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n-Color palindromic compositions with restricted subscripts

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Abstract. An n-color composition is one in which a part of size m can come in m colors (denoted by subscripts). Compositions that read the same when read forwards or backwards are said to be palindromic. In this paper, we study the number of n-color palindromic compositions whose parts have subscripts belonging to a particular arithmetic progression. That is, the subscripts are of the form $\ell a + b$, where ℓ and b are fixed positive integers and $a \geq 0$ is arbitrary. Among our results, we derive an explicit formula for the generating function and provide a connection with Riordan arrays. Finally, we describe bijections between certain restricted classes of palindromic n-color compositions and subsets of ordinary compositions and ternary words.

Keywords. n-color compositions; generating functions; combinatorial identities.

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1. Introduction

Given a positive integer ν, a *composition* of ν is a sequence of positive integers $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$, called *parts*, such that $\sigma_1 + \sigma_2 + \cdots + \sigma_\ell = \nu$. In this case, ν is referred to as the *weight* of σ. For example, the compositions of 4 are

$$\{4\}, \{3, 1\}, \{1, 3\}, \{2, 2\}, \{2, 1, 1\}, \{1, 2, 1\}, \{1, 1, 2\}, \{1, 1, 1, 1\}.$$

A *palindromic* (or *self-inverse*) *composition* is one whose sequence of parts is the same when read from left to right or from right to left. For example, the palindromic compositions of 4 are

$$\{4\}, \{2, 2\}, \{1, 2, 1\}, \{1, 1, 1, 1\}.$$

For further information on compositions, we refer the reader to the text by Heubach and Mansour [9].

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Agarwal [2] generalized the concept of a composition by allowing the parts to come in various colors. Specifically, he refers to compositions wherein each part of a given size has the same number of color options as *n-color compositions*, with the choice of color indicated by a subscript. That is, the m different colored parts of size m are denoted as $m_1, m_2, \ldots, m_m$ for each $m \geq 1$. Thus, for example, the possible n-color compositions of 4 are given by

$$\{4_1\}, \{4_2\}, \{4_3\}, \{4_4\}, \{3_1, 1_1\}, \{3_2, 1_1\}, \{3_3, 1_1\}, \{1_1, 3_1\}, \{1_1, 3_2\},$$
$$\{1_1, 3_3\}, \{2_1, 2_1\}, \{2_1, 2_2\}, \{2_2, 2_1\}, \{2_2, 2_2\}, \{2_1, 1_1, 1_1\},$$
$$\{2_2, 1_1, 1_1\}, \{1_1, 2_1, 1_1\}, \{1_1, 2_2, 1_1\}, \{1_1, 1_1, 2_1\},$$
$$\{1_1, 1_1, 2_2\}, \{1_1, 1_1, 1_1, 1_1\}.$$

The n-color palindromic compositions were first considered by Narang and Agarwal in [11], where various combinatorial properties were studied. The n-color palindromic compositions of 4, for example, are given by

$$\{4_1\}, \{4_2\}, \{4_3\}, \{4_4\}, \{2_1, 2_1\}, \{2_2, 2_2\}, \{1_1, 2_1, 1_1\},$$
$$\{1_1, 2_2, 1_1\}, \{1_1, 1_1, 1_1, 1_1\}.$$

It is well-known that the total number of n-color compositions of ν is equal to the Fibonacci number $F_{2\nu}$. Shapcott [14] has shown further that the number of n-color palindromic compositions of ν is given by $F_\nu + 2F_{\nu-1}$ if ν is odd, and by $3F_\nu$ if ν is even. For additional results about n-color compositions, see, e.g., [1–8,10–15]. In this paper, we study some new restrictions on n-color palindromic compositions that generalize previous results of Narang and Agarwal [11], among others.

This paper is organized as follows. In the next section, we introduce a restriction on n-color palindromic compositions wherein the subscript of each part is required to belong to a particular arithmetic sequence. We establish several enumerative results concerning this restricted class of compositions, including generating function and explicit formulas. Special attention is paid to the case when the subscripts are required to be of a particular parity. In the third section, it is shown that the sequence enumerating members of a general class of the aforementioned restricted compositions having a prescribed weight and number of parts is given equivalently by a Riordan array. In the final section, combinatorial proofs are provided for some of the prior results, including explicit bijections with ordinary compositions in which there are no rises of even index as well as with ternary words in which the 00 and 12 subwords are disallowed.

2. Generalized restricted *n*-color palindromic compositions

Recently, Acosta *et al.* [1] studied restricted n-color compositions where the possible subscripts were required to lie in an arithmetic progression. Specifically, given positive integers ℓ and b, let $\mathcal{C}_{\ell a+b}(\nu)$ denote the number of n-color compositions of ν whose parts all have subscripts of the form $\ell a + b$ for some $a \geq 0$. Furthermore, we will denote by $\mathcal{C}_{\ell a+b}(m, \nu)$ the number of such compositions of ν that have exactly m parts. For example,

$\mathcal{C}_{3a+2}(7) = 11$, the relevant compositions being

$$\{7_2\}, \{7_5\}, \{5_2, 2_2\}, \{5_5, 2_2\}, \{2_2, 5_2\}, \{2_2, 5_5\}, \{4_2, 3_2\},$$
$$\{3_2, 4_2\}, \{3_2, 2_2, 2_2\}, \{2_2, 3_2, 2_2\}, \{2_2, 2_2, 3_2\},$$

whereas $\mathcal{C}_{3a+2}(2, 7) = 6$.

