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The 2-rank of the class group of some real cyclic quartic number fields

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Abstract. In this paper, we investigate the 2-rank of the class group of some real cyclic quartic number fields. Precisely, we consider the case where the quadratic subfield is $\mathbb{Q}(\sqrt{\ell})$ with $\ell \equiv 5 \pmod 8$ is a prime.

Keywords. Real cyclic quartic number field; 2-rank; 2-class group; quadratic fields.

Mathematics Subject Classification. 11R16, 11R29, 11R11, 11R80.

1. Introduction

Let K be a number field and H its 2-class group, that is, the 2-Sylow subgroup of the ideal class group of K. The rank $r_2(H)$ of H is the number of cyclic 2-groups appearing in the decomposition of H whose orders are a 2 power, that is, the dimension of the $\mathbb{F}_2$-vector space H/H^2, where $\mathbb{F}_2$ is the field of 2 elements.

Many mathematicians are interested in determining $r_2(H)$ and the power of 2 dividing the class number of K. Hasse [12], Bauer [5] and others gave methods to determine the exact power of 2 dividing the class number of a quadratic number field. These methods were developed by Brown and Parry [6,7] and Parry [19–21] to determine $r_2(H)$ and the power of 2 dividing the class number of some imaginary cyclic quartic number field K having a quadratic subfield k with odd class number. For this, they needed a suitable genus theory convenient to their situation. Hence they showed that the theory firstly developed by Hilbert [14], assuming an imaginary base field k, can be adapted to the situation where K is a totally imaginary quartic cyclic extension of a totally real quadratic subfield k. This theory can be applied, with minor modifications, to any quartic number field K having a quadratic subfield k of odd class number.

The 2-rank of any biquadratic number field K is determined (partially or totally) in many papers [2,3,6,7,17,18] up to the case: K is a real quartic cyclic extension of the rational number field $\mathbb{Q}$. Denote by $k = \mathbb{Q}(\sqrt{\ell})$ its unique quadratic subfield, we aim to investigate this case, whenever k has odd class number and the norm of its fundamental unit equals -1, i.e., $\ell = 2$ or ℓ is a prime congruent to $1 \pmod 4$. In this paper, we restrict

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ourselves to the case where ℓ is a prime congruent to $5 \pmod 8$ for two reasons: to avoid a long paper and the technics used in this case are little bit different from those used in the other cases.

An outline of the paper is as follows. In § 2 we summarize some preliminary results on quartic cyclic number fields and the ambiguous class number formula, which we will use later. The main theorems are presented and proved in § 3. In § 4, we characterize all real cyclic quartic number fields $K = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$, with ℓ being a prime congruent to $5 \pmod 8$, having a 2-class group trivial, cyclic, of rank 2 or of rank 3.

Notations. Throughout this paper, we adopt the following notations:

- $\mathbb{Q}$: the rational field.
- ℓ: a prime integer congruent to 5 modulo 8.
- $k = \mathbb{Q}(\sqrt{\ell})$: a quadratic field.
- ϵ_0: the fundamental unit of k.
- n: a square-free positive integer relatively prime to ℓ.
- $\delta = 1$ or 2.
- $d = n\epsilon_0\sqrt{\ell}$.
- $\mathbb{K} = k(\sqrt{d})$: a real quartic cyclic number field.
- $\mathcal{O}_k$ (resp. $\mathcal{O}_{\mathbb{K}}$): the ring of integers of k (resp. $\mathbb{K}$).
- H: the 2-class group of $\mathbb{K}$.
- $\mathcal{Z}$: the prime ideal of k above 2.
- $r_2(H)$: the rank of H.
- $\mathbb{K}^*$, k^*: the nonzero elements of the fields $\mathbb{K}$ and k respectively.
- $N_{\mathbb{K}/k}(\mathbb{K})$: elements of k which are norm from $\mathbb{K}$.
- p, q, p_i, q_j: odd prime integers.
- $(\frac{x, y}{p})_k$: quadratic norm residue symbol over k.
- $[\frac{\alpha}{\beta}]$: quadratic residue symbol for k.
- $(\frac{a}{b})$: quadratic residue (Legendre) symbol.
- $(\frac{a}{b})_4$: rational 4-th power residue symbol.

2. Preliminary results

Let K be a cyclic quartic extension of the rational number field $\mathbb{Q}$. By [13, Theorem 1], it is known that K can be expressed uniquely in the form $K = \mathbb{Q}(\sqrt{a(\ell + b\sqrt{\ell})})$, where a, b, c and ℓ are integers satisfying the conditions: a is odd and square-free, $\ell = b^2 + c^2$ is square-free, positive and relatively prime to a, with $b > 0$ and $c > 0$. Note that K possesses a unique quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. Assuming the class number of k odd and $N_{k/\mathbb{Q}}(\epsilon_0) = -1$, then, by [22], this is equivalent to K/k has an integral basis, and by [15], this is equivalent to the existence of an integer n such that $K = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ with ϵ_0 being the fundamental unit of k, and

$$n = \begin{cases} 2a & \text{if } \ell \equiv 1 \pmod 4 \text{ and } b \equiv 1 \pmod 2 \\ a & \text{otherwise.} \end{cases}$$

Recall that the extensions $K/\mathbb{Q}$ were investigated by Hasse [11] prior to that of Leopoldt [16] on the arithmetic interpretation of the class number of real abelian fields. They were also investigated by Gras [8–10] and others. We have the following remark.

