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Automorphisms and the fundamental operators associated with the symmetrized tridisc

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Abstract. The automorphisms of the symmetrized polydisc $\mathbb{G}_n$ are well-known and are given in the coordinates of the polydisc in Edigarian and Zwonek (*Arch. Math.* **84** (2005) 364–374). We find an explicit formula for the automorphisms of $\mathbb{G}_n$ in its own coordinates. If τ is an automorphism of $\mathbb{G}_n$, then $\tau(S_1, \dots, S_{n-1}, P)$ is a Γ_n-contraction, where a Γ_n-contraction is a commuting n-tuple of Hilbert space operators for which the closed symmetrized polydisc Γ_n is a spectral set. Corresponding to every Γ_n-contraction $(S_1, \dots, S_{n-1}, P)$, there exist $n-1$ unique operators $A_1, \dots, A_{n-1}$ such that

$$S_i - S_{n-i}^* P = D_P A_i D_P, \quad D_P = (I - P^* P)^{1/2},$$

for $i = 1, \dots, n-1$. This unique $(n-1)$-tuple $(A_1, \dots, A_{n-1})$, which is called the fundamental operator tuple or $\mathcal{F}_O$-tuple of $(S_1, \dots, S_{n-1}, P)$ in the literature, plays central role in every section of operator theory on Γ_n. We find an explicit form of the $\mathcal{F}_O$-tuple of $\tau(S_1, \dots, S_{n-1}, P)$ when $n = 3$. We show by an example that a Γ_n-contraction may not have commuting $\mathcal{F}_O$-tuple. Also, we obtain a necessary and sufficient condition under which two Γ_n-contractions are unitarily equivalent.

Keywords. Symmetrized polydisc; automorphisms; Γ_n-contraction; fundamental operator tuple.

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1. Introduction

The open and closed *symmetrized polydisc* or, *symmetrized n-disc* for $n \geq 2$, are the following subsets of $\mathbb{C}^n$:

$$\mathbb{G}_n = \left\{ \left(\sum_{1 \leq i \leq n} z_i, \sum_{1 \leq i < j \leq n} z_i z_j, \dots, \prod_{i=1}^n z_i \right) : |z_i| < 1, i = 1, \dots, n \right\},$$

$$\Gamma_n = \left\{ \left(\sum_{1 \leq i \leq n} z_i, \sum_{1 \leq i < j \leq n} z_i z_j, \dots, \prod_{i=1}^n z_i \right) : |z_i| \leq 1, i = 1, \dots, n \right\}.$$

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The symmetrized polydisc has attracted considerable interests in recent decades because of its complex geometry [8,10], rich function theory [1,2,9,11,12] and associated beautiful operator theory [3–7,13–15,17]. Numerous interesting articles have been written on the symmetrized polydisc and only a few among them are mentioned here. More could be found in the reference lists of the articles cited here. Studying operators associated with a domain is always of interest in its own right. In this article, we describe an interrelation between automorphisms of the symmetrized polydisc and the tuples of commuting operators that have the closed symmetrized polydisc as a spectral set.

DEFINITION 1.1

A tuple of n commuting operators $(S_1, \ldots, S_{n-1}, P)$, defined on a Hilbert space $\mathcal{H}$, for which Γ_n is a spectral set is called a Γ_n-contraction.

An automorphism of a domain $\Omega \subseteq \mathbb{C}^n$ is a bijective and bi-holomorphic self-map of Ω. It is well-known that any automorphism τ of $\mathbb{G}_n$ (see [10]) is of the following form:

$$\tau\left(\sum_{i=1}^{n} z_i, \sum_{1\le i<j\le n} z_i z_j, \ldots, \prod_{i=1}^{n} z_i\right) = \left(\sum_{i=1}^{n} m(z_i), \sum_{1\le i<j\le n} m(z_i)m(z_j), \ldots, \prod_{i=1}^{n} m(z_i)\right), \tag{1.1}$$

where $z_1, \ldots, z_n$ are in the unit disk $\mathbb{D}$ and m is an automorphism of $\mathbb{D}$. An automorphism m of the unit disc $\mathbb{D}$ is a map

$$m(z) := \beta \frac{z-a}{1-\bar{a}z}, \text{ for some } a \in \mathbb{D} \text{ and } \beta \in \mathbb{T}, \tag{1.2}$$

where $\mathbb{T}$ is the unit circle in the complex plane. For obvious reason, an automorphism of $\mathbb{G}_n$ is often denoted by τ_m. In Lemma 2.1, we find an explicit form of an automorphism of $\mathbb{G}_n$ in its own coordinates. As a consequence of this result, we obtain a necessary and sufficient condition under which two Γ_n-contractions become unitarily equivalent. This is given in Theorem 3.3.