Acosta *et al.* [1] established the explicit formula

$$\mathcal{C}_{\ell a+b}(m, \nu) = \sum_{i=0}^{\left\lfloor \frac{\nu - bm}{\ell} \right\rfloor} \binom{m+i-1}{m-1} \binom{\nu - \ell i + m(1-b) - 1}{m-1}, \tag{1}$$

along with the recurrence $\mathcal{C}_{\ell a+b}(\nu) = \mathcal{C}_{\ell a+b}(\nu-1) + \mathcal{C}_{\ell a+b}(\nu-b) + \mathcal{C}_{\ell a+b}(\nu-\ell) - \mathcal{C}_{\ell a+b}(\nu-\ell-1)$ for $\nu > \max\{b, \ell+1\}$.

Let $\mathcal{GC}_{\ell a+b}(m, x)$ and $\mathcal{GC}_{\ell a+b}(x)$ denote the generating functions for the sequences $\mathcal{C}_{\ell a+b}(m, \nu)$ and $\mathcal{C}_{\ell a+b}(\nu)$, respectively. Then we have the formulas

$$\mathcal{GC}_{\ell a+b}(m, x) = \left(\frac{x^b}{(1-x)(1-x^\ell)} \right)^m, \tag{2}$$

$$\mathcal{GC}_{\ell a+b}(x) = \frac{x^b}{1 - x - x^\ell + x^{\ell+1} - x^b}. \tag{3}$$

We now introduce and enumerate restricted n-color palindromic compositions. Given positive integers ℓ and b, let $\mathcal{P}_{\ell a+b}(\nu)$ denote the number of n-color palindromic compositions of ν whose parts have subscripts of the form $\ell a + b$ for some $a \geq 0$ and denote by $\mathcal{P}_{\ell a+b}(m, \nu)$ the number of those compositions having exactly m parts. For example, $\mathcal{P}_{3a+2}(7) = 3$, the relevant compositions being

$$\{7_2\}, \{7_5\}, \{2_2, 3_2, 2_2\}.$$

We have the following generating function formulas for the preceding sequences.

Theorem 1. *Let $\mathcal{GP}_{\ell a+b}(x)$ and $\mathcal{GP}_{\ell a+b}(m, x)$ denote the generating functions for the sequences $\mathcal{P}_{\ell a+b}(\nu)$ and $\mathcal{P}_{\ell a+b}(m, \nu)$, respectively. Then for $m \geq 1$ and all $|x| < 1$, we have*

$$\mathcal{GP}_{\ell a+b}(x) = \frac{(1+x)(1+x^\ell)(1-x+x^b-x^\ell+x^{\ell+1})}{1-x^2-x^{2b}-x^{2\ell}+x^{2(\ell+1)}},$$

$$\mathcal{GP}_{\ell a+b}(2m, x) := \sum_{\nu \geq 1} \mathcal{P}_{\ell a+b}(2m, 2\nu) x^{2\nu} = \left(\frac{x^{2b}}{(1-x^2)(1-x^{2\ell})} \right)^m,$$

$$\mathcal{GP}_{\ell a+b}(2m-1, x) := \sum_{\nu \geq 1} \mathcal{P}_{\ell a+b}(2m-1, \nu) x^\nu$$
$$= \frac{x^b}{(1-x)(1-x^\ell)} \left(\frac{x^{2b}}{(1-x^2)(1-x^{2\ell})} \right)^{m-1}.$$

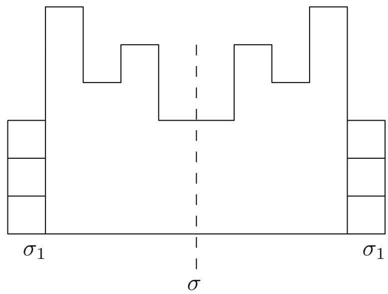

Figure 1. Decomposition of an n-color palindromic composition.

Proof. Let σ be an n-color palindromic composition where each subscript is of the form $\ell a + b$ for some $a \geq 0$. If the composition has only one part, then from (2) we may conclude that its generating function is given by

$$\frac{x^b}{(1-x)(1-x^\ell)}.$$

If the number of parts is greater than one, then the composition can be decomposed as $\sigma = \sigma_1 \cdot \sigma' \cdot \sigma_1$, where σ_1 is the first part (which must equal the last) and σ' is a palindromic colored composition (see Figure 1).

