum_{|\underline{i}|=p} \sum_{\ell=p}^{2k} \sum_{\substack{|\underline{s}|=\ell \\ s_1\geq i_1, s_2\geq i_2,\ldots,s_n\geq i_n}} \gamma_{\underline{i}}^{\underline{s}} D^{\ell-p}(a_{\underline{s}}) \alpha^{-\frac{|\underline{i}|}{2}} \tag{3.4}$$

such that

$$\gamma_{\underline{i}}^{\underline{s}} = (-1)^{\left[\frac{\ell-p+1}{2}\right]} \prod_{t=2}^{n} \frac{(-1)^{\psi(t)} \binom{\varphi(t)}{s_t - i_t}_s \Xi_{s_t,i_t}(\lambda_t)}{\binom{\left[\frac{\varphi(t)}{2}\right]}{\left[\frac{s_t-i_t}{2}\right]}\binom{\left[\frac{\varphi(t)+1}{2}\right]}{\left[\frac{s_t-i_t+1}{2}\right]}\binom{2\delta-p-1}{\left[\frac{\ell-p+1}{2}\right]}} \Xi_{s_1,i_1}(\lambda_1), \tag{3.5}$$

where the functions φ, ψ and Ξ are defined by

$$\varphi(t) = \sum_{j=1}^{t} s_t - i_t,\ \ \psi(t) = \sum_{j=1}^{t-1} s_j(s_{j+1} - i_{j+1}),\ \ \Xi_{s_t,i_t}(\lambda_t) = \binom{\left[\frac{s_t}{2}\right]}{\left[\frac{i_t}{2}\right]}\binom{2\lambda_t + \left[\frac{s_t-1}{2}\right]}{\left[\frac{2(s_t-i_t)+1+(-1)^{i_t}}{4}\right]}$$

and the notation $\binom{\nu}{q}$ stands for the binomial coefficient given by $\binom{\nu}{q} = \frac{\nu(\nu-1)\cdots(\nu-q+1)}{q!}$. Moreover, once the principal symbol is fixed, the symbol map $\sigma^{\varpi}_{\underline{\lambda},\mu}$ is unique.

Proof. We begin the proof by proving the $\mathfrak{osp}(1|2)$-equivariance of the map $\sigma^{\rm Id}_{\underline{\lambda},\mu}$. Indeed, let $X = X_F \in \mathcal{K}(1)$. We have $\sigma^{\rm Id}_{\underline{\lambda},\mu}((\mathfrak{L}^{\underline{\lambda},\mu}_X(A)) = \alpha^\delta \sum_{p=0}^{2k} \sum_{|\underline{i}|=p} \bar{a}^X_{\underline{i}} \alpha^{-\frac{|\underline{i}|}{2}}$. Then, we readily see that

$$\bar{a}_{\underline{i}}^{X} = \sum_{\ell=p}^{2k} \sum_{|\underline{s}|=\ell} \gamma_{\underline{i}}^{\underline{s}} D^{(\ell-p)}(a_{\underline{i}}^{X}), \;\; p = |\underline{i}|.$$

Thanks to Proposition 2.1, for all $0 \leq p = |\underline{i}| \leq k$, we get

$$\begin{aligned}\bar{a}_{\underline{i}}^{X} &= \sum_{\ell=p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|a_{\underline{s}}^{X}|(\ell-p)+\frac{(\ell-p)(\ell-p+1)}{2}} \gamma_{\underline{i}}^{\underline{s}} \bar{D}^{(\ell-p)}(a_{\underline{s}}^{X}) \\ &= \sum_{\ell=p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|a_{\underline{s}}^{X}|(\ell-p)+\frac{(\ell-p)(\ell-p+1)}{2}} \gamma_{\underline{i}}^{\underline{s}} \bar{D}^{(\ell-p)}\Bigg[\mathfrak{L}_{X_F}^{\delta-\frac{|\underline{s}|}{2}}(a_{\underline{s}}) \\ &\quad - \sum_{m=1}^{2k-|\underline{s}|} (-1)^{m(|F|+|a_{\underline{s}+\mathbf{1}_1}|)} \left(\binom{m+s_1}{m+2}_s - \tfrac{1}{2}(-1)^s \binom{m+s_1}{m+1}_s \right. \\ &\quad \left. +\lambda_1 \binom{m+s_1}{m}_s \right) \bar{D}^m(F') a_{\underline{s}+m\mathbf{1}_1} \\ &\quad - \sum_{m=1}^{2k-|\underline{s}|} \textstyle\sum_{j=2}^{n} (-1)^{m(|F|+|a_{\underline{s}+m\mathbf{1}_j}|+\sum_{t=1}^{j-1} s_t)} \\ &\quad \left(\binom{m+s_j}{m+2}_s - \tfrac{1}{2}(-1)^{s_j} \binom{m+s_j}{m+1}_s \right. \\ &\quad \left. +\lambda_j \binom{m+s_j}{m}_s \right) \bar{D}^m(F') a_{\underline{s}+m\mathbf{1}_j}\Bigg].\end{aligned}$$

Thus

$$\begin{aligned}\bar{a}_{\underline{i}}^{X} - \mathfrak{L}_{X}^{(\delta-\frac{p}{2})}(\bar{a}_{\underline{i}}) &= \sum_{\ell=p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|a_{\underline{s}}|+|F|+(\ell-p)} \gamma_{\underline{i}}^{\underline{s}} \Bigg[(\delta - \tfrac{l}{2}) \binom{\ell-p}{1}_s \\ &\quad + \tfrac{1}{2}(-1)^{\ell-p} \binom{\ell-p}{2}_s + \binom{\ell-p}{3}_s \Bigg] \bar{D}(F') D^{\ell-p-1}(a_{\underline{i}}) \\ &\quad - \sum_{\ell=p}^{2k} \sum_{|\underline{s}|=\ell} (-1)^{|a_{\underline{s}}|+|F|} \Bigg[\gamma_{\underline{i}}^{\underline{s}-\mathbf{1}_1} \left(\binom{s_1}{3}_s + \tfrac{1}{2}(-1)^{s_1} \binom{s_1}{2}_s \right. \\ &\quad \left. + \lambda_1 \binom{s_1}{3}_s \right) + \sum_{j=2}^{n} (-1)^{\sum_{t=1}^{j-1} s_t} \gamma_{\underline{i}}^{\underline{s}-\mathbf{1}_j} \left(\binom{s_j}{3}_s \right. \\ &\quad \left. + \tfrac{1}{2}(-1)^{s_j} \binom{s_j}{2}_s + \lambda_j \binom{s_j}{3}_s \right) \Bigg] \bar{D}(F') D^{\ell-p-1}(a_{\underline{i}}) \\ &\quad + (\text{higher terms in } \bar{D}^n(F'), n \geq 2).\end{aligned}$$

Now, through a simple calculation, one can check out that the scalars $\gamma_{\underline{i}}^{\underline{s}}$ satisfies the relationship

$$(-1)^{\ell-p}\Upsilon(\delta-\frac{\ell}{2},\ell-p)\gamma_{\underline{i}}^{\underline{s}}$$
$$-\Upsilon(\lambda_1,s_1)\gamma_{\underline{i}}^{\underline{s}-\mathbf{1}_1}-\sum_{j=2}^{n}(-1)^{s_1+s_2+\ldots+s_{j-1}}\Upsilon(\lambda_j,s_j)\gamma_{\underline{i}}^{\underline{s}-\mathbf{1}_j}=0,$$

where for $\lambda\in\mathbb{C}$ and $m\in\mathbb{N}$, we put

$$\Upsilon(\lambda,m)=\frac{1}{2}\Big(\big[\frac{m}{2}\big]+(1-(-1)^m)\lambda\Big).$$

Since the term in $\bar{D}(F')$ vanishes, we can clearly see that the map $\sigma_{\underline{\lambda},\mu}^{\mathrm{Id}}$ is $\mathfrak{osp}(1|2)$-equivariant.