Also, if τ is a $\mathbb{C}^n$-valued holomorphic map in a neighbourhood $N(\Gamma_n)$ of Γ_n that maps Γ_n into itself, then by functional calculus $\tau(S_1, \ldots, S_{n-1}, P)$ makes sense for any Γ_n-contraction $(S_1, \ldots, S_{n-1}, P)$. Indeed, if we denote

$$(S_{1\tau}, \ldots, S_{(n-1)\tau}, P_\tau) = \tau(S_1, \ldots, S_{n-1}, P),$$

then $(S_{1\tau}, \ldots, S_{(n-1)\tau}, P_\tau)$ is also a Γ_n-contraction. It was shown in [17] that corresponding to every Γ_n-contraction $(S_1, \ldots, S_{n-1}, P)$ there exists a unique $(n-1)$-tuple $(A_1, \ldots, A_{n-1})$ such that

$$S_i - S_{n-i}^* P = D_P A_i D_P,$$

where $D_P = (I - P^*P)^{1/2}$. The $(n-1)$-tuple $(A_1, \ldots, A_{n-1})$ is called the *fundamental operator tuple* or in short form the $\mathcal{F}_O$-tuple of the Γ_n-contraction $(S_1, \ldots, S_{n-1}, P)$. The reason behind carrying such a name is that it plays a pivotal role in all aspects of operator theory on Γ_n, e.g., [3,4,13,15–18]. The main aim of this paper is to find an explicit form of the $\mathcal{F}_O$-tuple of $(S_{1\tau}, \ldots, S_{(n-1)\tau}, P_\tau)$ in terms of the $\mathcal{F}_O$-tuple of $(S_1, \ldots, S_{n-1}, P)$. To avoid excessive complexities in the expressions, we restrict our attention to $n = 3$ here. We believe that analogous expressions could be obtained for an arbitrary n in a similar fashion, only one needs to deal with a bit more complicated calculations. The same program for $n = 2$ was carried out as part of the paper [4]. Since substantial differences have been witnessed between the operator theory on Γ_2 and Γ_3, we find it worth to determine the $\mathcal{F}_O$-pair of the Γ_3-contraction $(S_{1\tau}, S_{2\tau}, P_\tau)$ which is described in Theorem 4.3.

If $(s_1, \ldots, s_{n-1}, p)$ is a scalar Γ_n-contraction, that is, a point in Γ_n, then the $\mathcal{F}_O$-tuple of $(s_1, \ldots, s_{n-1}, p)$ is a point in Γ_{n-1}. This was obtained by Costara in [9] in a different format. So, a natural question arises when we deal with $\mathcal{F}_O$-tuple of a Γ_n-contraction: is the $\mathcal{F}_O$-tuple of a Γ_n-contraction a Γ_{n-1}-contraction? We shall show that the answer is negative. Indeed, in the last section we shall produce a Γ_n-contraction whose $\mathcal{F}_O$-tuple is not even commutative.

2. Automorphisms of $\mathbb{G}_n$ in its own coordinates

The automorphisms of the symmetrized polydisc $\mathbb{G}_n$ were explicitly determined in [10]. Here we determine the (same) formula in the coordinates of $\mathbb{G}_n$.

Lemma 2.1. *Suppose* $(s_1, \ldots, s_{n-1}, s_n) \in \Gamma_n$ *and* τ *is an automorphism of* $\mathbb{G}_n$. *Then* $\tau(s_1, \ldots, s_{n-1}, s_n) = (s_{1\tau}, \ldots, s_{(n-1)\tau}, s_{n\tau})$, *where*

$$s_{i\tau}=\beta^i \frac{\sum_{j=1}^{i-1}(-a)^{i-j}\left[\sum_{k=0}^{j}\left\{\binom{n-j}{i-j+k}\binom{j}{k}|a|^{2k}\right\}s_j\right]+\sum_{j=i}^{n}(-\bar{a})^{j-i}\left[\sum_{k=0}^{i}\binom{j}{i-k}\binom{n-j}{k}|a|^{2k}\right]s_j}{1+\sum_{i=1}^{n}(-1)^i(\bar{a})^i s_i}$$

$$+ \beta^i \frac{(-1)^i\binom{n}{i}a^i}{1+\sum_{i=1}^{n}(-1)^i(\bar{a})^i s_i}, \quad \textit{for any } a \in \mathbb{D} \textit{ and } \beta \in \mathbb{T}.$$

Proof. We prove by induction on n. Clearly, the lemma is true for $n = 2$. Suppose the lemma is true for n. Consider $(s_1, \ldots, s_{n+1}) \in \Gamma_{n+1}$. Then there exists $(z_1, \ldots, z_{n+1}) \in \bar{\mathbb{D}}^{n+1}$ such that $\pi_{n+1}(z_1, \ldots, z_{n+1}) = (s_1, \ldots, s_{n+1})$. Let $\pi_n(z_1, \ldots, z_n) = (s'_1, \ldots, s'_n)$. Then clearly,

$$\begin{aligned} s_1 &= s'_1 + z_{n+1} \\ s_i &= s'_i + s'_{i-1}z_{n+1}, \text{ for } 1 < i < n+1 \\ s_{n+1} &= s'_n z_{n+1}. \end{aligned}$$

