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A lower bound for the tail probability of partial maxima of dependent random variables and applications

FAKHREDDINE BOUKHARI

Department of Mathematics, Faculty of Sciences, Abou Bekr Belkaid University, BP 119, Tlemcen 13000, Algeria
E-mail: f_boukhari@yahoo.fr

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Abstract. In this paper, we establish a non-asymptotic lower bound for the running maximum of an asymptotically almost negatively associated (AANA) family of random variables. The obtained result is applied to characterize the Kolmogorov–Feller weak law of large numbers for these sequences. We also provide a simplified proof of the necessity part in the Baum–Katz law of large numbers in the AANA setting.

Keywords. AANA sequences; Kolmogorov–Feller weak law; complete convergence; Baum–Katz strong law; limit theorems.

Mathematics Subject Classification. 60F15, 60G50.

1. Introduction

To describe the dependence between random variables, several notions of dependence have been proposed in the mid-sixties. Association is one of these concepts that has received an increased attention in the literature. Following Esary *et al.* [10] (see also [1,14]), a finite family of random variables (rv's) $\{X_1, X_2, \ldots, X_n\}$ is said to be negatively associated (NA) if

$$\mathrm{Cov}\big(f(X_k,\ k \in A), g(X_k,\ k \in B)\big) \le 0,$$

for any disjoint subsets $A, B \subset \{1, 2, \ldots, n\}$ and any real coordinate-wise nondecreasing functions f on $\mathbb{R}^A$, g on $\mathbb{R}^B$ such that this covariance exists. An infinite family of rv's is NA if every finite subfamily is NA. For a complete exposition on the subject, we refer the reader to [5].

More recently, the above notion was further extended by Chandra and Ghosal [6] as follows.

DEFINITION 1.1

Let $\tilde{q} = \{q_n,\ n \ge 1\}$ be a sequence of nonnegative numbers with $q(n) \to 0$ as n tends to infinity. A sequence of r.v's $\mathbb{X} = \{X_n,\ n \ge 1\}$ is said to be asymptotically almost

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negatively associated (AANA) with $\tilde{q}$ as a sequence of mixing coefficients if

$$\begin{aligned}&\operatorname{Cov}\big(f(X_n), g(X_{n+1}, \ldots, X_{n+k})\big)\\&\quad\leq q(n)[\operatorname{Var}(f(X_n))\operatorname{Var}(g(X_{n+1}, \ldots, X_{n+k}))]^{1/2},\end{aligned} \tag{1.1}$$

for all $n, k \geq 1$ and for all coordinate-wise increasing continuous functions f and g whenever the right side of the above inequality is finite.

Thus, random variables from an AANA sequence are allowed to have positive correlations, provided they are small. It is clear from Definition 1.1 that an NA family of rv's (and *a fortiori* a family of independent rv's) is AANA with $q(n) = 0$ for all $n \geq 1$. The following is an example of an AANA sequence of rv's. We refer to [6] for details.

Example 1.2. Let $\tilde{\xi} = \{\xi_n,\ n \geq 1\}$ be a sequence of independent identically distributed, $\mathcal{N}(0, 1)$ rv's. Consider a sequence $\{a_n,\ n \geq 1\}$ of positive real numbers, satisfying $a_n \to 0$ as $n \to \infty$ and define

$$X_n = (1 + a_n^2)^{-1/2}(\xi_n + a_n\xi_{n+1}), \qquad n \geq 1.$$

Then, making use of some basic properties of $\tilde{\xi}$, Chandra and Ghosal [6] proved that (1.1) is verified for $\mathbb{X} = \{X_n,\ n \geq 1\}$ with $q(n) = a_n$, so that $\mathbb{X}$ is AANA. They also stressed that $\mathbb{X}$ is not NA.

Many precise results have been obtained for AANA sequences of rv's in the last decades. For instance, Chandra and Ghosal [6] established the Marcinkiewicz–Zygmund strong law of large numbers (SLLN) for an AANA collection of rv's. Yuan and An [22] obtained a Rosenthal-type inequality for these rv's. Shen and Wu [20] studied the rate of convergence in the SLLN for an AANA family of rv's. Finally, an extension of the divergent part in the Borel–Cantelli lemma has been derived in [4].