Now, we can easily adapt the proof of locality given in [8] for the unary case to our case and then use the locality property of an $\mathfrak{osp}(1|2)$-equivariant symbol map. Therefore, in addition, from the expression of the "normalized" symbol map $\sigma_{\underline{\lambda},\mu}^{\mathrm{Id}}$, we can suppose that a general symbol map $\sigma_{\underline{\lambda},\mu}^{\varpi}$ can be written as

$$\begin{aligned} A &= \sum_{p=0}^{2k}\sum_{|\underline{i}|=p} a_{|\underline{i}|}\bar{D}^{i_1}\otimes\bar{D}^{i_2}\otimes\cdots\otimes\bar{D}^{i_n} \\ &\longmapsto \alpha^{\delta}\sum_{p=0}^{2k}\sum_{|\underline{i}|=p}\sum_{\ell=p}^{2k}\sum_{|\underline{s}|=\ell}\varpi_{\underline{i}}^{\underline{s}}(x,\theta)D^{\ell-p}(a_{\underline{s}})\alpha^{-\frac{|\underline{i}|}{2}}. \end{aligned} \tag{3.6}$$

Obviously, to get the condition of $\mathfrak{osp}(1|2)$-equivariance, it is sufficient to impose invariance with respect to the vector fields $D=2X_\theta$ and $xD=2X_{x\theta}$ to meet the whole condition $\mathfrak{osp}(1|2)$-equivariance. Thus we have

(a) A symbol map (3.6) commutes with the action of D if and only if the coefficients $\varpi_{\underline{i}}^{\underline{s}}$ are constants (i.e., do not depend on x,θ),
(b) A symbol map (3.6) commutes with the action of xD if and only if the coefficients $\varpi_{\underline{i}}^{\underline{s}}$ satisfy the induction formula (3.3).

If $\delta=\mu-|\underline{\lambda}|$ is non-resonant, i.e., $\delta=\mu-|\underline{\lambda}|\neq\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},\ldots,k$, then it is easy to see the solution of Equation (3.3), and once the principal symbol σ^{ϖ} where $\varpi=(w_{\underline{i}}^{\underline{i}})_{|\underline{i}|=2k}$ is fixed, the symbol map $\sigma_{\underline{\lambda},\mu}^{\varpi}$ is unique. □

Remark 3.3. We can write the symbol map $\sigma_{\underline{\lambda},\mu}^{\mathrm{Id}}$ as in ([8], Theorem 6.1). Indeed, let $A=a_{\underline{i}}\bar{D}^{i_1}\otimes\bar{D}^{i_2}\otimes\cdots\otimes\bar{D}^{i_n}\in\mathfrak{D}_{\underline{\lambda},\mu}^{k}$ and $|\underline{i}|=2k$, then

$$\sigma_{\underline{\lambda},\mu}^{\mathrm{Id}}(A)=\alpha^{\delta}\sum_{\ell=0}^{2k}\sum_{|\underline{s}|=\ell}\chi_{\underline{i}}^{\underline{s}}D^{\ell}(a_{\underline{i}})\alpha^{\frac{|\underline{s}|-|\underline{i}|}{2}}, \tag{3.7}$$

where

$$\chi_{\underline{s}}^{\underline{i}} = \begin{cases} (-1)^{\left[\frac{\ell+1}{2}\right]} \prod_{t=2}^{n} \dfrac{(-1)^{\Delta(t)} \begin{pmatrix} \Gamma(t) \\ s_t \end{pmatrix}_s \Xi_{i_t, i_t - s_t}(\lambda_t)}{\begin{pmatrix} \left[\frac{\Gamma(t)}{2}\right] \\ \left[\frac{s_t}{2}\right] \end{pmatrix} \begin{pmatrix} \left[\frac{\Gamma(t)+1}{2}\right] \\ \left[\frac{s_t+1}{2}\right] \end{pmatrix} \begin{pmatrix} 2\delta + \varphi(n) - 1 \\ \left[\frac{\ell+1}{2}\right] \end{pmatrix}} \Xi_{i_1, i_1 - s_1}(\lambda_1) \\ \text{if} \quad i_t \geq s_t, t \in \{1, 2, \ldots, n\}, \\ 0 \quad \text{otherwise}. \end{cases}, \tag{3.8}$$

Here $\Gamma(t) = \sum_{j=1}^{t} s_j$ and $\Delta(t) = \sum_{j=1}^{t-1} s_j s_{j+1}$.

Now, by a direct computation, one can easily check the following explicit formula for the quantization map $Q_{\underline{\lambda},\mu}^{\mathrm{Id}}$.

PROPOSITION 3.4

The quantization map $Q_{\underline{\lambda},\mu}^{\mathrm{Id}}$, *i.e., the inverse of the symbol map* $\sigma_{\underline{\lambda},\mu}^{\mathrm{Id}}$ *given in Theorem 3.2 associates to a polynomial* $P = \alpha^{\delta} \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} \bar{b}_{\underline{i}} \alpha^{-\frac{|\underline{i}|}{2}} \in \mathcal{S}_{\delta}^{k}$ *the differential operator* $Q_{\underline{\lambda},\mu}^{\mathrm{Id}}(P) = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=p} \tilde{b}_{\underline{i}} \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n} \in \mathfrak{D}_{\underline{\lambda},\mu}^{k}$ *such that* $\tilde{b}_{\underline{i}} = \sum_{\ell=p}^{2k} \sum_{\underline{s}=\ell} \beta_{\underline{i}}^{\underline{s}} D^{\ell-p}(\bar{b}_{\underline{s}})$, *where*

$$\begin{cases} \beta_{\underline{i}}^{\underline{s}} = (-1)^{\left[\frac{\ell-p-1}{2}\right]} \prod_{t=2}^{n} \dfrac{(-1)^{\psi(t)} \begin{pmatrix} \varphi(t) \\ s_t - i_t \end{pmatrix}_s \Xi_{s_t, i_t}(\lambda_t)}{\begin{pmatrix} \left[\frac{\varphi(t)}{2}\right] \\ \left[\frac{s_t - i_t}{2}\right] \end{pmatrix} \begin{pmatrix} \left[\frac{\varphi(t)+1}{2}\right] \\ \left[\frac{s_t - i_t + 1}{2}\right] \end{pmatrix} \begin{pmatrix} 2\delta - l \\ \left[\frac{\ell-p+1}{2}\right] \end{pmatrix}} \Xi_{s_1, i_1}(\lambda_1), & \text{if } \ell = |\underline{s}| > p = |\underline{i}|, \\ \beta_{\underline{i}}^{\underline{s}} = \gamma_{\underline{i}}^{\underline{s}}, & \text{if } |\underline{s}| = |\underline{i}|. \end{cases} \tag{3.9}$$

4. Conjugation of n-ary differential operators

The aim of this section is to define and find explict formulas for conjugations of n-ary differential operators. We follow the same path as in [2,3,8]. These operators will be used later in the classification of modules of n-ary differential operators. Unfortunately, in our case, the computations are more complicated.