Remark 2.1. Keeping the notations above a cyclic quartic field, K is then real if and only if $a > 0$ (equivalently $n > 0$).

The field K also satisfies the following lemma.

Lemma 2.1 [23]. *Let a, b and c be positive integers, $\ell = b^2 + c^2$, with a and c odd, then* $\mathbb{Q}(\sqrt{2a(\ell + b\sqrt{\ell})}) = \mathbb{Q}(\sqrt{a(\ell + c\sqrt{\ell})})$.

We end this section by recalling the number of ambiguous ideal classes of a quadratic extension K/k.

Theorem 2.1 [1, 19]. *Let K/k be a cyclic extension of prime degree p. Denote by $A_{K/k}$ the number of ambiguous ideal classes of the number field K with respect to k. Then*

$$A_{K/k} = h(k)2^{\mu+r^*-(r+c+1)}$$

where r is the number of fundamental units of k, μ is the number of prime ideals of k (finite or infinite) which ramify in K, r^* is defined by $2^{r^*} = [N_{K/k}(K^*) \cap E_k : E_k^2]$ with E_k being the group of units of k, $c = 1$ if k contains a primitive p-th root of unity and $c = 0$ otherwise.

Furthermore, if $p = 2$ and the class number of k is odd, then the 2-rank of the class group of K is equal to

$$\mu + r^* - (r + c + 1).$$

The remark below explains how to get r^*.

Remark 2.2. Since the unit group of k is generated by -1 and ϵ_0, so

- $r^* = 0$, if -1, ϵ_0 and $-\epsilon_0$ are not in $N_{\mathbb{K}/k}(\mathbb{K}^*)$.
- $r^* = 1$, if (-1 is in $N_{\mathbb{K}/k}(\mathbb{K}^*)$ and ϵ_0 is not) or (-1 is not in $N_{\mathbb{K}/k}(\mathbb{K}^*)$ and ϵ_0 or $-\epsilon_0$ is).
- $r^* = 2$, if -1 and ϵ_0 are in $N_{\mathbb{K}/k}(\mathbb{K}^*)$.

3. The rank of *H*

Let ℓ be a prime integer congruent to $5 \pmod 8$ and n a square-free positive integer relatively prime to ℓ. Let $\mathbb{K} = k(\sqrt{n\epsilon_0\sqrt{\ell}})$ and $k = \mathbb{Q}(\sqrt{\ell})$, where ϵ_0 is the fundamental unit of k.

On one hand, as $\ell \equiv 1 \pmod 4$, so it is well known (e.g., [23]) that $\epsilon_0 = \frac{u+v\sqrt{\ell}}{2}$ and $N(\epsilon_0) = \frac{u+v\sqrt{\ell}}{2}\frac{u-v\sqrt{\ell}}{2} = \frac{u^2-v^2\ell}{4} = -1$. Since $u + v\sqrt{\ell} > 0$, then $u - v\sqrt{\ell} < 0$. On the other hand, the polynomial $f(x) = x^4 - mv\ell x^2 + m^2\ell$ is irreducible. Indeed, as ℓ divides both of $mv\ell$ and $m^2\ell$, and ℓ^2 does not divide $m^2\ell$ since m and ℓ are relatively prime, so the Eisentein characterization implies our claim. Thus f is the characteristic polynomial of $\mathbb{K}$. By the Kaplansky's theorem [4, page 68] and since $m^2\ell[(mv\ell)^2 - 4m^2\ell] = m^4\ell^2(v^2\ell - 4) = m^4\ell^2u^2 \in \mathbb{Q}^2$, the polynomial f defines a cyclic extension over $\mathbb{Q}$ of degree 4, i.e., $\mathbb{K}$ is a real cyclic quartic number field and k is its unique quadratic subfield.

To determine the exact value of r^*, we will use the norm residue symbol applied to primes of k ramifying in $\mathbb{K}$. Note that the infinite prime ideals of k do not ramify in the extension $\mathbb{K}$. Indeed, the discriminant of $g(X) = f(x^2) = X^2 - mv\ell X + m^2\ell$ is

$$\Delta = (mv\ell)^2 - 4m^2\ell = m^2\ell(v^2\ell - 4) = m^2\ell u^2 \geq 0,$$

hence the roots of g are

$$X_1 = x_1^2 = \frac{mv\ell - mu\sqrt{\ell}}{2} = -m\sqrt{\ell}\left(\frac{u - v\sqrt{\ell}}{2}\right) \geq 0$$

and

$$X_2 = x_2^2 = \frac{mv\ell + mu\sqrt{\ell}}{2} = m\sqrt{\ell}\left(\frac{u + v\sqrt{\ell}}{2}\right) = m\epsilon_0\sqrt{\ell} \geq 0,$$

from which we deduce the result claimed.

Note that, as n is relatively prime to ℓ, so the prime integers dividing n do not ramify in k. Likewise, 2 stay inert in k since $\ell \equiv 5 \pmod 8$.

To compute $r_2(H)$, the rank of the 2-class group H of $\mathbb{K}$, we will distinguish many cases. For this, let $p_1, p_2, \ldots, p_t, q_1, \ldots, q_s$ be positive prime integers. Put $\delta = 1$ or 2.