The present work is motivated by the weak law of large numbers (WLLN) established by Chandra [7] in the AANA setting. One of the main result of this article, Theorem 5.4 of Section 5, shows that Chandra's sufficient condition is also necessary for the WLLN to hold for AANA sequences. In this context, we also derive in Theorem 5.5, a generalized version of the Marcinkiewicz–Zygmund WLLN. The key ingredient in the proof of these two results is the new lower bound stated in Theorem 3.1 for the tail probability of the maximum of a finite collection of AANA rv's. This bound is obtained using a method which dates back to Paley and Zygmund [18].

The rest of the paper is organized as follows: Section 2 contains some useful properties of AANA families. In Section 3 we establish our main statement (Theorem 3.1). Leaning on this result, we prove in Section 4, two useful laws of large numbers for partial maxima of AANA sequences, extending previous results of Gut [12], Ch VI. In Section 5, we apply a corollary of Theorem 3.1, to derive a characterization of the Kolmogorov–Feller WLLN in the AANA framework, completing the work of Chandra [7]. In the last section, Theorem 3.1 is employed to supply an alternative and short proof of the necessity part in the Baum–Katz SLLN, established by Yuan and An [22] for AANA rv's.

2. Preliminaries

We begin by introducing the notation that will be used in the remainder of the paper.

Notation. Throughout this article, $\mathbb{X} = \{X_n,\ n \geq 1\}$ is a sequence of rv's defined on some probability space $(\Omega, \mathcal{F}, P)$. As usual S_n will denote its partial sum. When $\mathbb{X}$ is AANA with the sequence of mixing coefficients $\tilde{q} = \{q_n,\ n \geq 1\}$ in ℓ^2, we set

$$Q^2 = \sum_{n\geq 1} q_n^2 < \infty \quad \text{and} \quad \alpha(\tilde{q}) = \sqrt{2}\big(Q + (1+Q^2)^{1/2}\big). \tag{2.1}$$

Moreover, $\xrightarrow{P}$ means convergence in probability and the relation $\mathcal{L}(X) = \mathcal{L}(Y)$ indicates that X and Y are two rv's with the same law. The symbol C denotes a positive constant, possibly varying from place to place. Finally, the notion $a_n \asymp b_n$ stands for $\lambda_* \leq a_n/b_n \leq \lambda^*$ for some positive constants $\lambda_*,\ \lambda^*$ and large values of n.

In many situations, stating a maximal inequality is a crucial step in proving a limit theorem. The following extension of the celebrated Kolmogorov maximal inequality to AANA rv's was derived in Chandra and Ghosal [6].

Lemma 2.1. Let $\mathbb{X} = \{X_n,\ n \geq 1\}$ be an AANA sequence with the mixing coefficients $\tilde{q} = \{q_n,\ n \geq 1\} \in \ell^2$. Assume that each X_n is centered and square integrable, then

$$\forall \epsilon > 0, \qquad P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| > \epsilon\right) \leq \frac{\alpha^2(\tilde{q})}{\epsilon^2}\sum_{k=1}^{n}\mathbb{E}(X_k^2),$$

where $\alpha(\tilde{q})$ is given in (2.1).

The forthcoming lemma will be employed several times later.

Lemma 2.2 [22]. Let $\{f_n,\ n \geq 1\}$ be a sequence of nondecreasing (resp. nonincreasing) Borel functions. If $\mathbb{X} = \{X_n,\ n \geq 1\}$ is an AANA sequence with mixing coefficients $\tilde{q} = \{q_n,\ n \geq 1\}$, then so is $\{f_n(X_n),\ n \geq 1\}$.

The next useful inequality was stated in the proof of Theorem 2 of [6]. For a stronger version, we refer the reader to Theorem 2.3 of Yuan and An [22].

Lemma 2.3. Let $\mathbb{X} = \{X_n,\ n \geq 1\}$ be an AANA sequence of square integrable, mean zero rv's with mixing coefficients $\tilde{q} = \{q_n,\ n \geq 1\} \in \ell^2$. Then

$$\forall n \geq 1, \qquad \mathbb{E}\left(\sum_{k=1}^{n} X_k\right)^2 \leq \alpha^2(\tilde{q})\sum_{k=1}^{n}\mathbb{E}(X_k^2),$$

where $\alpha(\tilde{q})$ is defined in (2.1).