Let us denote by $\mathcal{B}$ the Berezin integral $\mathcal{B} : \mathfrak{F}_{\frac{1}{2}} \to \mathbb{C}$ given, for any $f = f_0 + \theta f_1$, by the formula [4]

$$\mathcal{B}(f\alpha^{\frac{1}{2}}) = \int_{S^1} f_1(x) \mathrm{d}x. \tag{4.1}$$

It is well known that the Berezin integral $\mathcal{B}$ is $\mathcal{K}(1)$-invariant, that is,

$$\mathcal{B}\left(\mathfrak{L}_{X_F}^{\frac{1}{2}}(f\alpha^{\frac{1}{2}})\right) = 0, \ \forall F, f \in C_{\mathbb{C}}^{\infty}(S^{1|1}). \tag{4.2}$$

So, the product of densities composed with $\mathcal{B}$ yields a bilinear $\mathcal{K}(1)$-invariant form

$$\langle .\,,\ .\rangle : \mathfrak{F}_\lambda \otimes \mathfrak{F}_{\frac{1}{2}-\lambda} \to \mathbb{C},\ \lambda \in \mathbb{C} \tag{4.3}$$

given by

$$\langle f\alpha^\lambda, g\alpha^{\frac{1}{2}-\lambda}\rangle = \int_{S^1} (f_1 g_0 + f_0 g_1)(x)\mathrm{d}x, \tag{4.4}$$

where $f = f_0 + \theta f_1 \in \mathfrak{F}_\lambda$ and $g = g_0 + \theta g_1 \in \mathfrak{F}_{\frac{1}{2}-\lambda}$.

Theorem 4.1. *For each value of $k \in \frac{1}{2}\mathbb{N}$ and for all j in $\{1, \ldots, n\}$, the $\mathcal{K}(1)$-modules $\mathfrak{D}^k_{\underline{\lambda},\mu}$ and $\mathfrak{D}^k_{\underline{\lambda}+(\frac{1}{2}-\mu-\lambda_j)1_j,\frac{1}{2}-\lambda_j}$ are isomorphic. The isomorphism is given by the $\mathcal{K}(1)$-invariant conjugation operator C_j defined, for a n-ary operator $A = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a_{\underline{i}} \bar{D}^{\underline{i}}$ by*

$$C_j(A) = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} \sum_{\substack{\underline{s}=(s_1,\cdots,s_n)\\ 0\leq s_{n-1}\leq\cdots\leq s_1\leq i_j}} H_j(\underline{s}) \bigotimes_{m=1}^{j-1} \bar{D}^{i_m+s_m-s_{m+1}} \otimes \bar{D}^{i_j-s_1} \circ a_{\underline{i}} \otimes \bigotimes_{m=j+1}^{n-1} \bar{D}^{i_m+s_{m-1}-s_m} \otimes \bar{D}^{i_n+s_{n-1}}, \tag{4.5}$$

where

$$H_j(\underline{s}) = (-1)^{\left[\frac{i_j+1}{2}\right]+(i_j+s_j)|a_{\underline{i}}|} (-1)^{\sum_{m=1}^{j-1}(s_1+s_{m+1})i_m} (-1)^{\sum_{m=j+1}^{n-1} s_m i_m} \binom{i_j}{s_1}_s \prod_{r=2}^{n-1} \binom{s_{r-1}}{s_r}_s. \tag{4.6}$$

Proof. Let $A \in \mathfrak{D}^k_{\underline{\lambda},\mu}$. Then, there exists a unique n-ary differential operator $C_j(A) \in \mathfrak{D}^k_{\underline{\lambda}+(\frac{1}{2}-\mu-\lambda_j)1_j,\frac{1}{2}-\lambda_j}$ such that

$$\langle A(F_1 \otimes F_2 \otimes \cdots \otimes F_n), \Psi\rangle = (-1)^{(|A|+\sum_{s=1}^{j-1}|f_s|)|f_j|+|\psi|(\sum_{s=j+1}^{n}|f_s|)} \langle F_j, C_j(A)(F_1 \otimes F_2 \otimes \cdots \otimes \Psi \otimes \cdots \otimes F_n)\rangle,$$

where $F_i = f_i\alpha^{\lambda_i} \in \mathfrak{F}_{\lambda_i}$, $f_i = (f_i)_0 + \theta(f_i)_1 \forall i \in \{1, 2, \ldots, n\}$ and $\Psi = \psi\alpha^{\frac{1}{2}-\mu} \in \mathfrak{F}_{\frac{1}{2}-\mu}$, $\psi = \psi_0 + \theta\psi_1$.

Thus we can easily show that we get a $\mathcal{K}(1)$-invariant linear bijective map

$$C_j : \mathfrak{D}^k_{\underline{\lambda},\mu} \to \mathfrak{D}^k_{\underline{\lambda}+(\frac{1}{2}-\mu-\lambda_j)1_j,\frac{1}{2}-\lambda_j}.$$

Let $A = a\bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_j} \otimes \cdots \otimes \bar{D}^{i_n} \in \mathfrak{D}_{\underline{\lambda},\mu}$ and suppose that $i_j = 2p+1$ is odd. Then

$$a\bar{D}^{i_1}(f_1)\bar{D}^{i_2}(f_2)\cdots\bar{D}^{i_j}(f_j)\cdots\bar{D}^{i_n}(f_n) = (-1)^p \left[a_0(f_j)_1^{(p)} \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \right.$$
$$+\theta \left(a_0(f_j)_1^{(p)} \sum_{t=2}^{n} \bar{D}^{i_t}(f_t)_1 \prod_{\substack{s=2\\ s\notin\{j,t\}}}^{n} \bar{D}^{i_s}(f_s)_0 + (a_1(f_j)_1^{(p)} \right.$$
$$\left.\left. - a_0(f_j)_0^{(p+1)}) \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \right) \right].$$

Thus

$$\langle A(F_1 \otimes F_2 \otimes \cdots \otimes F_n), \Psi \rangle$$
$$= \prod_{s=1}^{n} (-1)^{\sum_{r=s+1}^{n} i_r|f_s|} (-1)^p \int_{S^1} a_0(f_j)_1^{(p)} \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \psi_1 \mathrm{d}x$$
$$+ \prod_{s=1}^{n} (-1)^{\sum_{r=s+1}^{n} i_r|f_s|} (-1)^p$$
$$\int_{S^1} \left(a_0(f_j)_1^{(p)} \sum_{t=2}^{n} \bar{D}^{i_t}(f_t)_1 \prod_{\substack{s=2\\ s\notin\{j,t\}}}^{n} \bar{D}^{i_s}(f_s)_0 + (a_1(f_j)_1^{(p)} \right.$$
$$\left. - a_0(f_j)_0^{(p+1)}) \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \right) \psi_0(x) \mathrm{d}x$$
$$= \prod_{\substack{s=1\\ s\neq j}}^{n} (-1)^{\sum_{r=s+1}^{n} i_r|f_s|} \int_{S^1} \left((f_j)_0 (a_0 \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \psi_0)^{(p+1)} \right)(x) \mathrm{d}x$$
$$+ \prod_{\substack{s=1\\ s\neq j}}^{n} (-1)^{\sum_{r=s+1}^{n} i_r|f_s|} (-1)^{\sum_{r=j+1}^{n} i_r} \int_{S^1} (f_j)_1(x) \left(a_0 \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \psi_1 \right.$$
$$\left. + a_0 \sum_{t=2}^{n} \bar{D}^{i_t}(f_t)_1 \prod_{\substack{s=2\\ s\notin\{j,t\}}}^{n} \bar{D}^{i_s}(f_s)_0 \psi_0 + a_1 \prod_{\substack{s=1\\ s\neq j}}^{n} \bar{D}^{i_s}(f_s)_0 \psi_0 \right)^{(p)} (x) \mathrm{d}x.$$