3.1 *First case*: $n = \delta \prod_{i=1}^{t} p_i$ *and for all* i, $p_i \equiv 1 \pmod 4$

Theorem 3.1. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 5 \pmod 8$ *is a positive prime integer, n a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of the quadratic subfield* $k = \mathbb{Q}(\sqrt{\ell})$. *Let* $n = \delta \prod_{i=1}^{i=t} p_i$ *with* $p_i \equiv 1 \pmod 4$ *for all* $i \in \{1, \ldots, t\}$ *and t is a positive integer.*

(1) *If, for all* i, $(\frac{p_i}{\ell}) = -1$, *then* $r_2(H) = t$.
(2) *If, for all* i, $(\frac{p_i}{\ell}) = 1$, *then* $r_2(H) = 2t$.

Moreover, if $n = \delta \prod_{i=1}^{i=t_1} p_i \prod_{j=1}^{i=t_2} q_j$ *with* $(\frac{p_i}{\ell}) = -(\frac{q_j}{\ell}) = -1$ *and* $p_i \equiv q_j \equiv 1 \pmod 4$ *for all* $i \in \{1, \ldots, t_1\}$ *and* $j \in \{1, \ldots, t_2\}$, *then* $r_2(H) = t_1 + 2t_2$.

Proof. To prove the theorem assertions, we have to compute the integer r^* $(2^{r^*} = [N_{K/k}(K^*) \cap E_k : E_k^2])$ by applying Remark 2.2, and then we call Theorem 2.1.

(1) If $(\frac{p_i}{\ell}) = -1$ for all $i = 1, \ldots, t$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{z}$ and the prime ideals $\mathfrak{p}_i$, $i = 1, \ldots, t$, where $\mathfrak{z}$ (resp. $\mathfrak{p}_i$) is the prime ideal of k above 2 (resp. p_i). This implies that the number of primes of k ramifying in $\mathbb{K}$ is $\mu = t + 2$. Hence

$$\left(\frac{-1,\, d}{\mathfrak{p}_i}\right) = \left[\frac{-1}{\mathfrak{p}_i}\right] = \left(\frac{1}{p_i}\right) = 1 \text{ for all } i = 1, \ldots, t.$$

$$\left(\frac{-1,\, d}{(\sqrt{\ell})}\right) = \left[\frac{-1}{(\sqrt{\ell})}\right] = \left(\frac{-1}{\ell}\right) = 1.$$

$$\left(\frac{\epsilon_0,\, d}{\mathfrak{p}_i}\right) = \left[\frac{\epsilon_0}{\mathfrak{p}_i}\right] = \left(\frac{-1}{p_i}\right) = 1 \text{ for all } i = 1, \ldots, t.$$

$$\left(\frac{\epsilon_0,\, d}{(\sqrt{\ell})}\right) = \left[\frac{\frac{u}{2}}{(\sqrt{\ell})}\right] = \left[\frac{2u}{(\sqrt{\ell})}\right] = \left(\frac{2u}{\ell}\right) = \left(\frac{2}{\ell}\right)\left(\frac{u}{\ell}\right) = -\left(\frac{\ell}{u}\right) = -1,$$

indeed as

$$\epsilon_0 = \frac{u + v\sqrt{\ell}}{2}, \text{ so } -4 = u^2 - v^2\ell.$$

We summarize these results in the following table.

Unit\Character	$\sqrt{\ell}$	$\mathfrak{p}_i$	$\mathfrak{2}$
–1	+	+	+
ϵ_0	−	+	−
$-\epsilon_0$	−	+	−

So $r^* = 1$, from which we infer that

$$r_2(H) = \mu + r^* - 3 = t + 2 + 1 - 3 = t.$$

(2) If $(\frac{p_i}{\ell}) = 1$ for all $i = 1, \ldots, t$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{2}$ and the prime ideals $\wp_i$ and $\bar{\wp}_i$ with $p_i\mathcal{O}_k = \wp_i\bar{\wp}_i$, $i = 1, \ldots, t$, in the case $\mu = 2t + 2$. Hence

$$\left(\frac{-1,\, d}{\wp_i}\right) = \left(\frac{-1,\, d}{\bar{\wp}_i}\right) = \left[\frac{-1}{\wp_i}\right] = \left(\frac{-1}{p_i}\right) = 1, \text{ for all } i = 1, \ldots, t.$$
$$\left(\frac{\epsilon_0,\, d}{\wp_i}\right) = \left[\frac{\epsilon_0}{\wp_i}\right], \text{ for all } i = 1, \ldots, t.$$

To compute the last unity, put $p_i^{h_0} = \wp_i\bar{\wp}_i$ and $\wp_i = a_i + b_i\sqrt{\ell}$ and $\bar{\wp}_i = a_i - b_i\sqrt{\ell}$, for all i (note that $\mathcal{O}_k$ is a principal ring). According to [6], we have $\left[\frac{\epsilon_0\sqrt{\ell}}{\wp_i}\right] = \left(\frac{p_i}{\ell}\right)_4$. Thus

$$\left[\frac{\epsilon_0}{\wp_i}\right] = \left(\frac{p_i}{\ell}\right)_4 \left[\frac{\sqrt{\ell}}{\wp_i}\right].$$