We close this section by recalling that a sequence of rv's $\mathbb{X} = \{X_n,\ n \geq 1\}$, is *stochastically dominated* by a rv ξ (we note $\mathbb{X} \prec \xi$), if there exists a positive constant C, such that

$$\forall x \geq 0, \qquad \sup_{n\geq 1} P(|X_n| > x) \leq C P(|\xi| > x).$$

Obviously, a sequence $\mathbb{X} = \{X, X_n,\ n \geq 1\}$ of identically distributed rv's is stochastically dominated by X. Furthermore, we can check via the integration by parts formula, that if

$\mathbb{X} \prec \xi$, then

$$\mathbb{E}|X_n|^p \mathbb{1}(|X_n| \le x) \le C\Big[\mathbb{E}|\xi|^p \mathbb{1}(|\xi| \le x) + x^p P(|\xi| > x)\Big], \tag{2.2}$$

for all $n \ge 1,\ p > 0$ and $x \ge 0$.

3. The main result

In this section, we present the main result of this paper.

Theorem 3.1. *Let $t > 0,\ n \ge 1$ and consider an AANA sequence $\mathbb{X} = \{X_k,\ k \ge 1\}$ of rv's with $\tilde{q} = \{q_n,\ n \ge 1\} \in \ell^2$. Set*

$$I_n(t) = P\big(\max_{1 \le i \le n} |X_i| > t\big). \tag{3.1}$$

Then

$$\frac{1}{2} \cdot \frac{\sum_{k=1}^n P(|X_k| > t)}{2\alpha^2(\tilde{q}) + \sum_{k=1}^n P(|X_k| > t)} \le I_n(t) \le \sum_{k=1}^n P(|X_k| > t), \tag{3.2}$$

where $\alpha(\tilde{q})$ is given in (2.1).

Proof. The right-side inequality in (3.2) is obvious. To show the first one, we use an idea from Paley and Zygmund [18] and some properties of AANA rv's, displayed in the previous section.

Let $n \ge 1$ and $t > 0$ be fixed and set $A_j = A_j(t) = \{|X_j| > t\}$ for $j \ge 1$. First, observe that we may and do assume $\sum_{k=1}^n P(|X_k| > t) > 0$. Besides, if we put

$$Z_n = \sum_{j=1}^n \mathbb{1}(A_j),$$

then

$$\bigcap_{j=1}^n \bar{A}_j = \{Z_n = 0\} \qquad \text{and} \qquad I_n(t) = P(Z_n > 0).$$

Combining these relations with Cauchy–Schwarz's inequality, we get

$$\mathbb{E}^2(Z_n) = \mathbb{E}^2\big(Z_n \mathbb{1}(Z_n > 0)\big) \le \mathbb{E}(Z_n^2) P(Z_n > 0) = \mathbb{E}(Z_n^2) I_n(t). \tag{3.3}$$

Next, in view of Lemma 2.2, $\{\mathbb{1}(X_k > t) - P(X_k > t),\ 1 \le k \le n\}$ and $\{\mathbb{1}(X_k < -t) - P(X_k < -t),\ 1 \le k \le n\}$ are two AANA sequences of mean zero rv's, with mixing coefficients $\{q_n,\ n \ge 1\}$. So, by invoking Lemma 2.3 and the elementary inequality

$(a+b)^2 \le 2(a^2+b^2)$, we may write

$$\begin{aligned}\mathbb{E}\big(Z_n-\mathbb{E}(Z_n)\big)^2 &\le 2\left\|\sum_{k=1}^{n}\big(\mathbb{1}(X_k>t)-P(X_k>t)\big)\right\|_2^2\\ &\quad+2\left\|\sum_{k=1}^{n}\big(\mathbb{1}(X_k<-t)-P(X_k<-t)\big)\right\|_2^2\\ &\le 2\alpha^2(\widetilde{q})\sum_{k=1}^{n}P(|X_k|>t).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}(Z_n^2) &\le 2\mathbb{E}\big[\big(Z_n-\mathbb{E}(Z_n)\big)^2\big]+2\mathbb{E}^2(Z_n)\\ &\le 4\alpha^2(\tilde{q})\sum_{k=1}^{n}P(|X_k|>t)+2\left(\sum_{k=1}^{n}P(|X_k|>t)\right)^2.\end{aligned}$$