On the other hand, using (2.6) and (2.7), the operator (4.5) reads as

$$C_j(A) = (-1)^{p+1+|a_{\underline{i}}|} \sum_{0\le s_{n-1}\le\cdots\le s_1\le 2p+1} H_j(\underline{s}) \bigotimes_{m=1}^{j-1} \bar{D}^{i_j+s_m-s_{m+1}}$$

$$\otimes \bar{D}^{2p+1-s_1} \circ a \otimes \bigotimes_{m=j+1}^{n-1} \bar{D}^{i_m+s_{m-1}-s_m} \otimes \bar{D}^{i_n+s_{n-1}} = (-1)^{p+1}$$

$$\sum_{0\le s_{n-1}\le\cdots\le s_1\le p} \Bigg[H_j(2\underline{s}) \bigotimes_{m=1}^{j-1} \bar{D}^{i_m+2s_m-2s_{m+1}} \otimes \bar{D}^{2p-2s_1+1} \circ a$$

$$\otimes \bigotimes_{m=j+1}^{n-1} \bar{D}^{i_m+2s_{m-1}-2s_m} \otimes \bar{D}^{i_n+2s_{n-1}} + \sum_{r=1}^{j-1} H_j(2\underline{s}+\mathbf{1}_r)$$

$$\bigotimes_{m=1}^{r-1} \bar{D}^{i_m+2s_m-2s_{m+1}} \otimes \bar{D}^{i_r+2s_r-2s_{r+1}+1} \otimes \bigotimes_{m=r+1}^{j-1} \otimes \bar{D}^{2p-2s_1} \circ a$$

$$\otimes \bar{D}^{i_m+2s_m-2s_{m+1}} \otimes \bar{D}^{2p-2s_1} \circ a \otimes \bigotimes_{m=j+1}^{n-1} \bar{D}^{i_m+2s_{m-1}-2s_m} \otimes \bar{D}^{i_n+2s_{n-1}}$$

$$+ \sum_{r=j+1}^{n-1} H_j(2\underline{s}+\mathbf{1}_r) \bigotimes_{m=1}^{j-1} \bar{D}^{i_m+2s_m-2s_{m+1}} \otimes \bar{D}^{2p-2s_1} \circ a$$

$$\otimes \bigotimes_{m=j+1}^{r-1} \bar{D}^{i_m+2s_{m-1}-2s_m} \otimes \bar{D}^{i_r+2s_{r-1}-2s_r+1}$$

$$\otimes \bigotimes_{m=r+1}^{n-1} \bar{D}^{i_m+2s_{m-1}-2s_m} \bar{D}^{i_n+2s_{n-1}}$$

$$+ H_j(2\underline{s}+\mathbf{1}_{n-1}) \bigotimes_{m=1}^{j-1} \bar{D}^{i_m+2s_m-2s_{m+1}} \otimes \bar{D}^{2p-2s_1} \circ a$$

$$\otimes \bigotimes_{m=j+1}^{n-1} \bar{D}^{i_m+2s_{m-1}-2s_m} \otimes \bar{D}^{i_n+2s_{n-1}+1} \Bigg].$$

Let us denote by C_j^1 the first term (in the sum) in $C_j(A)$:

$$C_1^j = H_j(2\underline{s}) \bigotimes_{m=1}^{j-1} \bar{D}^{i_m+2s_m-2s_{m+1}}$$

$$\otimes \bar{D}^{2p-2s_1+1} \circ a \otimes \bigotimes_{m=j+1}^{n-1} \bar{D}^{i_m+2s_{m-1}-2s_m} \otimes \bar{D}^{i_n+2s_{n-1}},$$

given that

$$(-1)^{|f_s|} \bar{D}^{i_s}(f_s)_0 = (-1)^{i_s} \bar{D}^{i_s}(f_s)_0, \quad (-1)^{|f_s|} \bar{D}^{i_s}(f_s)_1 = (-1)^{i_s+1} \bar{D}^{i_s}(f_s)_1.$$

We get

$$
\begin{aligned}
& C_j^1(F_1 \otimes \cdots \otimes F_{j-1} \otimes \Psi \otimes F_{j+1} \otimes \cdots \otimes F_n) \\
& = (-1)^p H_j(2\underline{s}) \prod_{s=1, s\neq j}^{n} (-1)^{(\sum_{r=s+1}^{n} i_r)|f_s|} (-1)^{\sum_{r=j+1}^{n} |f_r||\psi|} \\
& \Bigg[(a_1\psi_0 + a_0\psi_1)^{(p-s_1)} \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})} \\
& \prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1} - s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})} + \theta(a_1\psi_0 + a_0\psi_1)^{(p-s_1)} \\
& \sum_{t=1}^{j-1} \bar{D}^{i_t}(f_t)_1^{(s_t - s_{t+1})} \prod_{m=1, m\neq t}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})} \\
& \prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1} - s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})} \\
& -(a_0\psi_0)^{(p-s_1+1)} \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})} \\
& \prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1} - s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})} + (a_1\psi_0 - a_0\psi_1)^{(p-s_1)} \\
& \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})} \sum_{t=j+1}^{n-1} \bar{D}^{i_t}(f_t)_1^{(s_{t-1} - s_t)} \\
& \prod_{m=j+1, m\neq t}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1} - s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})} \\
& +(a_1\psi_0 - a_0\psi_1)^{(p-s_1)} \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})} \\
& \prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1} - s_m)} \bar{D}^{i_n}(f_n)_1^{(s_{n-1})} \Bigg].
\end{aligned}
$$

Thus

$$
\begin{aligned}
& (-1)^{(|A| + \sum_{s=1}^{j-1} |f_s|)|f_j| + |\psi|(\sum_{s=j+1}^{n} |f_s|)} \langle F_j, C_j^1(F_1 \otimes \\
& \cdots \otimes F_{j-1} \otimes \Psi \otimes F_{j+1} \otimes \cdots \otimes F_n) \rangle \\
& = (-1)^p H_j(2\underline{s}) \prod_{s=1}^{j-1} (-1)^{(\sum_{r=s+1, r\neq j}^{n} i_r)|f_s|} \\
& \prod_{s=j+1}^{n-1} (-1)^{(\sum_{r=s+1}^{n} i_r)|f_s|} (-1)^{(\sum_{r=1, r\neq j}^{n} i_r)}
\end{aligned}
$$

$$\int_{S^1} f_j^1 (a_1\psi_0 + a_0\psi_1)^{(p-s_1)} \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})}$$
$$\prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1}-s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})} \mathrm{d}x$$
$$+(-1)^p \prod_{s=1}^{j-1} (-1)^{(\sum_{r=s+1, r\neq j}^{n} i_r)|f_s|}$$
$$\prod_{s=j+1}^{n-1} (-1)^{(\sum_{r=s+1}^{n} i_r)|f_s|} (-1)^{(\sum_{r=1}^{j-1} i_r)}$$
$$\sum_{t=1}^{j-1} \bar{D}^{i_t}(f_t)_1^{(s_t - s_{t+1})} \prod_{m=1, m\neq t}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})}$$
$$\prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1}-s_m)} \bar{D}^{i_n}(f_n)_1^{(s_{n-1})}$$
$$+(a_0\psi_0)^{(p-s_1+1)} \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})}$$
$$\prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1}-s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})} + (a_1\psi_0 - a_0\psi_1)^{(p-s_1)}$$
$$\prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})} \sum_{t=j+1}^{n-1} \bar{D}^{i_t}(f_t)_1^{(s_{t-1} - s_t)}$$
$$\prod_{m=j+1, m\neq t}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1}-s_m)} \bar{D}^{i_n}(f_n)_0^{(s_{n-1})}$$
$$+(a_1\psi_0 - a_0\psi_1)^{(p-s_1)} \prod_{m=1}^{j-1} \bar{D}^{i_m}(f_m)_0^{(s_m - s_{m+1})}$$
$$\prod_{m=j+1}^{n-1} \bar{D}^{i_m}(f_m)_0^{(s_{m-1}-s_m)} \bar{D}^{i_n}(f_n)_1^{(s_{n-1})} \mathrm{d}x.$$