On the other hand,

$$\left[\frac{\sqrt{\ell}}{\wp_i}\right] = \left[\frac{b_i^2\sqrt{\ell}}{\wp_i}\right] = \left[\frac{b_i(-a_i + a_i + b_i\sqrt{\ell})}{\wp_i}\right] = \left[\frac{-a_ib_i}{\wp_i}\right]$$
$$= \left(\frac{a_i}{p_i}\right)\left(\frac{b_i}{p_i}\right).$$

As $p_i^{h_0} = a_i^2 - b_i^2\ell$, so $b_i^2\ell \equiv a_i^2 \pmod{p_i}$. Note that ℓ and p_i are relatively prime, then $b_i^2\ell^2 \equiv \ell a_i^2 \pmod{p_i}$. This implies that $\left(\frac{b_i}{p_i}\right) = \left(\frac{b_i\ell}{p_i}\right) = \left(\frac{b_i^2\ell^2}{p_i}\right)_4 = \left(\frac{\ell a_i^2}{p_i}\right)_4 = \left(\frac{\ell}{p_i}\right)_4\left(\frac{a_i}{p_i}\right)$. Finally,

$$\left[\frac{\epsilon_0}{\wp_i}\right] = \left(\frac{p_i}{\ell}\right)_4 \left(\frac{a_i}{p_i}\right)\left(\frac{b_i}{p_i}\right) = \left(\frac{p_i}{\ell}\right)_4 \left(\frac{a_i}{p_i}\right)\left(\frac{\ell}{p_i}\right)_4 \left(\frac{a_i}{p_i}\right)$$
$$= \left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4.$$

Proceeding similarly, we get $\left[\frac{\epsilon_0}{\bar{\wp}_i}\right] = \left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4$. The two values $\left(\frac{-1, d}{(\sqrt{\ell})}\right)$ and $\left(\frac{\epsilon, d}{(\sqrt{\ell})}\right)$ are computed as above. These results are summarized in the following table:

Unit\Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{2}$
-1	+	+	+	+
ϵ_0	−	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	−
$-\epsilon_0$	−	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	−

So $r^* = 1$, from which we infer that

$$r_2(H) = \mu + r^* - 3 = 2t + 2 + 1 - 3 = 2t.$$

If $n = \delta \prod_{i=1}^{i=t_1} p_i \prod_{j=1}^{i=t_2} q_j$ with $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$ for all $i \in \{1, \ldots, t_1\}$ and $j \in \{1, \ldots, t_2\}$, then according to the two cases above, $r^* = 1$ and $r_2(H) = t_1 + 2t_2$. □

3.2 *Second case*: $n = 1$ *or* 2

Theorem 3.2. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 5 \pmod 8$ *is a positive prime integer*, n *a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of the quadratic subfield* $k = \mathbb{Q}(\sqrt{\ell})$. *If* $n = 1$ *or* 2, *then* $r_2(H) = 0$.

Proof. As in the first case, we compute r^* by applying Remark 2.2, and then we call Theorem 2.1. For the two cases $n = 1$ or 2, the prime ideals of k that ramify in $\mathbb{K}$ are $(\sqrt{\ell})$ and $\mathfrak{2}$, where $\mathfrak{2}$ is the prime ideal of k above 2, i.e., $\mu = 2$. Proceeding as in the first case, we get the following table:

Unit\Character	$\sqrt{\ell}$	$\mathfrak{2}$
-1	+	+
ϵ_0	−	−
$-\epsilon_0$	−	−

Hence $r^* = 1$, which implies that $r_2(H) = \mu + r^* - 3 = 2 + 1 - 3 = 0$. □

3.3 *Third case*: $n = \prod_{i=1}^{i=t} p_i$ *with* t *is odd and for all* i, $p_i \equiv 3 \pmod 4$

Theorem 3.3. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 5 \pmod 8$ *is a positive prime integer*, n *a square-free positive integer relatively prime to* ℓ

and ϵ_0 *the fundamental unit of the quadratic subfield* $k = \mathbb{Q}(\sqrt{\ell})$. *Assume* $n = \prod_{i=1}^{i=t} p_i$, $p_i \equiv 3 \pmod 4$ *for all* $i = 1, \ldots, t$ *and* t *is a positive odd integer.*

(1) *If, for all* i, $\left(\frac{p_i}{\ell}\right) = -1$, *then* $r_2(H) = t - 1$.
(2) *If, for all* i, $\left(\frac{p_i}{\ell}\right) = 1$, *then* $r_2(H) = 2t - 2$.

Moreover, if $n = \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$, *where* $p_i \equiv q_j \equiv 3 \pmod 4$ *and* $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, *for all* $i \in \{1, \ldots, t_1\}$, *and for all* $j \in \{1, \ldots, t_2\}$ *with* $t_1 + t_2$ *is odd, then* $r_2(H) = t_1 + 2t_2 - 2$.

Proof. As in the two cases above, we shall compute r^* by applying Remark 2.2, and then we call Theorem 2.1.