If $\sigma_1 = i$ with $i \geq b$, then σ_1 contributes to the generating function the term $w_i x^{2i}$, where

$$w_i = \left\lfloor \frac{i-b+\ell}{\ell} \right\rfloor,$$

while if $i < b$, then it fails to contribute. Note that w_i gives the number of possible colors of the part i and that the generating function of the sequence

$$\{w_i\}_{i\geq 0} = \left\{ \underbrace{0,\ldots,0}_{b}, \underbrace{1,\ldots,1}_{\ell}, \underbrace{2,\ldots,2}_{\ell}, \ldots \right\}$$

is given by

$$W_{\ell a+b}(x) := \sum_{n\geq 0} w_n x^n = \frac{x^b}{(1-x)(1-x^\ell)}. \tag{4}$$

Therefore, we have

$$\begin{aligned}\mathcal{GP}_{\ell a+b}(x) &= 1 + W_{\ell a+b}(x) + w_1 x^2 \mathcal{GP}_{\ell a+b}(x) \\ &\quad + w_2 x^4 \mathcal{GP}_{\ell a+b}(x) + w_3 x^6 \mathcal{GP}_{\ell a+b}(x) + \cdots \\ &= 1 + W_{\ell a+b}(x) + \left(\sum_{n\geq 0} w_{n+1} x^{2n+2} \right) \mathcal{GP}_{\ell a+b}(x)\end{aligned}$$

$$= 1 + W_{\ell a+b}(x) + \left(\frac{x^{2b}}{(1-x^2)(1-x^{2\ell})}\right) \mathcal{GP}_{\ell a+b}(x).$$

Solving this equation yields the desired result.

On the other hand, a palindromic n-color composition σ with exactly $2m$ parts may be written as $\sigma = \sigma'(\sigma')^r$, where σ' has m parts and $(\sigma')^r$ denotes the reverse of σ'. Therefore, by (2), we have

$$\mathcal{GP}_{\ell a+b}(2m, x) = \mathcal{GC}_{\ell a+b}(m, x^2) = \left(\frac{x^{2b}}{(1-x^2)(1-x^{2\ell})}\right)^m.$$

The generating function $\mathcal{GP}_{\ell a+b}(2m-1, x)$ is obtained in a similar manner. □

For example, taking $\ell = b = 1$ in Theorem 1 implies that the generating function for the total number $\mathcal{P}_\nu := \mathcal{P}_{a+1}(\nu)$ of n-color palindromic compositions is

$$\sum_{n\geq 0} \mathcal{P}_n x^n = \frac{1+x+x^3+x^4}{1-3x^2+x^4}$$
$$= 1 + x + 3x^2 + 4x^3 + 9x^4 + 11x^5 + 24x^6 + 29x^7 + 63x^8 + \cdots.$$

Considering the even and odd parts of the prior generating function implies that $\mathcal{P}_{2n} = 3F_{2n}$ for $n \geq 1$ and $\mathcal{P}_{2n+1} = L_{2n+1}$ for $n \geq 0$, where L_m denotes the m-th Lucas number. This corresponds to the result of Shapcott [14] mentioned above.

Taking $\ell = b = 2$ in Theorem 1 gives the generating function for the total number $\mathcal{P}_\nu^{(\text{even})} := \mathcal{P}_{2a+2}(\nu)$ of n-color palindromic compositions of ν with even subscripts:

$$\sum_{n\geq 0} \mathcal{P}_n^{(\text{even})} x^n = \frac{1+x^3+x^5+x^6}{1-x^2-2x^4+x^6}$$
$$= 1 + x^2 + x^3 + 3x^4 + 2x^5$$
$$+ 5x^6 + 4x^7 + 10x^8 + 7x^9 + \cdots.$$

Notice that the sequence $\{\mathcal{P}_{2n+1}^{(\text{even})}\}_{n\geq 1} = \{1, 2, 4, 7, 13, 23, 42, 75, 136, \ldots\}$ coincides with the absolute value of entry A078038 in the OEIS [17]. Our combinatorial interpretation differs however from the one given in [17] for this sequence. Recall that a *rise* within a composition consists of a part followed by a strictly larger part. Let $a(n)$ denote the number of compositions of n, where there is no rise between the $(2i)$-th and $(2i+1)$-st parts for any i (i.e., there is no rise of even index). Then it is seen that $a(n) = \mathcal{P}_{2n+1}^{(\text{even})}$. For example, $a(6) = 23 = \mathcal{P}_{2\cdot 6+1}^{(\text{even})}$, the enumerated compositions being

{1, 1, 1, 1, 1, 1}, {1, 1, 1, 2, 1}, {1, 1, 1, 3}, {1, 2, 1, 1, 1},
{1, 2, 1, 2}, {1, 2, 2, 1}, {1, 3, 1, 1}, {1, 3, 2}, {1, 4, 1}, {1, 5},
{2, 1, 1, 1, 1}, {2, 1, 1, 2}, {2, 2, 1, 1}, {2, 2, 2}, {2, 3, 1}, {2, 4},
{3, 1, 1, 1}, {3, 2, 1}, {3, 3}, {4, 1, 1}, {4, 2}, {5, 1}, {6}.

Note, for instance, that the compositions {1, $\underline{1}$, 2, 2} and {1, 1, 1, $\underline{1}$, 2} of 6 are not allowed because they have rises corresponding to the second and fourth positions, respectively. Also, we have that $a(n)$ enumerates, equivalently, compositions of n with no falls of even index, where a *fall* consists of a part followed by another that is strictly smaller. To realize this, simply interchange the order of the $(2i)$-th and $(2i+1)$-st parts for all i within a composition having no rises of even index.