A similar calculation can be done with the other terms of $C_j(A)$. We end up finding that $\langle A(F_1 \otimes F_2 \otimes \cdots \otimes F_n), \Psi\rangle = (-1)^{(|A| + \sum_{s=1}^{j-1} |f_s|)|f_j| + |\psi|(\sum_{s=j+1}^{n} |f_s|)} \langle F_j, C_j(A)(F_1 \otimes \cdots \otimes F_{j-1} \otimes \Psi \otimes F_{j+1} \otimes \cdots \otimes F_n)\rangle$.

Finally, a more easy calculation can be made when $i_j = 2p$ is even. □

The following definition is critical for later.

DEFINITION 4.2

A module $\mathfrak{D}^k_{\underline{\lambda},\mu}$ is said to be singular if either it is only isomorphic to itself or to the modules $\mathfrak{D}^k_{\underline{\lambda}+(\frac{1}{2}-\mu-\lambda_j)1_j,\frac{1}{2}-\lambda_j}$, $j=1,\ldots,n$.

5. Modules of n-ary differential operators of order ≤ 2

In this section, using the results proved in previous sections, we give a complete list of isomorphisms between distinct modules $\mathfrak{D}^k_{\underline{\lambda},\mu}$, $k\leq 2$ in the non-resonant case, i.e., $\delta\neq 1,\frac{3}{2},2,\frac{5}{2},\ldots,k$.

5.1 *Locality of isomorphisms and the invariant δ*

First, note that the difference $\delta=\mu-|\underline{\lambda}|$ of weight is an invariant in the condition $\mathcal{D}^k_{\underline{\lambda},\mu}\cong\mathcal{D}^k_{\underline{\rho},\nu}$ for every $k\in\frac{1}{2}\mathbb{N}$ which implies that $\mu-|\underline{\lambda}|=\nu-|\underline{\rho}|$. This is a consequence of the equivariance with respect to the vector field X_x.

We denote by $\sigma^{\mathrm{Id}}_{\underline{\lambda},\mu}$ the $\mathfrak{osp}(1|2)$-equivariant symbol map (associated with $\varpi=\mathrm{Id}$). Let us consider an isomorphism of $\mathcal{K}(1)$-modules $T:\mathfrak{D}^k_{\underline{\lambda},\mu}\to\mathfrak{D}^k_{\underline{\rho},\nu}$. As T is $\mathfrak{osp}(1|2)$-equivariant, it follows that the composition

$$T:\mathfrak{D}^k_{\underline{\lambda},\mu}\to\mathfrak{D}^k_{\underline{\rho},\nu}\xrightarrow{\sigma^{\mathrm{Id}}_{\underline{\rho},\nu}}\mathcal{S}^k_\delta=\bigoplus_{\ell=0}^{2k}\mathfrak{F}^{(\ell)}_{\delta-\frac{\ell}{2}},$$

is $\mathfrak{osp}(1|2)$-equivariant. Therefore, it coincides with the symbol map $\sigma^{\varpi}_{\underline{\lambda},\mu}$ for some ϖ. Namely, $\sigma^{\mathrm{Id}}_{\underline{\lambda},\mu}\circ T=\sigma^{\varpi}_{\underline{\lambda},\mu}$. It follows that

$$\sigma^{\mathrm{Id}}_{\underline{\lambda},\mu}\circ T\circ Q^{\mathrm{Id}}_{\underline{\lambda},\mu}=\sigma^{\varpi}_{\underline{\lambda},\mu}\circ Q^{\mathrm{Id}}_{\underline{\lambda},\mu}:\mathcal{S}^k_\delta\to\mathcal{S}^k_\delta.$$

Every isomorphism $T:\mathfrak{D}^k_{\underline{\lambda},\mu}\to\mathfrak{D}^k_{\underline{\rho},\nu}$ is *block diagonal* in terms of the $\mathfrak{osp}(1|2)$-equivariant symbols in the following sense.

Let $A=\sum_{p=0}^{2k}\sum_{|\underline{i}|=p}^{p}a_{\underline{i}}\bar{D}^{i_1}\otimes\bar{D}^{i_2}\otimes\cdots\otimes\bar{D}^{i_n}\in\mathfrak{D}^k_{\underline{\lambda},\mu}$. We denote by

$$\sigma^p_{\underline{\lambda},\mu}(A)=\alpha^\delta\sum_{i=0}^{p}\bar{a}_{\underline{i}}\alpha^{-\frac{p}{2}}\quad\text{and}\quad\sigma^p_{\underline{\lambda},\mu}(T(A))=\alpha^\delta\sum_{i=0}^{p}\bar{a}^T_{\underline{i}}\alpha^{-\frac{p}{2}},\tag{5.1}$$

the homogeneous components of order p in $\sigma^{\mathrm{Id}}_{\underline{\lambda},\mu}(A)$ and $\sigma^{\mathrm{Id}}_{\underline{\lambda},\mu}(T(A))$ respectively. Then, T is $\mathfrak{osp}(1|2)$-equivariant if and only if, for all $p\in\{0,1,\ldots,k\}$ the symbols $\sigma^p_{\underline{\lambda},\mu}(A)$ and $\sigma^p_{\underline{\lambda},\mu}(T(A))$ are proportional, that is, there exists a non singular matrix

$$\Upsilon_p=\left(\varepsilon^{\underline{m}}_{\underline{n}}\right)_{|\underline{n}|=|\underline{m}|=p}=\left(\varepsilon^{m_1\mathbf{1}_{j_1}+m_2\mathbf{1}_{j_2}+\ldots+m_p\mathbf{1}_{j_p}}_{n_1\mathbf{1}_{i_1}+n_2\mathbf{1}_{i_2}+\ldots+n_p\mathbf{1}_{i_p}}\right)_{|\underline{n}|=|\underline{m}|=p}\in GL_{n+p-1}(\mathbb{R}),$$

where $|\underline{n}| = n_1 + n_2 + \cdots + n_p$ and $|\underline{m}| = m_1 + m_2 + \cdots + m_p$ such that

$$\left(\bar{a}^T_{n_1\mathbf{1}_{i_1}+n_2\mathbf{1}_{i_2}+\cdots+n_p\mathbf{1}_{i_p}}\right)^t_{|\underline{n}|=p} = \Upsilon_p(\bar{a}_{n_1\mathbf{1}_{i_1}+n_2\mathbf{1}_{i_2}+\cdots+n_p\mathbf{1}_{i_p}})^t_{|\underline{n}|=p}, \tag{5.2}$$

where the notation u^t means the transpose of the vector $u \in \mathbb{R}^n$. Equivalently,

$$\bar{a}^T_{n_1\mathbf{1}_{i_1}+n_2\mathbf{1}_{i_2}+\cdots+n_p\mathbf{1}_{i_p}} = \sum_{|\underline{m}|=p} \varepsilon^{m_1\mathbf{1}_{j_1}+m_2\mathbf{1}_{j_2}+\cdots+m_p\mathbf{1}_{j_p}}_{n_1\mathbf{1}_{i_1}+n_2\mathbf{1}_{i_2}+\cdots+n_p\mathbf{1}_{i_p}} \bar{a}_{\underline{m}};\ \forall i \in \{0, 1, \ldots, p\}. \tag{5.3}$$

In other words, the map $\sigma^{\varpi}_{\underline{\lambda},\mu} \circ Q^{\mathrm{Id}}_{\underline{\lambda},\mu}$ on $\mathcal{S}^0_\delta \bigoplus \mathcal{S}^1_\delta \cdots \bigoplus \mathcal{S}^k_\delta$ multiplies each term of the direct sum by an invertible matrix.