Suppose $\tau_{n+1} \in \mathrm{Aut}(\mathbb{G}_{n+1})$ and

$$\begin{aligned} \tau_{n+1}(s_1, \ldots, s_{n+1}) &= (s_{1\tau}, \ldots, s_{(n+1)\tau}) \\ &= \pi_{n+1}\left(\beta\frac{z_1 - a}{1 - \bar{a}z_1}, \ldots, \beta\frac{z_{n+1} - a}{1 - \bar{a}z_{n+1}}\right), \end{aligned}$$

for some $a \in \mathbb{D}$ and $\beta \in \mathbb{T}$. Let $\tau_n \in \mathrm{Aut}(\mathbb{G}_n)$ and

$$\tau_n(s'_1, \ldots, s'_n) = (s'_{1\tau}, \ldots, s'_{n\tau}) = \pi_n\left(\beta\frac{z_1 - a}{1 - \bar{a}z_1}, \ldots, \beta\frac{z_n - a}{1 - \bar{a}z_n}\right).$$

Clearly, for any $n \in \mathbb{N}$,

$$(1-\bar{a}z_1)\dots(1-\bar{a}z_n) = 1 + \sum_{i=1}^{n}(-\bar{a})^i s_i$$

and

$$(z_1 - a)\dots(z_n - a) = \sum_{i=0}^{n}(-a)^i s_{n-i} \ ,$$

where $s_0 = 1$. Suppose the assertion is true for n. To complete the proof, we need to show that

$$\begin{aligned} s_{i\tau} = \frac{\beta^i}{(1-\bar{a}z_1)\dots(1-\bar{a}z_{n+1})}\Bigg[&\sum_{j=1}^{i-1}(-a)^{i-j}\left\{\sum_{k=0}^{j}\binom{n+1-j}{i-j+k}\binom{j}{k}|a|^{2k}\right\}s_j \\ &+ \sum_{j=i}^{n+1}(-\bar{a})^{j-i}\left\{\sum_{k=0}^{i}\binom{j}{i-k}\binom{n+1-j}{k}|a|^{2k}\right\}s_j + (-a)^i\binom{n+1}{j}\Bigg]. \end{aligned} \tag{2.1}$$

This requires merely a few steps of routine calculations and we skip it. □

The next result follows naturally and by Lemma 2.1, it provides transformation of a Γ_n-contraction under automorphisms of the symmetrized polydisc.

Lemma 2.2. *For* $(S_1, \dots, S_{n-1}, P)$ *and* τ *as above,* $(S_{1\tau}, \dots, S_{(n-1)\tau}, P_\tau)$ *is a* Γ_n*-contraction.*

Proof. We show that Γ_n is a spectral set of $(S_{1\tau}, \dots, S_{(n-1)\tau}, P_\tau)$. Let f be a polynomial over $\mathbb{C}$ in n-variables. Then

$$\begin{aligned} \|f(S_{1\tau}, \dots, S_{(n-1)\tau}, P_\tau)\| &= \|f \circ \tau(S_1, \dots, S_{n-1}, P)\| \\ &\leq \|f \circ \tau\|_{\infty, \Gamma_n} = \sup_{z \in \Gamma_n} |f(\tau(z))| \leq \|f\|_{\infty, \Gamma_n}, \end{aligned}$$

since $\tau(z) \in \Gamma_n$ for all $z \in \Gamma_n$ and hence $(S_{1\tau}, \dots, S_{(n-1)\tau}, P_\tau)$ is a Γ_n-contraction. □

3. Unitary equivalence of two Γ_n-contractions

In this section, we find a necessary and sufficient condition under which two Γ_n-contractions become unitarily equivalent. We begin with a result from the literature.

Lemma 3.1 ([7], *Lemma* 2.17). *For any commuting tuple* $(S_1, \dots, S_{n-1}, P)$ *of operators,* $\sigma(S_1, \dots, S_{n-1}, P) \subseteq \Gamma_n$ *if and only if*

$$I - \bar{a}S_1 + \bar{a}^2 S_2 + \dots + (-\bar{a})^{n-1}S_{n-1} + (-\bar{a})^n P$$

is invertible for all $a \in \mathbb{D}$.

Suppose $(S_1, \dots, S_{n-1}, P)$ is a Γ_n-contraction on a Hilbert space $\mathcal{H}$. For $a \in \mathbb{C}$, define $T_a = I + \sum_{i=1}^{n-1}(-\bar{a})^i S_i + (-\bar{a})^n P$. Consider the set $\Lambda_\Sigma = \{a \in \mathbb{C} : T_a \text{ is invertible}\}$. By the above lemma, it is clear that $\Lambda_\Sigma \supseteq \mathbb{D}$. Let us define

$$\Theta_\Sigma(a) = \left(S_{1\tau a}, \dots, S_{(n-1)\tau a}, S_{n\tau a}\right), \qquad \text{for all } a \in \Lambda_\Sigma,$$

where

$$\begin{aligned} S_{i\tau a} = {} & \left(\sum_{j=1}^{i-1}(-a)^{i-j}\left[\sum_{k=0}^{j}\left\{\binom{n-j}{i-j+k}\binom{j}{k}|a|^{2k}\right\} S_j\right]\right) T_a^{-1} \\ & + \left(\sum_{j=i}^{n}(-\bar{a})^{j-i}\left[\sum_{k=0}^{i}\binom{j}{i-k}\binom{n-j}{k}|a|^{2k}\right] S_j\right) T_a^{-1} + (-a)^i\binom{n}{i} T_a^{-1}. \end{aligned}$$