Substituting this into (3.3), we get

$$\begin{aligned}&\left(\sum_{k=1}^{n}P(|X_k|>t)\right)^2=\mathbb{E}^2(Z_n)\\ &\le 2\left(2\alpha^2(\tilde{q})\sum_{k=1}^{n}P(|X_k|>t)+\left(\sum_{k=1}^{n}P(|X_k|>t)\right)^2\right)I_n(t),\end{aligned}$$

which yields

$$I_n(t)\ge\frac{1}{2}\cdot\frac{\sum_{k=1}^{n}P(|X_k|>t)}{2\alpha^2(\tilde{q})+\sum_{k=1}^{n}P(|X_k|>t)}.$$

This proves the first inequality in (3.2) and achieves the proof of Theorem 3.1. □

Remark 3.2.

(i) Theorem 3.1 furnishes two non-asymptotic bounds, this may be helpful in applications.
(ii) For each $n \ge 1$ and $t > 0$, the lower bound in (3.2) lies in $[0, 1/2]$. Moreover, when $\mathbb{X}$ is an *NA* sequence of rv's (and *a fortiori* a sequence of independent rv's), then $q_n = 0$ for every $n \ge 1$, so that $\alpha(\tilde{q}) = \sqrt{2}$.
(iii) From the proof of Theorem 3.1, it appears clearly that this result remains true for a sequence of random variables with a different dependence structure, provided that it satisfies the properties given by Lemmas 2.2 and 2.3.

4. Two laws of large numbers for partial maxima

Our first application of Theorem 3.1 is related to a WLLN for partial maxima, obtained by Gut [11] for independent and identically distributed (i.i.d.) rv's.

For $n \geq 1$ and a sequence $\mathbb{X} = \{X_k,\ k \geq 1\}$ of rv's, we define

$$X_n^* = \max\{|X_k|,\ 1 \leq k \leq n\} \quad \text{and} \quad \mathbb{X}^* = \{X_k^*,\ k \geq 1\}. \tag{4.1}$$

Theorem 4.1. *Let $\mathbb{X} = \{X_k,\ k \geq 1\}$ be an AANA sequence with $\tilde{q} \in \ell^2$, $\mathbb{X}^*$ the sequence of its partial maxima defined in (4.1). If $\tilde{a} = \{a_k,\ k \geq 1\}$ is a sequence of positive numbers, then*

$$\frac{X_n^*}{a_n} \xrightarrow{P} 0 \quad as \quad n \to \infty,$$

if and only if

$$\forall \epsilon > 0, \qquad \lim_{n\to\infty} \sum_{k=1}^{n} P(|X_k| > \epsilon a_n) = 0. \tag{4.2}$$

Proof. For $n \geq 1$, let $I_n(\cdot)$ be defined as in (3.1). Notice that $X_n^*/a_n \to 0$ in probability iff $I_n(\epsilon a_n) \to 0$ as $n \to \infty$, for any $\epsilon > 0$. According to (3.2), this last assertion is equivalent to (4.2). □

As a second application of Theorem 3.1, we now generalize Theorem 6.11.1 of Gut [12], to the AANA setting. Remember that a sequence of rv's $\{Y_n,\ n \geq 1\}$ *converges completely* to a rv Y if

$$\forall \epsilon > 0, \qquad \sum_{n\geq 1} P(|Y_n - Y| > \epsilon) < \infty.$$

This concept was introduced by Hsu-Robbins [13]. The Borel–Cantelli lemma ensures that complete convergence is stronger that almost sure convergence.