5.2 *Classification theorems*

Now, we plan the cases of the modules $\mathfrak{D}^k_{\underline{\lambda},\mu}$, $k = \frac{1}{2}, 1, \frac{3}{2}, 2$. Let

$$A = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} a_{\underline{i}} \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n} \in \mathfrak{D}^k_{\underline{\lambda},\mu},\ P = \alpha^\delta \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} b_{\underline{i}} \alpha^{-\frac{|\underline{i}|}{2}} \in \mathcal{S}^k_\delta. \tag{5.4}$$

We set

$$\sigma^{\mathrm{Id}}_{\underline{\lambda},\mu}(A) = \alpha^\delta \sum_{p=0}^{2k} \sum_{|\underline{i}|=p} \bar{a}_{\underline{i}} \alpha^{-\frac{|\underline{i}|}{2}},\ Q^{\mathrm{Id}}_{\underline{\lambda},\mu}(P) = \sum_{\ell=0}^{2k} \sum_{|\underline{i}|=\ell} \tilde{b}_{\underline{i}} \bar{D}^{i_1} \otimes \bar{D}^{i_2} \otimes \cdots \otimes \bar{D}^{i_n}, \tag{5.5}$$

Theorem 5.1.

(*i*) *For* $\delta \neq \frac{1}{2}$, *all the* $\mathcal{K}(1)$-*modules* $\mathfrak{D}^{\frac{1}{2}}_{\underline{\lambda},\mu}$ *are isomorphic provided they have the same shift* δ.
(*ii*) *For* $\delta \neq \{\frac{1}{2}, 1\}$, *all the* $\mathcal{K}(1)$-*modules* $\mathfrak{D}^1_{\underline{\lambda},\mu}$ *are isomorphic provided they have the same shift* δ.

Proof.

(i) Let $T : \mathfrak{D}^{\frac{1}{2}}_{\underline{\lambda},\mu} \to \mathfrak{D}^{\frac{1}{2}}_{\underline{\rho},\nu}$ be an isomorphism of $\mathcal{K}(1)$-modules. Then obviously $\delta = \mu - |\underline{\lambda}| = \nu - |\underline{\rho}|$. Let us denote by $\widetilde{T} := \sigma^{\mathrm{Id}}_{\underline{\rho},\nu} \circ T \circ Q^{\mathrm{Id}}_{\underline{\lambda},\mu} : \mathcal{S}^{\frac{1}{2}}_\delta \to \mathcal{S}^{\frac{1}{2}}_\delta$. Since $\widetilde{T}$ is diagonal, there exist $\varepsilon_0 \in \mathbb{R}^*$, $\Upsilon_1 = \left(\varepsilon^{\mathbf{1}_j}_{\mathbf{1}_i}\right)_{\substack{0\le i\le n\\ 0\le j\le n}} \in GL_n(\mathbb{R})$ such that

$$\bar{a}^T_{\underline{0}} = \varepsilon_0 \bar{a}_{\underline{0}} \, , \, \big(\bar{a}^T_{\mathbf{1}_i}\big)^t_{(1\le i\le n)} = \Upsilon_1 \big(\bar{a}_{\mathbf{1}_i}\big)^t_{(1\le i\le n)}. \tag{5.6}$$

Following (2.25), (3.5) and (3.4), we have

$$\bar{a}^X_{\underline{0}} = \mathfrak{L}^{\delta}_{X_F}(\bar{a}_{\underline{0}}), \bar{a}^X_{\mathbf{1}_i} = \mathfrak{L}^{\delta-\frac{1}{2}}_{X_F}(\bar{a}_{\mathbf{1}_i}), \tag{5.7}$$

$$\begin{cases} \bar{a}_{\underline{0}} = a_{\underline{0}} - \displaystyle\sum_{i=1}^{n} \frac{2\lambda_i}{2\delta-1} D(a_{\mathbf{1}_i}), \\ \bar{a}_{\mathbf{1}_i} = a_{\mathbf{1}_i} \end{cases} \text{and} \quad \begin{cases} \tilde{b}_{\underline{0}} = b_{\underline{0}} + \displaystyle\sum_{i=1}^{n} \frac{2\lambda_i}{2\delta-1} D(b_{\mathbf{1}_i}), \\ \tilde{b}_{\mathbf{1}_i} = b_{\mathbf{1}_i}. \end{cases} \tag{5.8}$$

Thus, the conclusion can easily be stated.

(ii) This follows by similar reasoning.

□

Theorem 5.2.

(*i*) *All the* $\mathcal{K}(1)$*-modules* $\mathfrak{D}^{\frac{3}{2}}_{\underline{\lambda},\mu}$ *with* $\delta \neq \{\frac{1}{2}, 1\ , \frac{3}{2}\}$ *and* $(\underline{\lambda}, \mu) \notin \{(\lambda \mathbf{1}_i, \frac{1}{2}), (\underline{0}, \frac{1}{2}-\lambda), i \in \{1, 2, \ldots, n\} \lambda \neq 0, -\frac{1}{2}, -1\}$ *are isomorphic.*

(*ii*) *The modules of the form*

$$\mathfrak{D}^{\frac{3}{2}}_{\lambda\mathbf{1}_i,\frac{1}{2}} \cong \mathfrak{D}^{\frac{3}{2}}_{\lambda\mathbf{1}_j,\frac{1}{2}} \cong \mathfrak{D}^{\frac{3}{2}}_{\underline{0},\frac{1}{2}-\lambda} \; ; \, for \ each \ i \neq j \ \ ; \lambda \neq 0, -\frac{1}{2}, -1 \tag{5.9}$$

are singular.

Proof. Let $T : \mathfrak{D}^{\frac{3}{2}}_{\underline{\lambda},\mu} \to \mathfrak{D}^{\frac{3}{2}}_{\underline{\rho},\nu}$ be an isomorphim of $\mathcal{K}(1)$-modules. The diagonally property of $\widetilde{T}$ reads

$$\bar{a}^T_{\underline{0}} = \varepsilon_0 \bar{a}_{\underline{0}} \, , \, \big(\bar{a}^T_{\mathbf{1}_i}\big)^t_{(1\le i\le n)} = \Upsilon_1 \big(\bar{a}_{\mathbf{1}_i}\big)^t_{(1\le i\le n)} \, , \, (\bar{a}^T_{\mathbf{1}_i+\mathbf{1}_j})^t_{(0\le i,j\le n)} = \Upsilon_2 (\bar{a}_{\mathbf{1}_i+\mathbf{1}_j})^t_{(1\le i,j\le n)} \text{ and}$$

$$(\bar{a}^T_{\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_r})^t_{(1\le i,j,r\le n)} = \Upsilon_3 (\bar{a}_{\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_r})^t_{(1\le i,j,r\le n)},$$