DEFINITION 3.2

Let $\Sigma = (S_1, \dots, S_{n-1}, P)$ and $\Sigma' = (S'_1, \dots, S'_{n-1}, P')$ be two Γ_n-contractions on $\mathcal{H}$ and $\mathcal{H}'$ respectively. Then we say that Θ_Σ and $\Theta_{\Sigma'}$ coincide if for each $i = 1, \dots, n$ and $a \in \Lambda_\Sigma \cap \Lambda_{\Sigma'}$, $S_{i\tau a}$ is unitarily equivalent to $S'_{i\tau a}$ by the same unitary.

Theorem 3.3. *Let $\Sigma = (S_1, \dots, S_{n-1}, P)$ and $\Sigma' = (S'_1, \dots, S'_{n-1}, P')$ be two Γ_n-contractions on $\mathcal{H}$ and $\mathcal{H}'$ respectively. Then $(S_1, \dots, S_{n-1}, P)$ is unitarily equivalent to $(S'_1, \dots, S'_{n-1}, P')$ if and only if Θ_Σ and $\Theta_{\Sigma'}$ coincide.*

Proof. Suppose Θ_Σ and $\Theta_{\Sigma'}$ coincide. Then by definition there exists a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $US_{i\tau a} = S'_{i\tau a}U$ for all $a \in \Lambda_\Sigma \cap \Lambda_{\Sigma'}$ and $i = 1, \dots, n$. In particular, if $a = 0$, then $\Theta_\Sigma(0) = \Sigma$ and $\Theta_{\Sigma'}(0) = \Sigma'$. Therefore, $US_i = S'_iU$ for all $i = 1, \dots, n-1$ and $UP = P'U$. Hence Σ is unitarily equivalent to Σ'.

Conversely, suppose Σ is unitarily equivalent to Σ'. Then there exists a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $US_i = S'_iU$ and $UP = P'U$. To prove $US_{i\tau a} = S'_{i\tau a}U$ for all $a \in \Lambda_\Sigma \cap \Lambda_{\Sigma'}$ it suffices to prove $UT_a^{-1} = T_a'^{-1}U$ for all $a \in \Lambda_\Sigma \cap \Lambda_{\Sigma'}$. Now it is clear that $UT_a = T'_aU$ for all $a \in \Lambda_\Sigma \cap \Lambda_{\Sigma'}$ and hence $UT_a^{-1} = T_a'^{-1}U$. □

4. The automorphisms and the fundamental operator pair

Before presenting the main result of this paper, we recall few results from the literature which we shall use in the proof the main theorem.

Theorem 4.1 ([17], Theorem 6.4). *If (S_1, S_2, P) is a Γ_3-contraction, then $\left(\frac{1}{3}S_1 + \frac{\beta}{3}S_2, \beta P\right)$ is a Γ-contraction, for all $\beta \in \mathbb{T}$.*

Theorem 4.2 ([6], Theorem 2.2). *Suppose $(S_1, \dots, S_{n-1}, P)$ is a Γ_n-contraction with commuting fundamental operator tuple $(A_1, \dots, A_{n-1})$. Then for each $i, j = 1, \dots, n-1$, we have*

$$S_i^* S_j - S_{n-j}^* S_{n-i} = D_P(A_i^* A_j - A_{n-j}^* A_{n-i})D_P.$$

Now we state and prove the main result of this paper.

Theorem 4.3. *Let (S_1, S_2, P) be a Γ_3-contraction with commuting fundamental operator pair (A_1, A_2). Then*

(i) $(\bar{a}I - aA_2^*)\,(aI - \bar{a}A_2) \le ((1+|a|^2)I - aA_1^*)((1+|a|^2)I - \bar{a}A_1)$, *for all* $a \in \bar{\mathbb{D}}$;
(ii) *if* $r(A_1) \le 2$, *then* $\|(aI - \bar{a}A_2)\big((1+|a|^2)I - \bar{a}A_1\big)^{-1}\| \le 1$, *for all* $a \in \mathbb{D}$.

Moreover, if the inequality above is strict then for any $\tau = \tau_m \in \mathrm{Aut}(\mathbb{G}_3)$, *there exists a unitary* $\mathcal{U} : \mathcal{D}_{P_\tau} \to \mathcal{D}_P$ *such that the fundamental operator pair* $(A_{1\tau}, A_{2\tau})$ *of* $(S_{1\tau}, S_{2\tau}, P_\tau)$ *is given by*

$$\begin{aligned} A_{1\tau} =& \mathcal{U}^* T^{-1/2}\beta((1+3|a|^2)A_1 + a^2(3+|a|^2)A_1^* - 2\bar{a}A_2 \\ &- 2a^3A_2^* - a(1+|a|^2)(A_1^*A_1 - A_2^*A_2) - 3a(1+|a|^2)I)T^{-1/2}\mathcal{U}, \\ A_{2\tau} =& \mathcal{U}^* T^{-1/2}\beta^2(-2(aA_1 + a^3A_1^*) + A_2 + a^4A_2^* \\ &+ a^2(A_1^*A_1 - A_2^*A_2) + 3a^2I)T^{-1/2}\mathcal{U}, \end{aligned}$$

where

$$\begin{aligned} T =& (1+|a|^2+|a|^4)I - (1+|a|^2)(aA_1^* + \bar{a}A_1) + (a^2A_2^* + \bar{a}^2A_2) \\ &+ |a|^2(A_1^*A_1 - A_2^*A_2) \end{aligned}$$

and $\tau = \tau_m$ *as in* (1.1) *and* (1.2).