Analogously, the generating function for the total number $\mathcal{P}_{\nu}^{(\text{odd})} := \mathcal{P}_{2a+1}(\nu)$ of n-color palindromic compositions with odd subscripts is given by

$$\sum_{n\geq 0} \mathcal{P}_n^{(\text{odd})} x^n = \frac{1+x+x^3+x^6}{1-2x^2-x^4+x^6}$$
$$= 1+x+2x^2+3x^3+5x^4+7x^5$$
$$+ 12x^6+16x^7+27x^8+\cdots.$$

In this case, we have that the sequence $\{\mathcal{P}_{2n+1}^{(\text{odd})}\}_{n\geq 0} = \{1, 3, 7, 16, 36, 81, 182, 409, 919, \ldots\}$ coincides with entry A033303 in [17]. Let $e(n)$ be the number of ternary words of length n in which the subwords 00 and 12 are not allowed. The previous sequences coincide, that is, $e(n) = \mathcal{P}_{2n+1}^{(\text{odd})}$. For example, $e(3) = 16 = \mathcal{P}_{2\cdot 3+1}^{(\text{odd})}$, the enumerated ternary words being

{0, 1, 0}, {0, 1, 1}, {0, 2, 0}, {0, 2, 1}, {0, 2, 2},
{1, 0, 1}, {1, 0, 2}, {1, 1, 0}, {1, 1, 1}, {2, 0, 1}, {2, 0, 2},
{2, 1, 0}, {2, 1, 1}, {2, 2, 0}, {2, 2, 1}, {2, 2, 2}.

In the final section of this paper, the aforementioned equalities $a(n) = \mathcal{P}_{2n+1}^{(\text{even})}$ and $e(n) = \mathcal{P}_{2n+1}^{(\text{odd})}$ are shown via direct bijections between the relevant structures.

Let $c_{\ell a+b}(\nu)$ and $d_{\ell a+b}(\nu)$ be the number of n-color restricted palindromic compositions of ν into an even or an odd number of parts, respectively. We have the following generating function formulas for these sequences.

COROLLARY 2

Let $\mathcal{GQ}_{\ell a+b}^{(\text{even})}(x)$ and $\mathcal{GQ}_{\ell a+b}^{(\text{odd})}(x)$ denote the generating functions for the sequences enumerating the n-color restricted palindromic compositions having an even or an odd number of parts, respectively. Then we have

$$\mathcal{GQ}_{\ell a+b}^{(\text{even})}(x) := \sum_{n\geq 1} c_{\ell a+b}(n)x^n = \frac{x^{2b}}{1-x^2-x^{2b}-x^{2\ell}+x^{2\ell+2}},$$
$$\mathcal{GQ}_{\ell a+b}^{(\text{odd})}(x) := \sum_{n\geq 1} d_{\ell a+b}(n)x^n = \frac{x^{b}(1+x)(1+x^{\ell})}{1-x^2-x^{2b}-x^{2\ell}+x^{2\ell+2}}.$$

Moreover,

$$\begin{aligned} c_{\ell a+b}(n) &= c_{\ell a+b}(n-2) + c_{\ell a+b}(n-2b) + c_{\ell a+b}(n-2\ell) \\ &\quad - c_{\ell a+b}(n-(2\ell+2)) \end{aligned}$$

and

$$\begin{aligned} d_{\ell a+b}(n) &= d_{\ell a+b}(n-2) + d_{\ell a+b}(n-2b) + d_{\ell a+b}(n-2\ell) \\ &\quad - d_{\ell a+b}(n-(2\ell+2)), \end{aligned}$$

where $n > \max\{2b, 2\ell+2\}$.

Proof. From Theorem 1, we have

$$\mathcal{GQ}^{(\text{even})}_{\ell a+b}(x) = \sum_{m\geq 1}\left(\frac{x^{2b}}{(1-x^2)(1-x^{2\ell})}\right)^m = \frac{x^{2b}}{1-x^2-x^{2b}-x^{2\ell}+x^{2\ell+2}}.$$

The second case is analogous. □

For example, taking $\ell = b = 1$ in Corollary 2 implies that the generating function for the total number of n-color palindromic compositions with an even number of parts is (see Theorem 4.1 of [11])

$$\begin{aligned} \mathcal{GQ}^{(\text{even})}_{a+1}(x) &= \sum_{n\geq 0} c_{a+1}(n)x^n = \frac{x^2}{1-3x^2+x^4} \\ &= x^2 + 3x^4 + 8x^6 + 21x^8 + 55x^{10} + 144x^{12} + \cdots. \end{aligned}$$

Note that for all $n \geq 1$, $c_{a+1}(2n) = F_{2n}$. Similarly, the generating function for the total number of n-color palindromic compositions with an odd number of parts is (see Theorem 4.1 of [11])

$$\begin{aligned} \mathcal{GQ}^{(\text{odd})}_{a+1}(x) &= \sum_{n\geq 0} d_{a+1}(n)x^n = \frac{x(1+x)^2}{1-3x^2+x^4} \\ &= x + 2x^2 + 4x^3 + 6x^4 + 11x^5 + 16x^6 + 29x^7 + \cdots. \end{aligned}$$

One can show from this that

$$d_{a+1}(n) = \begin{cases} 2F_n, & \text{if } n \text{ is even;} \\ L_n, & \text{if } n \text{ is odd.} \end{cases}$$

From the combinatorial identity (1), we obtain the following result.