(1) If $\left(\frac{p_i}{\ell}\right) = -1$ for all $i \in \{1, \ldots, t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{p}_i$ and $(\sqrt{\ell})$, where $\mathfrak{p}_i$ is the prime ideal of k above p_i. Hence, for all $i \in \{1, \ldots, t\}$, we have

$$\left(\frac{-1,\, d}{\mathfrak{p}_i}\right) = \left[\frac{-1}{\mathfrak{p}_i}\right] = \left(\frac{1}{p_i}\right) = 1 \quad \text{and} \quad \left(\frac{\epsilon_0,\, d}{\mathfrak{p}_i}\right) = \left[\frac{\epsilon_0}{\mathfrak{p}_i}\right] = \left(\frac{-1}{p_i}\right) = -1.$$

The two values $\left(\frac{-1,\, d}{(\sqrt{\ell})}\right)$ and $\left(\frac{\epsilon,\, d}{(\sqrt{\ell})}\right)$ are computed as in the first case. We summarize these results in the following table:

Unit\Character	$(\sqrt{\ell})$	$\mathfrak{p}_i$
-1	+	+
ϵ_0	−	−
$-\epsilon_0$	−	−

Hence $r^* = 1$, which implies that $r_2(H) = \mu + r^* - 3 = t + 1 + 1 - 3 = t - 1$.

(2) If $\left(\frac{p_i}{\ell}\right) = 1$ for all $i \in \{1, \ldots, t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\wp_i$ and $\bar{\wp}_i$, where $p_i \mathcal{O}_k = \wp_i \bar{\wp}_i$, $i = 1, \ldots, t$. Hence, for all $i \in \{1, \ldots, t\}$, we have

$$\left(\frac{-1,\, d}{\wp_i}\right) = \left(\frac{-1,\, d}{\bar{\wp}_i}\right) = \left[\frac{-1}{\wp_i}\right] = \left(\frac{-1}{p_i}\right) = -1.$$

Proceeding as in the proof of Theorem 3.1, we get

$$\left(\frac{\epsilon_0,\, d}{\wp_i}\right) = \left[\frac{\epsilon_0}{\wp_i}\right] = \left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4.$$

According to [6], we have

$$\left[\frac{\epsilon\sqrt{\ell}}{\bar{\wp}_i}\right] = -\left(\frac{p}{\ell}\right)_4, \quad \text{then} \quad \left[\frac{\epsilon_0}{\bar{\wp}_i}\right] = -\left(\frac{p}{\ell}\right)_4 \left[\frac{\sqrt{\ell}}{\bar{\wp}_i}\right].$$

On the other hand,

$$\left[\frac{\sqrt{\ell}}{\bar{\wp}_i}\right] = -\left[\frac{-b_i^2\sqrt{\ell}}{\bar{\wp}_i}\right] = -\left[\frac{b_i(-a_i + a_i - b_i\sqrt{\ell})}{\bar{\wp}_i}\right] = -\left[\frac{-a_i b_i}{\bar{\wp}_i}\right]$$

$$= \left(\frac{a_i}{p_i}\right)\left(\frac{b_i}{p_i}\right),$$

which implies (as in the proof of Theorem 3.1) that

$$\left[\frac{\epsilon}{\bar{\wp}_i}\right] = -\left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4 \left(\frac{a}{p_i}\right)^2 = -\left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4.$$

The two values $\left(\frac{-1,\, d}{(\sqrt{\ell})}\right)$ and $\left(\frac{\epsilon_0,\, d}{(\sqrt{\ell})}\right)$ are computed as above. We summarize these results in the following table:

Unit/Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$
–1	+	–	–
ϵ_0	–	$\left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4$
$-\epsilon_0$	–	$-\left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4 \left(\frac{\ell}{p_i}\right)_4$

Hence $r^* = 0$, which implies that

$$r_2(H) = \mu + r^* - 3 = 2t + 1 + 0 - 3 = 2t - 2.$$

Finally, if $n = \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$, with $p_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$ for all $i \in \{1, \ldots, t_1\}$ and $j \in \{1, \ldots, t_2\}$ with $t_1 + t_2$ is odd, then according to the two cases above, there are $t_1 + 2t_2 + 1$ prime ideals of k which ramify in $\mathbb{K}$ and $r^* = 0$. Thus $r_2(H) = t_1 + 2t_2 + 1 + 0 - 3 = t_1 + 2t_2 - 2$. □

3.4 *Fourth case*: $n = \delta \prod_{i=1}^{i=t} p_i$ *with t is even or* $n = 2 \prod_{i=1}^{i=t} p_i$ *with t is odd and for all* i, $p_i \equiv 3 \pmod 4$

Theorem 3.4. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 5 \pmod 8$ *is a positive prime integer, n a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of the quadratic subfield* $k = \mathbb{Q}(\sqrt{\ell})$. *Assume* $n = \delta \prod_{i=1}^{i=t} p_i$ *and t is even or* $n = 2 \prod_{i=1}^{i=t} p_i$ *and t is odd with* $p_i \equiv 3 \pmod 4$, $i = 1, \ldots, t$.

(1) *If, for all i,* $\left(\frac{p_i}{\ell}\right) = -1$, *then* $r_2(H) = t$.
(2) *If, for all i,* $\left(\frac{p_i}{\ell}\right) = 1$, *then* $r_2(H) = 2t - 1$.

Moreover, if $n = \delta \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$, *with* $t_1 + t_2$ *is even, or* $n = 2 \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$ *with* $t_1 + t_2$ *is odd, where* $p_i \equiv q_j \equiv 3 \pmod 4$ *and* $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, *for all* $i \in \{1, \ldots, t_1\}$ *and* $j \in \{1, \ldots, t_2\}$, *then* $r_2(H) = t_1 + 2t_2 - 1$.