Proof. We apply Lemma (2.1) for $n = 3$ and get

$$\begin{aligned} \tau(s_1, s_2, p) &= \tau(z_1 + z_2 + z_3, z_1z_2 + z_1z_3 + z_2z_3, z_1z_2z_3) \\ &= \Bigg(\beta\sum_{i=1}^{3}\frac{z_i - a}{1 - \bar{a}z_i}, \beta^2\sum_{1\le i<j\le 3}\frac{(z_i - a)(z_j - a)}{(1-\bar{a}z_i)(1-\bar{a}z_j)}, \\ &\qquad \beta^3\prod_{i=1}^{3}\frac{(z_i - a)}{(1-\bar{a}z_i)}\Bigg) \\ &= \Bigg(\beta\frac{((1+2|a|^2)s_1 - \bar{a}(2+|a|^2)s_2 + 3\bar{a}^2p - 3a)}{1 - \bar{a}s_1 + \bar{a}^2s_2 - \bar{a}^3p}, \\ &\qquad \beta^2\frac{(-a(2+|a|^2)s_1 + (1+2|a|^2)s_2 - 3\bar{a}p + 3a^2)}{1 - \bar{a}s_1 + \bar{a}^2s_2 - \bar{a}^3p}, \\ &\qquad \beta^3\frac{(p - as_2 + a^2s_1 - a^3)}{1 - \bar{a}s_1 + \bar{a}^2s_2 - \bar{a}^3p}\Bigg). \end{aligned}$$

It is obvious that τ can be defined on the open set

$$\Gamma_{3a} = \left\{\left(\sum_{i=1}^{3}z_i, \sum_{1\le i<j\le 3}z_iz_j, \prod_{i=1}^{3}z_i\right) : |z_i| < 1/|a|, i = 1, 2, 3\right\},$$

which contains Γ_3. Clearly

$$(S_{1\tau}, S_{2\tau}, P_\tau) = \tau(S_1, S_2, P),$$

where

$$\begin{aligned} S_{1\tau} &= \beta((1+2|a|^2)S_1 - \bar{a}(2+|a|^2)S_2 \\ &\quad + 3\bar{a}^2 P - 3aI)(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1}, \\ S_{2\tau} &= \beta^2(-a(2+|a|^2)S_1 + (1+2|a|^2)S_2 - 3\bar{a}P \\ &\quad + 3a^2 I)(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1}, \end{aligned}$$

and

$$P_\tau = \beta^3\big(P - aS_2 + a^2 S_1 - a^3 I\big)\big(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P\big)^{-1}.$$

Here

$$\begin{aligned} D_{P_\tau}^2 &= (I - P_\tau^* P_\tau) \\ &= I - (I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1}(P^* - \bar{a}S_2^* + \bar{a}^2 S_1^* - \bar{a}^3 I)(P - aS_2 \\ &\quad + a^2 S_1 - a^3 I)(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1} \\ &= (I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1}\{(I - aS_1^* + a^2 S_2^* - a^3 P^*)(I - \bar{a}S_1 + \bar{a}^2 S_2 \\ &\quad - \bar{a}^3 P) - (P^* - \bar{a}S_2^* + \bar{a}^2 S_1^* - \bar{a}^3 I)(P - aS_2 + a^2 S_1 - a^3 I)\}(I - \bar{a}S_1 \\ &\quad + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1} \\ &= (1 - |a|^2)(I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1} D_P\{(1 + |a|^2 + |a|^4)I - (1 + |a|^2) \\ &\quad (aA_1^* + \bar{a}A_1) + (a^2 A_2^* + \bar{a}^2 A_2) + |a|^2(A_1^* A_1 - A_2^* A_2)\}D_P(I - \bar{a}S_1 + \bar{a}^2 S_2 \\ &\quad - \bar{a}^3 P)^{-1}. \end{aligned}$$