Theorem 3. *The sequence* $\mathcal{P}_{\ell a+b}(m, \nu)$, *where* $m, \nu \geq 1$, *is given by*

$$\begin{aligned} &\mathcal{P}_{\ell a+b}(2m-1, 2\nu-1) \\ &= \sum_{j=1}^{\nu}\left\lfloor\frac{2j-1-b+\ell}{\ell}\right\rfloor \sum_{i=0}^{\left\lfloor\frac{\nu-j-b(m-1)}{\ell}\right\rfloor}\binom{m+i-2}{m-2}\binom{\nu-j-\ell i+(m-1)(1-b)-1}{m-2}, \end{aligned}$$

$$\mathcal{P}_{\ell a+b}(2m-1,2\nu) = \sum_{j=1}^{\nu}\left\lfloor \frac{2j-b+\ell}{\ell}\right\rfloor \sum_{i=0}^{\left\lfloor \frac{\nu-j-b(m-1)}{\ell}\right\rfloor} \binom{m+i-2}{m-2}\binom{\nu-j-\ell i+(m-1)(1-b)-1}{m-2},$$

$$\mathcal{P}_{\ell a+b}(2m,2\nu) = \sum_{i=0}^{\left\lfloor \frac{\nu-bm}{\ell}\right\rfloor} \binom{m+i-1}{m-1}\binom{\nu-\ell i+m(1-b)-1}{m-1}.$$

Proof. Let σ be a restricted n-color palindromic composition of $2\nu-1$ with $2m-1$ parts. Such a composition may be written as $\sigma=\sigma'\sigma^*(\sigma')^r$, where σ^* is an odd part and σ' is a composition with $m-1$ parts. The part $\sigma^*=2j-1$ has

$$w_{2j-1} = \left\lfloor \frac{2j-1-b+\ell}{\ell}\right\rfloor$$

possible colors. Therefore, we have

$$\mathcal{P}_{\ell a+b}(2m-1,2\nu-1) = \sum_{j=1}^{\nu} w_{2j-1}\mathcal{C}_{\ell a+b}(m-1,\nu-j),$$

from which the first identity follows from (1). The proofs of the second and third identities are similar. □

Taking $\ell=b=1$ in Theorem 3, and comparing with prior results, one obtains the following combinatorial identities for the Fibonacci and Lucas numbers for $\nu\geq 1$:

$$2F_{2\nu} = \sum_{m=1}^{\nu}\sum_{j=1}^{\nu}\sum_{i=0}^{\nu-j-m+1} 2j\binom{m+i-2}{m-2}\binom{\nu-i-j-1}{m-2},$$

$$L_{2\nu-1} = \sum_{m=1}^{\nu}\sum_{j=1}^{\nu}\sum_{i=0}^{\nu-j-m+1} (2j-1)\binom{m+i-2}{m-2}\binom{\nu-i-j-1}{m-2},$$

$$F_{2\nu} = \sum_{m=1}^{\nu}\sum_{i=0}^{\nu-m}\binom{m+i-1}{m-1}\binom{\nu-i-1}{m-1},$$

where it is understood that $\binom{r}{-1}=0$ if $r\geq 0$, with $\binom{-1}{-1}=1$.

3. Connections with Riordan arrays

In this section, we introduce the matrix $\mathcal{F}_\ell := \left[\mathcal{P}_{\ell a+1}(2m+1,2n+1)\right]_{n,m\geq 0}$. For example, the first few rows of the array $\mathcal{F}_3$ are

$$\mathcal{F}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 8 & 7 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 12 & 16 & 11 & 5 & 1 & 0 & 0 & 0 & 0 \\ 5 & 18 & 30 & 28 & 16 & 6 & 1 & 0 & 0 & 0 \\ 5 & 27 & 52 & 61 & 45 & 22 & 7 & 1 & 0 & 0 \\ 6 & 36 & 88 & 120 & 110 & 68 & 29 & 8 & 1 & 0 \\ 7 & 48 & 138 & 225 & 241 & 183 & 98 & 37 & 9 & 1 \end{pmatrix}.$$

As will be seen, the family of matrices $\mathcal{F}_\ell$ is an example of what is called a Riordan array. Recall that an infinite lower triangular matrix is referred to as a *Riordan array* [16] if its k-th column has generating function $g(x)\,(f(x))^k$ for $k \geq 0$, where $g(x)$ and $f(x)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. For example, the Pascal matrix is given by the Riordan array

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right).$$

The matrix corresponding to the pair $f(x)$, $g(x)$ is denoted by $(g(x),\ f(x))$. If we multiply $(g(x),\ f(x))$ by a column vector $(c_0, c_1, \dots)^T$ whose generating function is $h(x)$, the resulting column vector has generating function $g(x)h(f(x))$. This property is known as the fundamental theorem of Riordan arrays. Recall that the product of two Riordan arrays $(g(x),\ f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x),\ f(x)) * (h(x), l(x)) = (g(x)h\left(f(x)\right), l\left(f(x)\right))$$

and that the set of all Riordan arrays forms a group under this operation [16].