Theorem 4.2. *Let $\mathbb{X}$, $\mathbb{X}^*$ and $\{a_n,\ n \geq 1\}$ be as in Theorem 4.1 and assume that $\mathcal{L}(X_k) = \mathcal{L}(X_1)$ for any $k \geq 1$. If $\tilde{u} = \{u_n,\ n \geq 1\}$ is a nondecreasing sequence of nonnegative numbers, then*

$$\sum_{n\geq 1} u_n P(X_n^* > a_n) < \infty \iff \sum_{n\geq 1} n u_n P(|X_1| > a_n) < \infty.$$

In particular, for any $r > 0$ and $\beta \geq 2$,

$$\sum_{n\geq 1} n^{\beta-2} P(X_n^* > n^{1/r}) < \infty \iff \mathbb{E}(|X_1|^{r\beta}) < \infty$$

and

$$\lim_{n\to\infty} \frac{X_n^*}{n^{1/r}} = 0 \ \ completely \iff \mathbb{E}(|X_1|^{2r}) < \infty.$$

Proof. For the first equivalence, let $t > 0$ and remember that $I_n(t)$ was defined in (3.1). Thereby, $P(X_n^* > a_n) = I_n(a_n)$ and

$$\frac{1}{2} \cdot \frac{nu_n P(|X_1| > a_n)}{2\alpha^2(\tilde{q}) + nP(|X_1| > a_n)} \le u_n P(X_n^* > a_n) \le nu_n P(|X_1| > a_n). \tag{4.3}$$

Now if $\sum_{n\ge1} u_n P(X_n^* > a_n) < \infty$, we infer from (4.3) and the assumption on $\tilde{u}$ that $nP(|X| > a_n) \le \alpha^2(\tilde{q})$, for large values of n. Combining this with the first inequality in (4.3), we conclude that $u_n P(X_n^* > a_n) \ge nu_n P(|X_1| > a_n)/18\alpha^2(\tilde{q})$ for n sufficiently large, so that $\sum_{n\ge1} nu_n P(|X_1| > a_n) < \infty$. The converse implication follows from the second inequality in (4.3).

For the second equivalence, we use the first point with $a_n = n^{1/r}$ and $u_n = n^{\beta-2}$ for $n \ge 1,\ r > 0$ and $\beta \ge 2$. Thus, $\sum_{n\ge1} n^{\beta-2} P(X_n^* > n^{1/r}) < \infty$ iff $\sum_{n\ge1} n^{\beta-1} P(|X_1| > n^{1/r}) < \infty$. But, taking into consideration Theorem 2.12.1(iv) of [12], this last assertion is equivalent to $\mathbb{E}(|X_1|^{r\beta}) < \infty$.

Finally, for the last statement of Theorem 4.2 it is enough to apply the second equivalence to the AANA sequence $\{X_k/\epsilon,\ k \ge 1\}$ with $\epsilon > 0$. □

5. A Kolmogorov–Feller weak law of large numbers

For a sequence of i.i.d. random variables to obey the SLLN, the existence of the first absolute moment is necessary and sufficient. The situation in the WLLN is rather different, the following Kolmogorov–Feller theorem provides a necessary and sufficient condition, for the WLLN to hold in this case.

Theorem 5.1 ([12], p. 279). *Assume that $\mathbb{X} = \{X, X_n,\ n \ge 1\}$ is a sequence of i.i.d. random variables. Then*

$$\frac{S_n - n\mathbb{E}\big(X\mathbb{1}(|X| \le n)\big)}{n} \xrightarrow{P} 0 \quad as\ n \to \infty$$

if and only if

$$xP(|X| > x) \longrightarrow 0 \quad as\ x \to \infty. \tag{5.1}$$

In the seventies, Klass and Teicher [15] generalized this WLLN by showing that, under some restrictions on the normalizing sequence $\tilde{a} = \{a_n,\ n \ge 1\}$ (see Conditions (5.3) below), the relation

$$\lim_{n\to\infty} nP(|X| > a_n) = 0, \tag{5.2}$$

is necessary and sufficient for

$$\left\{\frac{S_n - n\mathbb{E}X\mathbb{1}(|X| \le a_n)}{a_n},\ n \ge 1\right\},$$

to converges to zero in probability. It is worthwhile noticing here that the sufficiency part in the WLLN due to Gut [11], is a particular case of Klass–Teicher'result, see for instance [16].