Proof. We proceed as above, we first compute r^* by applying Remark 2.2, and then we apply Theorem 2.1.

(1) If $\left(\frac{p_i}{\ell}\right) = -1$ for all $i \in \{1, \ldots, t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{2}$, $\mathfrak{p}_i$ and $(\sqrt{\ell})$, where $\mathfrak{p}_i$ (resp. $\mathfrak{2}$) is the prime ideal of k above p_i (resp. 2). Proceeding

as in the cases above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{2}$	$\mathfrak{p}_i$
-1	$+$	$+$	$+$
ϵ_0	$-$	$(-1)^{t+1}$	$-$
$-\epsilon_0$	$-$	$(-1)^{t+1}$	$-$

Hence $r^* = 1$, from which we deduce that

$$r_2(H) = \mu + r^* - 3 = t + 2 + 1 - 3 = t.$$

(2) If $\left(\frac{p_i}{\ell}\right) = 1$ for all $i \in \{1, \ldots, t\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{2}$, $\wp_i$, and $\bar{\wp}_i$, where $p\mathcal{O}_k = \wp_i\bar{\wp}_i$, $i = 1, \ldots, t$. Proceeding as in the cases above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{2}$
-1	$+$	$-$	$-$	$+$
ϵ_0	$-$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$(-1)^{t+1}$
$-\epsilon_0$	$-$	$-\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$(-1)^{t+1}$

Hence $r^* = 0$, from which we deduce that

$$r_2(H) = \mu + r^* - 3 = 2t + 2 + 0 - 3 = 2t - 1.$$

Assume $n = \delta \prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$ with t_1+t_2 is even, or $n = 2\prod_{i=1}^{t_1} p_i \prod_{j=1}^{t_2} q_j$ with t_1+t_2 is odd, where $p_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, for all $i \in \{1, \ldots, t_1\}$ and $j \in \{1, \ldots, t_2\}$. Then according to the previous discussion, we obtain $r^* = 0$, and thus $r_2(H) = t_1 + 2t_2 - 1$. □

3.5 *Fifth case*: $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, $p_i \equiv -q_j \equiv 1 \pmod 4$ $\forall(i, j)$ *and* s *is odd.*

Theorem 3.5. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 5 \pmod 8$ *is a positive prime integer,* n *a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of the quadratic subfield* $k = \mathbb{Q}(\sqrt{\ell})$*. Assume* $n = \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ *with* $p_i \equiv -q_j \equiv 1 \pmod 4$ *for all* $(i, j) \in \{1, \ldots, t\} \times \{1, \ldots, s\}$ *and* s *is odd. Denote by* h *the number of prime ideals of* k *dividing all the* $p_i's$, $i \in \{1, \ldots, t\}$.

(1) *If, for all* j, $\left(\frac{q_j}{\ell}\right) = -1$*, then* $r_2(H) = h + s - 1$.
(2) *If, for all* j, $\left(\frac{q_j}{\ell}\right) = 1$*, then* $r_2(H) = h + 2s - 2$.

Moreover, if $\prod_{j=1}^{j=s} q_j = \prod_{i=1}^{i=s_1} q_i \prod_{j=s_1+1}^{j=s} q_j$ *with* s *odd,* $q_i \equiv q_j \equiv 3 \pmod 4$ *and* $\left(\frac{q_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$*, for all* $i \in \{1, \ldots, s_1\}$ *and* $j \in \{s_1 + 1, \ldots, s\}$*, then* $r_2(H) = h + s_1 + 2(s - s_1) - 2$.

Proof. There are four cases to discuss:

(1) If $\left(\frac{p_i}{\ell}\right) = \left(\frac{q_j}{\ell}\right) = -1$, for all $i = 1, \ldots, t$ and for all $j = 1, \ldots, s$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{p}_i$ and $\mathfrak{q}_j$, the prime ideals of k above p_i and q_j respectively, i.e., $\mu = t + s + 1$. Proceeding as above, we get the following table.

Unit/Character	$\sqrt{\ell}$	$\mathfrak{p}_i$	$\mathfrak{q}_j$
–1	+	+	+
ϵ_0	–	+	–
$-\epsilon_0$	–	+	–

Thus $r^* = 1$, from which we deduce that

$$r_2(H) = \mu + r^* - 3 = t + s + 1 + 1 - 3 = t + s - 1.$$

(2) If $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = 1$, for all $i = 1, \ldots, t$ and for all $j = 1, \ldots, s$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\wp_i$, $\bar{\wp}_i$ and $\mathfrak{q}_j$, where $p\mathcal{O}_k = \wp_i\bar{\wp}_i$ and $\mathfrak{q}_j$ is the prime ideal of k above q_j, i.e., $\mu = 2t + s + 1$. Proceeding as above, we get the following results:

Unit/Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{q}_j$
–1	+	+	+	+
ϵ_0	–	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	–
$-\epsilon_0$	–	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	–

Thus $r^* = 1$, this result implies that

$$r_2(H) = \mu + r^* - 3 = 2t + s + 1 + 1 - 3 = 2t + s - 1.$$

(3) If $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, for all $i = 1, \ldots, t$ and for all $j = 1, \ldots, s$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\mathfrak{p}_i$, ρ_j, and $\bar{\rho}_j$, where $q_j\mathcal{O}_k = \rho_j\bar{\rho}_j$ and $\mathfrak{p}_j$ is the prime ideal of k above p_j, i.e., $\mu = t + 2s + 1$. Proceeding as above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{p}_i$	ρ_j	$\bar{\rho}_j$
–1	+	+	–	–
ϵ_0	–	+	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$
$-\epsilon_0$	–	+	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$