As a consequence of Theorem 1 above, we obtain the following formula for F_ℓ.

Theorem 4. *The matrix $\mathcal{F}_\ell$ is a Riordan array given by*

$$\mathcal{F}_\ell = \left(g_\ell(x),\ \frac{x}{(1-x)(1-x^\ell)}\right),$$

where

$$g_\ell(x) := \begin{cases} \dfrac{1}{(1-x)(1-x^{\ell/2})}, & \text{if } \ell \text{ is even;} \\ \dfrac{1+x^{(\ell+1)/2}}{(1-x)(1-x^\ell)}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Proof. If ℓ is even, then the (n, m)-th entry of the Riordan array is given by

$$[x^n]\frac{1}{(1-x)(1-x^{\ell/2})}\left(\frac{x}{(1-x)(1-x^\ell)}\right)^m$$
$$= [x^n]\frac{x^m}{(1-x)^{m+1}(1-x^{\ell/2})(1-x^\ell)^m}.$$

On the other hand, by Theorem 1, we have

$$\mathcal{GP}_{\ell a+b}(2m-1,x) := \sum_{\nu\geq 1} \mathcal{P}_{\ell a+b}(2m-1,\nu)x^{\nu} = \frac{x^b}{(1-x)(1-x^{\ell})}\left(\frac{x^{2b}}{(1-x^2)(1-x^{2\ell})}\right)^{m-1}.$$

Therefore,

$$\frac{1}{2x^{1/2}}\left(\mathcal{GP}_{\ell a+1}(2m+1,x^{1/2}) - \mathcal{GP}_{\ell a+1}(2m+1,-x^{1/2})\right) = \frac{x^m}{(1-x)^{m+1}(1-x^{\ell/2})(1-x^{\ell})^m}.$$

By comparing coefficients, we conclude

$$[x^n]\frac{1}{(1-x)(1-x^{\ell/2})}\left(\frac{x}{(1-x)(1-x^{\ell})}\right)^m = \mathcal{P}_{\ell a+1}(2m+1,2n+1).$$

The proof in the odd case is analogous. □

Let $G_{\ell} = \left[\mathcal{P}_{\ell a+1}(2m+1,2n+2)\right]_{n,m\geq 0}$. Then proceeding as before, but considering the even instead of the odd part of the generating function $\mathcal{GP}_{\ell a+b}(2m+1,x)$, yields the following result.

Theorem 5. *The matrix G_{ℓ} is a Riordan array given by*

$$G_{\ell} = \left(h_{\ell}(x), \frac{x}{(1-x)(1-x^{\ell})}\right),$$

where

$$h_{\ell}(x) := \begin{cases} \dfrac{1}{(1-x)(1-x^{\ell/2})}, & \text{if } \ell \text{ is even;} \\ \dfrac{1+x^{(\ell-1)/2}}{(1-x)(1-x^{\ell})}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Remark. Note that Riordan array formulas comparable to those in Theorems 4 and 5 cannot be given if $b>1$, for in this case, functions f and g simultaneously satisfying $f'(0)\neq 0$ and $g(0)\neq 0$ cannot be found. If the number of parts is to be even, then there are no palindromic compositions of odd weight, whereas the matrix $\left[\mathcal{P}_{\ell a+1}(2m,2n)\right]_{n,m\geq 0}$ is seen to correspond to a Riordan array with $g(x)=1$ and $f(x)=\frac{x}{(1-x)(1-x^{\ell})}$.

4. Combinatorial proofs

In this section, we provide bijective proofs of the previously noted equalities $a(m) = \mathcal{P}_{2m+1}^{(\text{even})}$ and $e(m) = \mathcal{P}_{2m+1}^{(\text{odd})}$ as well as of the recurrences for the sequences $c_{\ell a+b}(m)$ and $d_{\ell a+b}(m)$.

Combinatorial proof of the equality $a(m) = \mathcal{P}_{2m+1}^{(\text{even})}$. Given $m \geq 1$, let $\mathcal{A}_m$ denote the set of compositions of m in which there are no rises of even index, and let $\mathcal{C}_m$ be the set of palindromic n-color compositions of $2m+1$ in which each part has even subscript. Suppose $\pi \in \mathcal{A}_m$ contains either $2j-1$ or $2j$ parts for some $j \geq 1$. Then we may represent π as $\pi = c_1c_2\cdots c_{2j}$, where c_{2j} is taken to be zero if π contains $2j-1$ parts. Since there are no rises of even index in π, we have $c_{2i} \geq c_{2i+1}$ for $1 \leq i \leq j-1$.