where $\varepsilon_0 \in \mathbb{R}^*$ and $\Upsilon_i \in GL\Big(\binom{i+n-1}{n-1}, \mathbb{R}\Big)$, $i \in \{1, 2, 3\}$. Unlike the cases $k = \frac{1}{2}$ and $k = 1$, we obtain here an additional condition that expresses a relationship between ε_0 and Υ_3, namely,

$$\Upsilon_3 \Gamma(\underline{\rho}, \nu)^t = \varepsilon_0 \Gamma(\underline{\lambda}, \mu)^t$$

where $\Gamma(\underline{\lambda}, \mu)$ stands for the vector in $\mathbb{R}^{\binom{i+n-1}{n-1}}$ defined by

$$\Gamma(\underline{\lambda}, \mu) = (\gamma_{\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_r}(\underline{\lambda}, \mu))_{(1\le i,j,r\le n)},$$

with

$$\begin{cases} \gamma_{\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_r}(\underline{\lambda},\mu) = \dfrac{\lambda_i(2\delta+2\lambda_i-1)}{2(\delta-1)} & \text{if } i=j=r, \\ \gamma_{\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_r}(\underline{\lambda},\mu) = \dfrac{\lambda_i\lambda_j}{2(\delta-1)} & \text{if } i\neq\{j,r\},\, j=r, \\ \gamma_{\mathbf{1}_i+\mathbf{1}_j+\mathbf{1}_r}(\underline{\lambda},\mu) = 0 & \text{otherwise.} \end{cases}$$

Since T is an isomorphism, two cases arise:

- $\Gamma(\underline{\rho},\nu)\neq 0$ and $\Gamma(\underline{\lambda},\mu)\neq 0$, that is, $(\underline{\rho},\nu)$, $(\underline{\lambda},\mu)\notin A$. Thus we get a family of $\mathcal{K}(1)$-isomorphisms given by the conditions

$$\varepsilon_0=1,\ \Upsilon_1\in GL(n,\mathbb{R}),\ \Upsilon_2\in GL\left(\binom{n+1}{n-1},\mathbb{R}\right)\quad\text{and}\quad \Upsilon_3\Gamma(\underline{\rho},\nu)^t=\Gamma(\underline{\lambda},\mu)^t.$$

- $\Gamma(\underline{\rho},\nu)=\Gamma(\underline{\lambda},\mu)=0$, then $(\underline{\rho},\nu)$, $(\underline{\lambda},\mu)\in A$ and (ii) is clearly obtained.

This concludes the proof of the theorem. □

Theorem 5.3. *Let* $\mathbb{B}=\mathbb{B}_1\cup\mathbb{B}_2$, *where* $\mathbb{B}_1=\{(\lambda\mathbf{1}_i,\frac{1}{2}),(\underline{0},\frac{1}{2}-\lambda);\, i\in\{1,2,\dots,n\},\lambda\neq 0,-\frac{1}{2},-1,-\frac{3}{2}\}$ *and* $\mathbb{B}_2=\{(\lambda\mathbf{1}_i,\frac{1}{2}-\lambda),(\lambda\mathbf{1}_i+\lambda\mathbf{1}_j,\frac{1}{2});\, i\neq j;\lambda\neq 0,-\frac{1}{4},-\frac{1}{2},-\frac{3}{4}\}$. *Then*

(*i*) *All the* $\mathcal{K}(1)$-*modules* $\mathfrak{D}^2_{\underline{\lambda},\mu}$ *with* $\delta\neq\{\frac{1}{2},1,\frac{3}{2},2\}$ *and* $(\underline{\lambda},\mu)\notin\mathbb{B}$ *are isomorphic.*
(*ii*) *The modules of the form*

$$\mathfrak{D}^2_{\lambda\mathbf{1}_i,\frac{1}{2}}\cong\mathfrak{D}^2_{\lambda\mathbf{1}_j,\frac{1}{2}}\cong\mathfrak{D}^2_{\underline{0},\frac{1}{2}-\lambda};\ \forall i\neq j\ ;\ \lambda\neq 0,-\frac{1}{2},-1,-\frac{3}{2} \tag{5.10}$$

and

$$\mathfrak{D}^2_{\lambda\mathbf{1}_i,\frac{1}{2}-\lambda}\cong\mathfrak{D}^2_{\lambda\mathbf{1}_j,\frac{1}{2}-\lambda}\cong\mathfrak{D}^2_{\lambda\mathbf{1}_i+\lambda\mathbf{1}_j,\frac{1}{2}};\ \forall i\neq j\ ;\ \lambda\neq 0,-\frac{1}{4},-\frac{1}{2},-\frac{3}{4} \tag{5.11}$$

are singular.

Proof. In this case, the isomorphism $T:\mathfrak{D}^2_{\underline{\lambda},\mu}\to\mathfrak{D}^2_{\underline{\rho},\nu}$ satisfy the following equations:

$$\begin{cases} \Upsilon_3\Gamma_1(\underline{\rho},\nu)^t=\varepsilon_0\Gamma_1(\underline{\lambda},\mu)^t, \\ \Upsilon_4\Gamma_2(\underline{\rho},\nu)^t=\varepsilon_0\Gamma_2(\underline{\lambda},\mu)^t, \\ \Upsilon_4\Gamma_3(\underline{\rho},\nu)^t=\varepsilon^{1,0,\dots,0}_{1,0,\dots,0}\Gamma_3(\underline{\lambda},\mu)^t+\varepsilon^{0,1,\dots,0}_{1,0,\dots,0}\Gamma_4(\underline{\lambda},\mu)^t+\cdots+\varepsilon^{0,0,\dots,1}_{1,0,\dots,0}\Gamma_{n+2}(\underline{\lambda},\mu)^t, \\ \Upsilon_4\Gamma_4(\underline{\rho},\nu)^t=\varepsilon^{1,0,\dots,0}_{0,1,\dots,0}\Gamma_3(\underline{\lambda},\mu)^t+\varepsilon^{0,1,\dots,0}_{0,1,\dots,0}\Gamma_4(\underline{\lambda},\mu)^t+\cdots+\varepsilon^{0,0,\dots,1}_{0,1,\dots,0}\Gamma_{n+2}(\underline{\lambda},\mu)^t, \\ \vdots \\ \Upsilon_4\Gamma_{n+2}(\underline{\rho},\nu)^t=\varepsilon^{1,0,\dots,0}_{0,0,\dots,1}\Gamma_3(\underline{\lambda},\mu)^t+\varepsilon^{0,1\dots,0}_{0,0,\dots,1}\Gamma_4(\underline{\lambda},\mu)^t+\cdots+\varepsilon^{0,0,\dots,1}_{0,0,\dots,1}\Gamma_{n+2}(\underline{\lambda},\mu)^t, \end{cases}$$

where $\varepsilon_0 \in \mathbb{R}^*$ and $\Upsilon_i \in GL\left(\binom{i+n-1}{n-1}, \mathbb{R}\right)$, $i = 1, 2, 3, 4$ and the vector $\Gamma_i, i = 1, \ldots, n+2$ are respectively given by

$$\Gamma_1(\underline{\lambda}, \mu) = \left(\gamma^1_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}}(\underline{\lambda}, \mu)\right)_{(1\le i_1,i_2,i_3\le n)},$$

where

$$\begin{cases} \gamma^1_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}}(\underline{\lambda}, \mu) = \dfrac{\lambda_{i_1}(2\delta+2\lambda_{i_1}-1)}{2(\delta-1)} & \text{if } i_1 = i_2 = i_3, \\ \gamma^1_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}}(\underline{\lambda}, \mu) = \dfrac{\lambda_{i_1}\lambda_{i_2}}{2(\delta-1)} & \text{if } i_1 \neq \{i_2, i_3\}, i_2 = i_3, \\ \gamma^1_{\mathbf{1}_{i_1}+\mathbf{1}_{i_3}+\mathbf{1}_{i_3}}(\underline{\lambda}, \mu) = 0 & \text{otherwise,} \end{cases}$$