Thus $r^* = 0$, this implies that

$$r_2(H) = \mu + r^* - 3 = t + 2s + 1 + 0 - 3 = t + 2s - 2.$$

(4) If $\left(\frac{p_i}{\ell}\right) = \left(\frac{q_j}{\ell}\right) = 1$, for all $i = 1, \dots, t$ and for all $j = 1, \dots, s$, then the prime ideals of k which ramify in $\mathbb{K}$ are $(\sqrt{\ell})$, $\wp_i$, $\bar{\wp}_i$, ρ_j and $\bar{\rho}_j$, where $p_i\mathcal{O}_k = \wp_i\bar{\wp}_i$ and $q_j\mathcal{O}_k = \rho_j\bar{\rho}_j$, i.e., $\mu = 2t + 2s + 1$. Proceeding as above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\wp_i$	$\bar{\wp}_i$	ρ_j	$\bar{\rho}_j$
–1	+	+	+	–	–
ϵ_0	–	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$
$-\epsilon_0$	–	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$

From this, we deduce that $r^* = 0$, and thus

$$r_2(H) = \mu + r^* - 3 = 2t + 2s + 1 + 0 - 3 = 2t + 2s - 2.$$

In general, if $\prod_{j=1}^{j=s} q_j = \prod_{i=1}^{i=s_1} q_i \prod_{j=s_1+1}^{j=s} q_j$, with s being an odd integer, $q_i \equiv q_j \equiv 3 \pmod 4$ and $\left(\frac{q_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, $i \in \{1, \dots, s_1\}$, $j \in \{1, \dots, s_2\}$. Then taking into account the discussions above one gets $r^* = 0$, and thus

$$r_2(H) = h + s_1 + 2s_2 + 1 + 0 - 3 = h + s_1 + 2(s - s_1) - 2,$$

where h is always the number of the prime divisors of all the p_i's, $p_i \equiv 1 \pmod 4$, $i \in \{1, \dots, t\}$ in k. □

3.6 *Sixth case*: $n = \delta \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, *s is even, or* $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, *s is odd, where* $p_i \equiv -q_j \equiv 1 \pmod 4$ *for all* (i, j)

Theorem 3.6. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ *be a real cyclic quartic number field, where* $\ell \equiv 5 \pmod 8$ *is a positive prime integer, n a square-free positive integer relatively prime to* ℓ *and* ϵ_0 *the fundamental unit of the quadratic subfield* $k = \mathbb{Q}(\sqrt{\ell})$. *Assume* $n = \delta \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ *with s even or* $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ *with s odd, where* $p_i \equiv -q_j \equiv 1 \pmod 4$, *for all* $(i, j) \in \{1, \dots, t\} \times \{1, \dots, s\}$, *are prime integers. Denote by h the number of the prime ideals of k above all the* p_i'*s*, $i \in \{1, \dots, t\}$.

(1) *If, for all* j, $\left(\frac{q_j}{\ell}\right) = -1$, *then* $r_2(H) = h + s$.
(2) *If, for all* j, $\left(\frac{q_j}{\ell}\right) = 1$, *then* $r_2(H) = h + 2s - 1$.

Moreover, if $\prod_{j=1}^{j=s} q_j = \prod_{i=1}^{i=s_1} q_i \prod_{j=s_1+1}^{j=s} q_j$ *with* $\left(\frac{q_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, *for all* $i \in \{1, \dots, s_1\}$ *and* $j \in \{s_1 + 1, \dots, s\}$, *assuming s even if* $n = \delta \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ *and odd if* $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, *then* $r_2(H) = h + s_1 + 2(s - s_1) - 1$.

Proof. There are also four cases to distinguish:

(1) If $\left(\frac{p_i}{\ell}\right) = \left(\frac{q_j}{\ell}\right) = -1$, for all $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{2}$, $(\sqrt{\ell})$, $\mathfrak{p}_i$ and $\mathfrak{q}_j$. Proceeding as in the first cases above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{2}$	$\mathfrak{p}_i$	$\mathfrak{q}_j$
–1	+	+	+	+
ϵ_0	–	$(-1)^{s-1}$	+	–
$-\epsilon_0$	–	$(-1)^{s-1}$	+	–

Hence $r^* = 1$, and

$$r_2(H) = \mu + r^* - 3 = t + s + 2 + 1 - 3 = t + s.$$

(2) If $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = 1$, for all $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{2}$, $(\sqrt{\ell})$, $\wp_i$, $\bar{\wp}_i$ and $\mathfrak{q}_j$, where $p_i\mathcal{O}_k = \wp_i\bar{\wp}_i$. Proceeding as in the first cases above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{2}$	$\wp_i$	$\bar{\wp}_i$	$\mathfrak{q}_j$
–1	+	+	+	+	+
ϵ_0	–	$(-1)^{s-1}$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	–
$-\epsilon_0$	–	$(-1)^{s-1}$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	–