Now we show that the operator

$$\begin{aligned} &(1 + |a|^2 + |a|^4)I - (1 + |a|^2)(aA_1^* + \bar{a}A_1) \\ &\quad + (a^2 A_2^* + \bar{a}^2 A_2) + |a|^2(A_1^* A_1 - A_2^* A_2) \end{aligned}$$

defined on $\mathcal{D}_p$ is positive. Suppose $T = (1 + |a|^2 + |a|^4)I - (1 + |a|^2)(aA_1^* + \bar{a}A_1) + (a^2 A_2^* + \bar{a}^2 A_2) + |a|^2(A_1^* A_1 - A_2^* A_2)$. Now for $h \in \mathcal{H}$ we have

$$\begin{aligned} \langle TD_P h, D_P h\rangle &= \langle D_P T D_P h, h\rangle \\ &= \frac{1}{1 - |a|^2}\langle (I - aS_1^* + a^2 S_2^* - a^3 P^*) D_{P_\tau}^2 (I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)h, h\rangle \\ &= \frac{1}{1 - |a|^2}\langle D_{P_\tau}^2 (I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)h, (I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)h\rangle \\ &= \frac{1}{1 - |a|^2}\langle D_{P_\tau}(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)h, D_{P_\tau}(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)h\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-|a|^2}\|D_{P_\tau}(I-\bar{a}S_1+\bar{a}^2S_2-\bar{a}^3P)h\|^2 \\
&\geq 0.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&(1+|a|^2+|a|^4)I-(1+|a|^2)(aA_1^*+\bar{a}A_1) \\
&\quad +(a^2A_2^*+\bar{a}^2A_2)+|a|^2(A_1^*A_1-A_2^*A_2) \\
&= ((1+|a|^2)I-aA_1^*)((1+|a|^2)I-\bar{a}A_1)-(\bar{a}I-aA_2^*)(aI-\bar{a}A_2) \\
&\geq 0.
\end{aligned}
$$

This implies

$$(\bar{a}I-aA_2^*)(aI-\bar{a}A_2)\leq((1+|a|^2)I-aA_1^*)((1+|a|^2)I-\bar{a}A_1)$$

for all $a\in\mathbb{D}$. By continuity, we have that

$$(\bar{a}I-aA_2^*)(aI-\bar{a}A_2)\leq((1+|a|^2)I-aA_1^*)((1+|a|^2)I-\bar{a}A_1)$$

for all $a\in\bar{\mathbb{D}}$.

Claim. If $r(A_1)\leq 2$, then $\big((1+|a|^2)I-\bar{a}A_1\big)$ is invertible for all $a\in\mathbb{D}$. If $a=0$, then it is obvious. If $0<|a|<1$, then $\dfrac{1}{|a|}+|a|>2$. Now

$$r\left(\frac{\bar{a}A_1}{1+|a|^2}\right)=\frac{|a|}{1+|a|^2}r(A_1)<\frac{1}{2}\,.\,2=1.$$

Therefore, $\big((1+|a|^2)I-\bar{a}A_1\big)$ is invertible. This completes the proof of the claim.
So we have

$$((1+|a|^2)I-aA_1^*)^{-1}(\bar{a}I-aA_2^*)(aI-\bar{a}A_2)((1+|a|^2)I-\bar{a}A_1)^{-1}\leq I$$

and consequently

$$\|(aI-\bar{a}A_2)((1+|a|^2)I-\bar{a}A_1)^{-1}\|\leq 1,\ \text{for all } a\in\mathbb{D}\,.$$

Suppose $\|(aI-\bar{a}A_2)((1+|a|^2)I-\bar{a}A_1)^{-1}\|<1$. Then T is invertible. Let $X=(1-|a|^2)^{1/2}T^{1/2}D_P(I-\bar{a}S_1+\bar{a}^2S_2-\bar{a}^3P)^{-1}$. Then X is an operator from $\mathcal{H}$ to $\mathcal{D}_P$. Also $D_{P_\tau}^2=X^*X$ and $\overline{\text{Ran}}\,X=\mathcal{D}_P$ as T is invertible. Now define

$$
\begin{aligned}
&\mathcal{U}:\mathcal{D}_{P_\tau}\to\overline{\text{Ran}}\,X=\mathcal{D}_P \\
&D_{P_\tau}h\mapsto Xh.
\end{aligned}
$$

Clearly $\mathcal{U}$ is onto. Moreover,

$$\|\mathcal{U}D_{P_\tau}h\|^2=\|Xh\|^2=\langle X^*Xh,h\rangle=\langle D_{P_\tau}^2h,h\rangle=\|D_{P_\tau}h\|^2.$$