In the light of these two statements, it is natural to inquire whether these WLLNs may be generalized to dependent rv's. A first answer to this question was given by Kruglov [16]. Let $\mathbb{X} = \{X, X_n,\ n \geq 1\}$ be an NA sequence of rv's with a common law, $\tilde{a} = \{a_n,\ n \geq 1\}$ a sequence of positive real numbers and consider the following requirements:

$$\text{(i)} \quad \sum_{k=1}^{n} \frac{a_k^2}{k^2} = O\left(\frac{a_k^2}{k}\right), \qquad \text{(ii)} \quad \lim_{n\to\infty} \frac{a_n}{n} = +\infty. \tag{5.3}$$

Theorem 1 in [16] guarantees that if $\tilde{a}$ fulfills (5.3)(i), then

$$\left\{\frac{1}{a_n} \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} \big(X_i - \mathbb{E}X\mathbb{1}(|X| \leq a_n)\big)\right|,\ n \geq 1\right\},$$

converges to zero in probability as $n \to \infty$ if and only if (5.2) holds true. The sufficiency part of Kruglov's WLLN has been extended by Chandra [7], to AANA collections of rv's. However, unlike the above two WLLNs, the result of Chandra does not provide any necessary condition for the WLLN to hold. In this section, we shall try to fill this gap using Theorem 4.1. We emphasize that our method is completely different from those employed in [11,15,16].

Some comments on Condition (5.3)(i) are in order.

Remark 5.2 [16]. Let $\tilde{a} = \{a_n,\ n \geq 1\}$ be a nondecreasing sequence of positive reals.

(a) If (5.3)(i) is verified by $\tilde{a}$, then $\lim_{n\to\infty} a_n^2/n = +\infty$.

(b) Let $r > 1/2$ and consider a positive, slowly varying function l (see e.g. [12], p. 566), defined on $\mathbb{R}_+$. Put $u(x) = x^r l(x)$ for $x \geq 0$, then (5.3)(i) obtains for $\{u(n),\ n \geq 1\}$.

Consider a sequence of positive real numbers $\tilde{a} = \{a_i,\ i \geq 1\}$ and define for $1 \leq k \leq n$,

$$m_k(n) = \mathbb{E}(X_k\mathbb{1}(|X_k| \leq a_n)),$$

$$\sigma_n^* = \frac{1}{a_n} \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} \big(X_i - m_k(n)\big)\right| \quad \text{and} \quad \sigma^* = \{\sigma_n^*,\ n \geq 1\}.$$

We start by treating the necessity part in the Kolmogorov–Feller WLLN.

Theorem 5.3. *Let $\tilde{a} = \{a_k,\ k \geq 1\}$ be an unbounded sequence of positive real numbers and consider an AANA sequence $\mathbb{X} = \{X_k,\ k \geq 1\}$ of rv's with the sequence of mixing coefficients $\tilde{q} \in \ell^2$. Assume that $\mathbb{X} \prec \xi$ for some rv ξ and σ^* converges in probability to zero, then*

$$\forall \epsilon > 0, \qquad \lim_{n\to\infty} \sum_{k=1}^{n} P(|X_k| > \epsilon a_n) = 0. \tag{5.4}$$

Proof. Let $1 \leq k \leq n$ and observe that

$$\frac{1}{a_n} \max_{1\leq k\leq n} |X_k - m_k(n)| \leq \frac{2}{a_n} \max_{1\leq k\leq n} \left|\sum_{i=1}^{k} \big(X_i - m_i(n)\big)\right|.$$

Joining this with the assumption made on σ^* leads to

$$\frac{1}{a_n} \max_{1\leq k\leq n} |X_k - m_k(n)| \xrightarrow{P} 0, \quad \text{as } n \to \infty. \tag{5.5}$$

Besides, as $\mathbb{X} \prec \xi$, we have via (2.2),

$$\frac{1}{a_n} \max_{1 \le k \le n} |m_k(n)| \le C \left[\frac{1}{a_n} \mathbb{E}(|\xi| \mathbb{1}(|\xi| \le a_n)) + P(|\xi| > a_n) \right].$$

But $\tilde{a}$ is unbounded, hence both terms in the right-hand side of the last inequality approach zero as n goes to infinity. Consequently,

$$\lim_{n \to \infty} \frac{1}{a_n} \max_{1 \le k \le n} |m_k(n)| = 0,$$

which, in conjunction with (5.5), yields

$$\frac{1}{a_n} \max_{1 \le k \le n} |X_k| \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Invoking Theorem 4.1, we conclude that (5.4) is met. This ends the proof. □

Now we are in position to supply a characterization of the WLLN for AANA families of rv's. This statement completes Theorem 1.3 of [7] and improves similar results from [11, 15, 16].