Given π, we form $\lambda = \lambda_\pi \in \mathcal{C}_m$ as follows. First, let λ have (necessarily odd) central part $2(c_1 + c_{2j}) + 1$, with subscript $2c_1$. Let the i-th part of λ away from the central part, either to the left or to the right, be given by $c_{2i} + c_{2i+1}$, with subscript $2c_{2i+1}$. Note that this assignment of subscripts is permitted since π has no rises of even index. Let $f(\pi) = \lambda_\pi$. For example, if $m = 14$ and $\pi = (3, 2, 2, 6, 1) \in \mathcal{A}_{14}$, then the central part of λ_π is 7_6. The first part away from the central is given by 4_4, whereas the second is 7_2. This results in $\lambda_\pi = (7_2, 4_4, 7_6, 4_4, 7_2) \in \mathcal{C}_{14}$. One may verify that the mapping f is reversible and hence a bijection between $\mathcal{A}_m$ and $\mathcal{C}_m$. □

Proof of $e(m) = \mathcal{P}_{2m+1}^{(\text{odd})}$. Given $m \geq 1$, let $\mathcal{E}_m$ denote the set of ternary words of length m in which there are no occurrences of 00 or 12, and let $\mathcal{F}_m$ be the set of palindromic n-color compositions of $2m+1$ wherein each part has odd subscript. We define f between $\mathcal{E}_m$ and $\mathcal{F}_m$ as follows. Let $\sigma \in \mathcal{E}_m$. If $\sigma = 2^a1^b$ where $0 \leq b \leq m$, then let $f(\sigma) = (2m+1)_{2b+1} \in \mathcal{F}_m$, so assume σ contains at least one zero. Then we have

$$\sigma = 2^a1^b0\sigma_10\sigma_2\cdots0\sigma_k,$$

where $k \geq 1$, the σ_i consist of 1's and 2's and $a, b \geq 0$. Note that $\sigma \in \mathcal{E}_m$ implies σ_i is non-empty for $1 \leq i \leq k-1$, with σ_k possibly empty. Furthermore, we must have $\sigma_i = 2^{a_i}1^{b_i}$, where $a_i, b_i \geq 0$ are not both zero if $1 \leq i \leq k-1$, with σ_k of the same form if non-empty. We create $f(\sigma)$ in this case using σ of the stated form above as follows. To do so, first express $f(\sigma)$ as $\lambda s \lambda^r$, where $s \geq 1$ and λ is a non-empty sequence of parts. Put $s = (2(a+b)+1)_{2b+1}$.

To obtain λ, first assume σ_k is non-empty in the decomposition of σ above and write $\lambda = \lambda_1\lambda_2\cdots\lambda_k$, where λ_i denotes either a single part or a pair of consecutive parts which we form using the segment $0\sigma_i$ for $1 \leq i \leq k$ as follows. If a_i is odd, then let $\lambda_i = (a_i+b_i+1)_x$, where $x = 2\lceil b_i/2\rceil+1$. If a_i is even and positive, let $\lambda_i = (a_i+b_i)_x1_1$. If $a_i = 0$ and $b_i > 0$ is even, then let $\lambda_i = (b_i+1)_1$. Finally, if $a_i = 0$ and b_i is odd, then let $\lambda_i = (b_i)_11_1$. Concatenating the various λ_i yields λ, and thus $f(\sigma) = \lambda s \lambda^r$, as desired. Finally, if σ_k is empty, apply the transformations described above to σ_i for $1 \leq i \leq k-1$ to obtain $\lambda' = \lambda_1\cdots\lambda_{k-1}$. Then define $f(\sigma)$ using $\lambda = 1_1\lambda'$ and $s \geq 1$, where s is determined as before. For example, if $m = 37$ and $\sigma = 21^302^202^51^202^51^601^301^40 \in \mathcal{E}_{37}$, then $f(\sigma) = \lambda 9_7 \lambda^r \in \mathcal{F}_{37}$, where $\lambda = (1_1, 2_1, 1_1, 8_3, 12_7, 3_1, 1_1, 5_1)$.

The mapping f is reversed by first considering whether or not $\rho \in \mathcal{F}_m$ contains a single part. If it contains more than one part, then write $\rho = \lambda s \lambda^r$, where λ and s are as above, and consider cases based on whether or not λ starts with a run of the part 1_1 of odd length. For example, if $m = 40$ and $\rho = \lambda s \lambda^r \in \mathcal{F}_{40}$, where $\lambda = (1_1, 1_1, 1_1, 3_3, 1_1, 1_1, 1_1, 9_5, 8_7, 4_1, 1_1, 1_1)$ and $s = 17_7$, then

$$f^{-1}(\rho) = 2^5 1^3 0102^2 10102^5 1^3 021^6 02^3 010 \in \mathcal{E}_{40}.$$

Note that λ starting with an odd number (here, three) of 1_1 parts implies $f^{-1}(\rho)$ ends in 0. Furthermore, after every part a_b in λ that is not 1_1, one needs to consider the length of the run (possibly empty) of 1_1 parts that directly follows a_b in reconstructing the corresponding section of the inverse image. Thus, for example, the $02^2 1$ section of $f^{-1}(\rho)$ corresponds to $(3_3, 1_1)$ in λ, the 01 that follows to $(1_1, 1_1)$ directly prior to 9_5 in λ, and the $02^5 1^3$ section of $f^{-1}(\rho)$ to the part 9_5 itself. □

We now provide a combinatorial explanation for the recurrences given above for the sequences $c_{\ell a+b}(m)$ and $d_{\ell a+b}(m)$.