So $\mathcal{U}$ is a surjective isometry, i.e., a unitary. Also,

$$
\begin{aligned}
&S_{1\tau} - S_{2\tau}^* P_\tau \\
&\quad = \beta\{((1+2|a|^2)S_1 - \bar{a}(2+|a|^2)S_2 + 3\bar{a}^2 P - 3aI\big)\big(I - \bar{a}S_1 + \bar{a}^2 S_2 \\
&\qquad - \bar{a}^3 P)^{-1} - (I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1}(-\bar{a}(2+|a|^2)S_1^* + (1+2|a|^2)S_2^* \\
&\qquad - 3aP^* + 3\bar{a}^2 I)(P - aS_2 + a^2 S_1 - a^3 I)(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1}] \\
&\quad = \beta(I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1} D_P\{(1+2|a|^2 - 3|a|^4)A_1 + a^2(3 - 2|a|^2 \\
&\qquad - |a|^4)A_1^* - \bar{a}(2 - 2|a|^2)A_2 - a^3(2 - 2|a|^2)A_2^* - a(1 - |a|^4)(A_1^* A_1 - \\
&\qquad A_2^* A_2) - 3a(1 - |a|^4)I\} D_P (I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1} \\
&\quad = (1 - |a|^2)(I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1} \beta D_P\{(1+3|a|^2)A_1 \\
&\qquad + a^2(3+|a|^2)A_1^* - 2\bar{a}A_2 - 2a^3 A_2^* - a(1+|a|^2)(A_1^* A_1 - A_2^* A_2) - 3a(1+|a|^2)I\} \\
&\qquad D_P(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1} \\
&\quad = X^* T^{-1/2} \beta\{(1+3|a|^2)A_1 + a^2(3+|a|^2)A_1^* - 2\bar{a}A_2 - 2a^3 A_2^* \\
&\qquad - a(1+|a|^2)(A_1^* A_1 - A_2^* A_2) - 3a(1+|a|^2)I\} T^{-1/2} X \\
&\quad = D_{P_\tau} \mathcal{U}^* T^{-1/2} \beta\{(1+3|a|^2)A_1 + a^2(3+|a|^2)A_1^* - 2\bar{a}A_2 - 2a^3 A_2^* \\
&\qquad - a(1+|a|^2)(A_1^* A_1 - A_2^* A_2) - 3a(1+|a|^2)I\} T^{-1/2} \mathcal{U} D_{P_\tau}.
\end{aligned}
$$

Again, since $S_{1\tau} - S_{2\tau}^* P_\tau = D_{P_\tau} A_{1\tau} D_{P_\tau}$ and $A_{1\tau}$ is unique, we have

$$
\begin{aligned}
A_{1\tau} = &\, \mathcal{U}^* T^{-1/2} \beta\{(1+3|a|^2)A_1 + a^2(3+|a|^2)A_1^* - 2\bar{a}A_2 - 2a^3 A_2^* \\
&- a(1+|a|^2)(A_1^* A_1 - A_2^* A_2) - 3a(1+|a|^2)I\} T^{-1/2} \mathcal{U}.
\end{aligned}
$$

Again,

$$
\begin{aligned}
S_{2\tau} - S_{1\tau}^* P_\tau &= \beta^2\{(-a(2+|a|^2)S_1 + (1+2|a|^2)S_2 - 3\bar{a}P + 3a^2 I)(I - \bar{a}S_1 + \bar{a}^2 S_2 \\
&\quad - \bar{a}^3 P)^{-1} - (I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1}((1+|a|^2)S_1^* - a(2+|a|^2)S_2^* \\
&\quad + 3a^2 P^* - 3\bar{a}I)(P - aS_2 + a^2 S_1 - a^3 I)(I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1}\} \\
&= \beta^2(1 - |a|^2)(I - aS_1^* + a^2 S_2^* - a^3 P^*)^{-1} D_P\{-2(aA_1 + a^3 A_1^*) + A_2 \\
&\quad + a^4 A_2^* + a^2(A_1^* A_{12}{}^* A_2) + 3a^2 I\} D_P (I - \bar{a}S_1 + \bar{a}^2 S_2 - \bar{a}^3 P)^{-1} \\
&= D_{P_\tau} \mathcal{U}^* T^{-1/2} \beta^2\{-2(aA_1 + a^3 A_1^*) + A_2 + a^4 A_2^* + a^2(A_1^* A_1 - A_2^* A_2) \\
&\quad + 3a^2 I\} T^{-1/2} \mathcal{U} D_{P_\tau}.
\end{aligned}
$$

Since $S_{2\tau} - S_{1\tau}^* P_\tau = D_{P_\tau} A_{2\tau} D_{P_\tau}$ and $A_{2\tau}$ is unique, we have that

$$
\begin{aligned}
A_{2\tau} = &\, \mathcal{U}^* T^{-1/2} \beta^2\{-2(aA_1 + a^3 A_1^*) + A_2 + a^4 A_2^* \\
&+ a^2(A_1^* A_1 - A_2^* A_2) + 3a^2 I\} T^{-1/2} \mathcal{U}.
\end{aligned}
$$

The proof is now complete. $\square$

5. Non-commuting fundamental operator tuple: An example

In Theorem 3.6 in [9], Costara has shown that corresponding to every point $(s_1, \dots, s_{n-1}, p) \in \Gamma_n$, there is a unique point $(c_1, \dots, c_{n-1}) \in \Gamma_{n-1}$ such that

$$s_i = c_i + \bar{c}_{n-i} p, \text{ for each } i = 1, \dots, n-1.$$

In the development of the theory of Γ_n-contraction, when we introduced the $\mathcal{F}_O$-tuple associated with a Γ_n-contraction, it became clear that for a scalar Γ_n-contraction $(s_1, \dots, s_{n-1}, p)$ (which is nothing but a point in Γ_n), the $\mathcal{F}_O$-tuple is just the Γ_{n-1}-contraction $(c_1, \dots, c_{n-1})$. Therefore, it is naturally asked whether the same result holds for a Γ_n-contraction. Also, in the hypothesis of Theorem 4.3, we assumed the fact that the fundamental operator pair (A_1, A_2) of the concerned Γ_3-contraction is commutative and that $r(A_1) \leq 2$. In this section, we shall show that the $\mathcal{F}_O$-tuple of a Γ_n-contraction may not be a Γ_{n-1}-contraction, indeed it may not even be a commuting $n-1$ tuple. Here we provide a Γ_3-contraction (S_1, S_2, P) on $\mathbb{C}^2$ that has non-commuting $\mathcal{F}_O$-pair (A_1, A_2) and that $r(A_1) > 2$. Consider the following 2×2 matrix:

$$A = \begin{pmatrix} 0 & \frac{3}{4} \\ 1 & 0 \end{pmatrix}.$$

Since A is a contraction which dilates to a unitary U and consequently we have that $\bar{\mathbb{D}}^3$ is a complete spectral set for (A, A, A) which further implies that $\bar{\mathbb{D}}^3$ is a spectral set for (A, A, A). Therefore, their symmetrization $(3A, 3A^2, A^3)$ is a Γ_3-contraction. Now consider

$$(S_1, S_2, P) = \left(\begin{pmatrix} 0 & \frac{9}{4} \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix}, \begin{pmatrix} 0 & \frac{9}{16} \\ \frac{3}{4} & 0 \end{pmatrix} \right).$$

One can easily check that

$$D_P = (I - P^*P)^{1/2} = \begin{pmatrix} \frac{\sqrt{7}}{4} & 0 \\ 0 & \frac{\sqrt{175}}{16} \end{pmatrix},$$

$$S_1 - S_2^* P = \begin{pmatrix} 0 & \frac{63}{64} \\ \frac{21}{16} & 0 \end{pmatrix} = D_P \begin{pmatrix} 0 & \frac{9}{5} \\ \frac{12}{5} & 0 \end{pmatrix} D_P$$

and

$$S_2 - S_1^* P = \begin{pmatrix} 0 & 0 \\ 0 & \frac{63}{64} \end{pmatrix} = D_P \begin{pmatrix} 0 & 0 \\ 0 & \frac{36}{25} \end{pmatrix} D_P.$$

This implies that

$$(A_1, A_2) = \left(\begin{pmatrix} 0 & \frac{9}{5} \\ \frac{12}{5} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \frac{36}{25} \end{pmatrix} \right)$$

is the $\mathcal{F}_O$-pair of (S_1, S_2, P). It is clear that $A_1 A_2 \neq A_2 A_1$ and $r(A_1) = \frac{\sqrt{108}}{5} > 2$.

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References

[1] Agler J and Young N J, The magic functions and automorphisms of a domain, *Complex Anal. Oper. Theory* **2** (2008) 383–404

[2] Agler J, Lykova Z and Young N J, Algebraic and geometric aspects of rational Γ-inner functions, *Adv. Math.* **328** (2018) 133–159

[3] Bhattacharyya T, Pal S and Shyam Roy S, Dilations of Γ-contractions by solving operator equations, *Adv. Math.* **230** (2012) 577–606

[4] Bhattacharyya T and Pal S, A functional model for pure Γ-contractions, *J. Operator Thoery* **71** (2014) 327–339

[5] Bisai B and Pal S, Structure theorems for operators associated with two domains related to μ-synthesis, *Bull. Sci. Math.* **159** (2020) 102822, 30 pp.

[6] Bisai B and Pal S, The fundamental operator tuples associated with the symmetrized polydisc, preprint

[7] Biswas S and Roy Subrata S, Functional models of Γ_n-contractions and characterization of Γ_n-isometries, *J. Funct. Anal.* **266** (2014) 6224–6255

[8] Costara C, The symmetrized bidisc and Lempert's theorem, *Bull. London Math. Soc.* **36** (2004) 656–662

[9] Costara C, On the spectral Nevanlinna-Pick problem, *Studia Math.* **170** (2005) 23–55

[10] Edigarian A and Zwonek W, Geometry of the symmetrized polydisc, *Arch. Math.* **84** (2005) 364–374

[11] Jarnicki M and Pflug P, On automorphisms of the symmetrized bidisc, *Arch. Math. (Basel)* **83** (2004) 264–266

[12] Nikolov N, Pflug P and Zwonek W, The Lempert function of the symmetrized polydisc in higher dimensions is not a distance, *Proc. Amer. Math. Soc.* **135** (2007) 2921–2928

[13] Pal S, From Stinespring dilation to Sz.-Nagy dilation on the symmetrized bidisc and operator models, *New York J. Math.* **20** (2014) 545–564

[14] Pal S, On decomposition of operators having Γ_3 as a spectral set, *Oper. Matrices* **11** (2017) 891–899

[15] Pal S, Canonical decomposition of operators associated with the symmetrized polydisc, *Complex Anal. Oper. Theory* **12** (2018) 931–943

[16] Pal S and Shalit O M, Spectral sets and distinguished varieties in the symmetrized bidisc, *J. Funct. Anal.* **266** (2014) 5779–5800

[17] Pal S, Operator theory and distinguished varieties in the symmetrized n-disk. arXiv:1708.06015

[18] Sarkar J, Operator theory on symmetrized bidisc, *Indiana Univ. Math. J.* **64** (2015) 847–873

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