Theorem 5.4. *Let $\mathbb{X} = \{X, X_k, \ k \ge 1\}$ be an AANA sequence of identically distributed rv's, with the sequence of mixing coefficients $\tilde{q} \in \ell^2$. Let also $\tilde{a} = \{a_k, \ k \ge 1\}$ be an unbounded sequence of positive real numbers fulfilling one of the conditions in* (5.3)*. Then σ^* converges to zero in probability if and only if* (5.2) *is satisfied.*

Proof. The necessity of Assumption (5.2) follows from Theorem 5.3, so we only show the sufficiency part. Suppose that (5.2) is met. From Remark (a) after Theorem 1.3 in Chandra [7], σ^* converges to zero in probability if

$$\lim_{n \to \infty} \frac{n}{a_n^2} \mathbb{E}(X^2 \mathbb{1}(|X| \le a_n)) = 0.$$

But this last relation is verified when $\tilde{a}$ fulfills one of the requirements in (5.3), see [15], p. 863. This proves the sufficiency and completes the proof. □

For $n \ge 1$ and $0 < p < 2$, set $b_n = n^{1/p}$ and $\tilde{b} = \{b_k, \ k \ge 1\}$. Clearly, $\tilde{b}$ satisfies (5.3)(ii) when $0 < p < 1$. Furthermore, taking account of Remark 5.2(b), we deduce that Condition (5.3)(i) holds for $\tilde{b}$ if $1 \le p < 2$. Thus, the next generalized Marcinkiewicz–Zygmund weak law of large numbers follows immediately from Theorem 5.4.

Theorem 5.5. *Let $0 < p < 2$ and consider an AANA sequence $\mathbb{X} = \{X, X_k, \ k \ge 1\}$, of identically distributed rv's, with the sequence of mixing coefficients $\tilde{q} \in \ell^2$. Then*

$$\frac{1}{n^{1/p}} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} \left(X_i - \mathbb{E} X \mathbb{1}(|X|^p \le n) \right) \right| \xrightarrow{P} 0$$

as $n \to \infty$, *if and only if*

$$\lim_{n\to\infty} nP(|X|^p > n) = 0.$$

This last statement improves the corresponding ones obtained in [3] and Theorem 6.4.2 of [12].

6. The Baum–Katz strong law of large numbers

The study of the rate of convergence is a classical issue in probability theory and statistics. One of the fundamental results in this area was stated by Baum and Katz [2] and strengthened by Chow [8]. This theorem ensures that, for $0 < p < 2$, $r \geq p$ and a sequence $\mathbb{X} = \{X_k,\ k \geq 1\}$ of i.i.d. random variables, with $\mathbb{E}(X_1) = 0$ when $p \geq 1$, the assertion

$$\mathbb{E}(|X_1|^r) < \infty \tag{6.1}$$

is equivalent to

$$\sum_{n\geq 1} n^{(r/p)-2} P(\max_{1\leq k\leq n} |S_k| > n^{1/p}\epsilon) \qquad \text{for all } \epsilon > 0. \tag{6.2}$$

This statement has been extended to several classes of dependent rv's, see Peligrad [19] and the references therein.

More recently, Yuan and An [22] established the equivalence between (6.1) and (6.2) in the AANA setting.