$$\Gamma_2(\underline{\lambda}, \mu) = \left(\Omega^2_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu)\right)_{(1\le i_1,i_2,i_3,i_4\le n)},$$

where

$$\begin{cases} \gamma^2_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{3\lambda_{i_1}(2\delta+2\lambda_{i_1}-1)}{(2\delta-1)(2\delta-4)} & \text{if } i_1 = i_2 = i_3 = i_4, \\ \gamma^2_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{6\lambda_{i_1}\lambda_{i_2}}{(2\delta-1)(2\delta-4)} & \text{if } i_1 \neq \{i_2, i_3, i_4\}, i_2{=}i_3{=}i_4 \text{ or } i_1{=}i_3{\neq} i_2{=}i_4, \\ \gamma^2_{\mathbf{1}_{i_1}+\mathbf{1}_{i_3}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = 0 & \text{otherwise} \end{cases}$$

and

$$\Gamma_j(\underline{\lambda}, \mu) = \left(\gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu)\right)_{(1\le i_1,i_2,i_3,i_4\le n)}, \quad j \in \{3, \ldots, n+2\},$$

where

$$\begin{cases} \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{-(2\delta+4\lambda_{i_1}-1)}{2(2\delta-3)} & \text{if } i_1 = i_2 = i_3 = i_4 = j-2, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{\lambda_{i_1}(2\delta+2\lambda_{i_1}-2)}{(2\delta-3)} & \text{if } i_1 = i_2 = i_3; i_4 = j-2 \text{ and } i_1 < i_4, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{-\lambda_{i_1}(2\delta+2\lambda_{i_1}-2)}{(2\delta-3)} & \text{if } i_1 = i_2 = i_3; i_4 = j-2 \text{ and } i_1 > i_4, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{\lambda_{i_4}(1+2\lambda_{i_1})}{(2\delta 3)} & \text{if } i_1 = i_2 = i_3 = j-2 \text{ and } i_1 < i_4, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{-\lambda_{i_4}(1+2\lambda_{i_1})}{(2\delta-3)} & \text{if } i_1 = i_2 = i_3 = j-2 \text{ and } i_1 > i_4, \end{cases}$$

$$\begin{cases} \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{-\lambda_{i_1}}{(2\delta-3)} & \text{if } i_1 = i_2; i_3 = i_4 = j-2, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{2\lambda_{i_1}\lambda_{i_3}}{(2\delta-3)} & \text{if } i_3 \neq \{i_1, i_2, i_4\}, i_1 = i_2; i_4 = j-2 \text{ and } i_3 > i_4, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = \dfrac{-2\lambda_{i_1}\lambda_{i_3}}{(2\delta-3)} & \text{if } i_3 \neq \{i_1, i_2, i_4\}, i_1 = i_2; i_4 = j-2 \text{ and } i_3 < i_4, \\ \gamma^j_{\mathbf{1}_{i_1}+\mathbf{1}_{i_2}+\mathbf{1}_{i_3}+\mathbf{1}_{i_4}}(\underline{\lambda}, \mu) = 0 & \text{otherwise.} \end{cases}$$

Thus, the following cases arise:

• $\Gamma_i(\underline{\rho}, \nu) \neq 0$ and $\Gamma_i(\underline{\lambda}, \mu) \neq 0$, $\forall i = 1, 2, 3, 4, 5$, that is, $(\underline{\rho}, \nu), (\underline{\lambda}, \mu) \notin B$. Then we get a family of $\mathcal{K}(1)$-isomorphisms given by the conditions

$$\varepsilon_0 = 1, \Upsilon_1 = I_n, \Upsilon_2 \in GL\left(\binom{n+1}{n-1}, \mathbb{R}\right)$$

and

$$\Upsilon_3\Gamma_1(\underline{\rho}, \nu)^t = \Gamma_1(\underline{\lambda}, \mu)^t, \Upsilon_4\Gamma_i(\underline{\rho}, \nu)^t = \Gamma_i(\underline{\lambda}, \mu)^t, \quad i = 2, 3, \ldots, n+2.$$

- $\Gamma_1(\underline{\rho}, \nu) = \Gamma_1(\underline{\lambda}, \mu) = \Gamma_2(\underline{\rho}, \nu) = \Gamma_2(\underline{\lambda}, \mu) = 0$. Then $(\underline{\rho}, \nu), (\underline{\lambda}, \mu) \in B_1$, which leads to (5.10).
- $\Gamma_3(\underline{\lambda}, \mu) = 0$, that is, $(\underline{\lambda}, \mu) \in B_2$. Then $\Gamma_{i+2}(\underline{\lambda}, \mu) \neq 0$ for all $i \in \{2, \ldots, n\}$ and $\det(\Upsilon_1) = \sum_{j=1}^{n}(-1)^{j+1}\varepsilon_{\mathbf{1}_{i_j}}^{\mathbf{1}_1} \det(\Upsilon_1^j)$ where $\det(\Upsilon_1^j)$ is the determinant of the submatrix of Υ_1:

$$\Upsilon_1^{-1}\begin{pmatrix} \Upsilon_4\Gamma_3(\underline{\rho}, \nu)^t \\ \Upsilon_4\Gamma_4(\underline{\rho}, \nu)^t \\ \vdots \\ \Upsilon_4\Gamma_{n+2}(\underline{\rho}, \nu)^t \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma_4(\underline{\lambda}, \mu)^t \\ \vdots \\ \Gamma_{n+2}(\underline{\lambda}, \mu)^t \end{pmatrix}.$$

  We take the first line and we get

$$\sum_{j=1}^{n}(-1)^{j+1}\det(\Upsilon_1^j)\Gamma_{j+2}(\underline{\rho}, \nu)^t = 0.$$

  Since Υ_1 is an invertible matrix, there exist at least $j_0 \in \{3, 4, \ldots, n+2\}$

  where $\det(\Upsilon_1^{j_0}) \neq 0$. Then, we obtain the following two situations:

  ⋆ If $\det(\Upsilon_1^j) = 0$ for all $j \neq j_0$, then $\Gamma_{j_0}(\underline{\rho}, \nu)^t = 0$, so $(\underline{\rho}, \nu) = (\rho\mathbf{1}_{j_0}, \frac{1}{2} - \rho)$.

  ⋆ If $\det(\Upsilon_1^j) = 0$ for all $j \notin \{j_1, j_2\}$ and $\det(\Upsilon_1^{j_1}), \det(\Upsilon_1^{j_2}) \neq 0$, where $j_1 \neq j_2 \in \{3, 4, \ldots, n+2\}$, then we get $(-1)^{j_1}\det(\Upsilon_1^{j_1})\Gamma_{j_1+2}(\underline{\rho}, \nu)^t + (-1)^{j_2}\det(\Upsilon_1^{j_2})\Gamma_{j_2+2}(\underline{\rho}, \nu)^t = 0$ and finally $(\underline{\rho}, \nu) = (\rho\mathbf{1}_{j_1} + \rho\mathbf{1}_{j_2}, \frac{1}{2})$.

- If $\Gamma_{r+2}(\underline{\lambda}, \mu) = 0$, where $r \in \{2, \ldots, n\}$, then $\Gamma_{i+2}(\underline{\lambda}, \mu) \neq 0$ for all $i \neq r$. By a similar reasoning, we end up with same results. □

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