Combinatorial proof of recurrences in Corollary 2. Let $\mathcal{V}_m$ and $\mathcal{O}_m$ denote the class of n-color compositions enumerated by $c_{\ell a+b}(m)$ and $d_{\ell a+b}(m)$, respectively. First suppose $\sigma \in \mathcal{V}_m$, where $m > \max\{2b, 2\ell + 2\}$, and write $\sigma = \lambda\lambda^r$ for some composition λ. Note that there are $c_{\ell a+b}(m-2)$ possibilities for σ in which the first part of λ is of the form x_y, where $x > y$. This can be realized by adding 1 to the first (and last) part of a composition enumerated by $c_{\ell a+b}(m-2)$, keeping the subscript the same. Next, there are clearly $c_{\ell a+b}(m-2b)$ possibilities for σ in which the first part of λ is b_b. Finally, by subtraction, there are $c_{\ell a+b}(m-2\ell) - c_{\ell a+b}(m-2\ell-2)$ compositions $\sigma' = \lambda'(\lambda')^r$ in $\mathcal{V}_{m-2\ell}$ in which the first part of λ' is of the form x_x for some x. Upon replacing this part with $(x+\ell)_{x+\ell}$, it is seen that there are the same number of $\sigma = \lambda\lambda^r$ in $\mathcal{V}_m$ in which the first part of λ is of the form y_y, where $y > b$. Combining the previous cases then yields all members of $\mathcal{V}_m$ and implies the recurrence for $c_{\ell a+b}(m)$.

A similar argument applies to the recurrence for $d_{\ell a+b}(m)$. Note however that it is possible for a member of $\mathcal{O}_m$ to consist of a single part. Thus, there are $d_{\ell a+b}(m-2)$ possibilities in which the first part of σ is of the form x_y, where $x > y$ with σ containing three or more parts altogether or in which $\sigma = m_y$, where $m > y+1$. So, by subtraction, there are $d_{\ell a+b}(m-2\ell) - d_{\ell a+b}(m-2\ell-2)$ members $\sigma' \in \mathcal{O}_{m-2\ell}$ in which the first (and last) part is of the form x_x with σ' containing at least three parts altogether or in which $\sigma' = (m-2\ell)_{m-2\ell}$ or $(m-2\ell)_{m-2\ell-1}$. In the first case, increase both the base and the subscript of the first (and last) part of σ' by ℓ, while in the second, increase the base and the subscript of the sole part by 2ℓ. Note that the second case does not apply if $m \not\equiv b, b+1 \pmod{\ell}$, with only one of the options applying for a given m. Combining all of the previous cases then yields the recurrence of $d_{\ell a+b}(m)$. □

Remark. Note that a comparable combinatorial proof may be given for the recurrence satisfied by the sequence $\mathcal{C}_{\ell a+b}(m)$ stated above.

We now state explicitly the initial values where $1 \le i \le \max\{2b, 2\ell+2\}$ for the recurrences above satisfied by $c_{\ell a+b}(i)$ and $d_{\ell a+b}(i)$. We find these values directly, considering cases on b and ℓ. First, assume $b > \ell \ge 1$. Then we have $c_{\ell a+b}(i) = 0$ for $1 \le i \le 2b-1$, with $c_{\ell a+b}(2b) = 1$. If $1 \le i \le 2b$, then $d_{\ell a+b}(i) = [\frac{i-b}{\ell}$ is a non-negative integer], where $[P]$ equals 1 or 0 depending on the truth or falsity of the statement P.

Next, suppose $\ell \ge b \ge 2$. Then

$$c_{\ell a+b}(i) = \begin{cases} 0, & \text{if } i \neq 2rb \text{ for some } r \ge 1; \\ 1, & \text{otherwise,} \end{cases}$$

for $1 \le i \le 2\ell+2$, and $d_{\ell a+b}(i) = f_b(i)$ for $1 \le i \le 2\ell+1$, where $f_b(i) := [\frac{i}{b}$ is odd$] + [\frac{i-\ell}{b}$ is odd and positive], with $d_{\ell a+b}(2\ell+2) = f_b(2\ell+2) + \delta_{b,2}$. Finally, if $b = 1$, then we have

$$c_{\ell a+1}(i) = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ 1, & \text{if } i \text{ is even,} \end{cases}$$

for $1 \leq i \leq 2\ell + 1$, with $c_{\ell a+1}(2\ell + 2) = 2$. If $1 \leq i \leq 2\ell$, then $d_{\ell a+1}(i) = f_1(i)$, with $d_{\ell a+1}(2\ell + 1) = 2 + [\ell \text{ is even}]$ and $d_{\ell a+1}(2\ell + 2) = [\ell \text{ is odd}]$.

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