Theorem 6.1 ([22], Theorem 5.1). *Let* r, p *be as above and set* $s = (r/p - 1)/(1/p - 1/2) + r + 2$. *Let also* $\mathbb{X} = \{X_k,\ k \geq 1\}$ *be an AANA collection of identically distributed, centered rv's, such that*

$$\sum_{n\geq 1} q^{2/(3s)}(n) < \infty. \tag{6.3}$$

Then (6.1) *and* (6.2) *are equivalent.*

A perusal of Yuan and An's demonstration of Theorem 6.1, reveals that the proof of the assertion (6.2) $\Rightarrow$ (6.1) is based on an idea from Erdös [9] and a Rosenthal type inequality (Theorem 2.1 in [22]). Furthermore, Condition (6.3) was only used to show that (6.1) $\Rightarrow$ (6.2). Finally, taking into account the range of s, it is easy to see that Assumption (6.3) is stronger than $\tilde{q} \in \ell^2$.

Appealing once again to Theorem 3.1, we shall supply a new and much shorter proof of the implication (6.2) $\Rightarrow$ (6.1). Moreover, we also complete the result of Yuan and An, by showing that the rv's must be centered when $p \geq 1$. More precisely, we have

Theorem 6.2. *Let* p, r *be as above and let* $\mathbb{X} = \{X_k,\ k \geq 1\}$ *be an AANA sequence of identically distributed rv's with* $\tilde{q} \in \ell^2$. *if* (6.2) *is satisfied, then so is* (6.1). *If, in addition* $p \geq 1$, *then* $\mathbb{E}(X_1) = 0$.

Proof. Let $\epsilon > 0,\ t > 0$ and $n \geq 1$. Recall that we have set

$$I_n(t) = P\left(\max_{1\leq k\leq n} |X_k| > t\right).$$

A direct application of Theorem 3.1 yields

$$\frac{1}{2}\cdot\frac{nP(|X_1| > t)}{2\alpha^2(\widetilde{q}) + nP(|X_1| > t)} \leq I_n(t) \leq nP(|X_1| > t). \tag{6.4}$$

Besides, as $|X_k| \leq |S_k| + |S_{k-1}|$, then

$$I_n(t) \leq P\Big(\max_{1\leq k\leq n} |S_k| > t/2\Big),$$

which, via (6.2), ensures that

$$\sum_{n\geq 1} n^{\frac{r}{p}-2} I_n(n^{\frac{1}{p}}\epsilon) < \infty. \tag{6.5}$$

But

$$n^{\frac{r}{p}-1} I_n(n^{\frac{1}{p}}\epsilon) = O\left(\sum_{k=n}^{2n} k^{\frac{r}{p}-2} I_k(k^{\frac{1}{p}} 2^{-\frac{1}{p}}\epsilon)\right),$$

so that $I_n(n^{1/p}\epsilon) = o(1)$. By (6.4), this last relation leads to $nP(|X_1| > n^{1/p}\epsilon) = o(1)$ and

$$I_n(n^{\frac{1}{p}}\epsilon) \asymp nP(|X_1| > n^{\frac{1}{p}}\epsilon).$$

Combining this with (6.5), we deduce that

$$\sum_{n\geq 1} n^{\frac{r}{p}-1} P(|X_1| > n^{\frac{1}{p}}\epsilon) < \infty,$$

and this yields $\mathbb{E}(|X_1|^r) < \infty$, via Theorem 2.12.1 (iv) of [12].

Now suppose that $p \geq 1$, then Condition (6.2) implies

$$\sum_{n\geq 1} \frac{1}{n} P\left(\max_{1\leq k\leq n} |S_k| > n^{1/p}\epsilon\right) < \infty \qquad \text{for all } \epsilon > 0,$$

which, in the light of Lemma 2.4 of [21], entails

$$\frac{S_n}{n^{1/p}} \longrightarrow 0 \quad \text{almost surely},$$

as $n \to \infty$. Since $X \in L^r \subset L^p$, the Marcinkiewics–Zygmund SLLN obtained in [4,6] guarantees that $\mathbb{E}(X_1) = 0$. The proof of Theorem 6.2 is achieved. □

Remark 6.3. As observed in the first section, a sequence of independent rv's is AANA with $\alpha(\tilde{q}) = 0$. Accordingly, the above approach provides an alternative way to prove the corresponding implication in the classical Baum–Katz theorem. This simplified method has also allowed us to avoid a tedious argument of Erdös [9], which is usually employed to show the necessity part in several extensions of the Baum–Katz SLLN, see for instance [17,19,22].

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