Hence $r^* = 1$, and

$$r_2(H) = \mu + r^* - 3 = 2t + s + 2 + 1 - 3 = 2t + s.$$

(3) If $\left(\frac{p_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$; for all $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{2}$, $(\sqrt{\ell})$, $\mathfrak{p}_i$, ρ_j, and $\bar{\rho}_j$, where $q_j\mathcal{O}_k = \rho_j\bar{\rho}_j$. Proceeding as in the first cases above, we get the following table:

Unit/Character	$\sqrt{\ell}$	$\mathfrak{2}$	$\mathfrak{p}_i$	ρ_j	$\bar{\rho}_j$
–1	+	+	+	–	–
ϵ_0	–	$(-1)^{s-1}$	+	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$
$-\epsilon_0$	–	$(-1)^{s-1}$	+	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$

Hence $r^* = 0$, and

$$r_2(H) = \mu + r^* - 3 = t + 2s + 2 + 0 - 3 = t + 2s - 1.$$

(4) If $\left(\frac{p_i}{\ell}\right) = \left(\frac{q_j}{\ell}\right) = 1$, for all $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, then the prime ideals of k which ramify in $\mathbb{K}$ are $\mathfrak{2}$, $(\sqrt{\ell})$, $\wp_i$, $\bar{\wp}_i$, ρ_j and $\bar{\rho}_j$, where $p_i\mathcal{O}_k = \wp_i\bar{\wp}_i$, $q_j\mathrm{O}_k = \rho_j\bar{\rho}_j$.

Proceeding as in the first cases above, we get the following table:

Unit/ Character	$\sqrt{\ell}$	2	$\wp_i$	$\bar{\wp}_i$	ρ_j	$\bar{\rho}_j$
-1	$+$	$+$	$+$	$+$	$-$	$-$
ϵ_0	$-$	$(-1)^{s-1}$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$
$-\epsilon_0$	$-$	$(-1)^{s-1}$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$\left(\frac{p_i}{\ell}\right)_4\left(\frac{\ell}{p_i}\right)_4$	$-\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$	$\left(\frac{q_j}{\ell}\right)_4\left(\frac{\ell}{q_j}\right)_4$

Hence $r^* = 0$, and

$$r_2(H) = \mu + r^* - 3 = 2t + 2s + 2 + 0 - 3 = 2t + 2s - 1.$$

In general, if $\prod_{j=1}^{j=s} q_j = \prod_{i=1}^{i=s_1} q_i \prod_{j=s_1+1}^{j=s} q_j$ with $\left(\frac{q_i}{\ell}\right) = -\left(\frac{q_j}{\ell}\right) = -1$, for all $i \in \{1, \ldots, s_1\}$ and $j \in \{s_1+1, \ldots, s\}$, assuming s even if $n = \delta \prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$ and odd if $n = 2\prod_{i=1}^{i=t} p_i \prod_{j=1}^{j=s} q_j$, then taking into account the discussions above one gets $r^* = 0$, and thus $r_2(H) = h + s_1 + 2(s - s_1) + 2 + 0 - 3 = h + s_1 + 2(s - s_1) - 1$. □

4. Applications

In this section, we shall determine the integers n satisfying $r_2(H)$, the rank of the 2-class group H of $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$, which is equal to 0, 1, 2 or 3. For this, we adopt the following notations: p and p_i (resp. q and q_i), $i \in \mathbb{N}^*$, are prime integers congruent to 1 (resp. 3) modulo 4. $\delta = 1$ or 2. The following theorems are simple deductions from the results of the previous sections.

Theorem 4.1. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 5 \pmod 8$ is a positive prime integer, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. The class number of $\mathbb{K}$ is odd if and only if one of the following assertions holds*:

(1) $n = 1$ *or* 2.
(2) n *is a prime integer congruent to* $3 \pmod 4$.

Numerical Examples 4.1. For all the examples below, we use PARI/GP calculator version 2.9.1 (64 bit), Nov. 22, 2016.

(1) For $n = 1$ and $\ell = 173 \equiv 5 \pmod 8$, H is trivial, the class number of the class group of $\mathbb{K} = \mathbb{Q}(\sqrt{\epsilon_0\sqrt{\ell}})$ is 5. For $n = 2$ and $\ell = 197 \equiv 5 \pmod 8$, H is trivial, in reality the class group of $\mathbb{K} = \mathbb{Q}(\sqrt{2\epsilon_0\sqrt{\ell}})$ is of type $(3, 3)$.
(2) For $n = q = 67 \equiv 3 \pmod 4$ and $\ell = 53 \equiv 5 \pmod 8$, we have $(\frac{q}{\ell}) = -1$ and H is trivial, the class number of the class group of $\mathbb{K} = \mathbb{Q}(\sqrt{67\epsilon_0\sqrt{\ell}})$ is 17. For $n = q = 79 \equiv 3 \pmod 4$ and $\ell = 13 \equiv 5 \pmod 8$, we have $(\frac{q}{\ell}) = 1$ and the class group of $\mathbb{K} = \mathbb{Q}(\sqrt{79\epsilon_0\sqrt{\ell}})$ is trivial.

Theorem 4.2. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 5 \pmod 8$ is a prime, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. H is cyclic if and only if one of the following assertions holds*:

(1) $n = \delta p$ *with* $(\frac{p}{\ell}) = -1$.
(2) $n = 2q$.
(3) $n = pq$ *with* $(\frac{p}{\ell}) = -1$.

Numerical Examples 4.2. Here are some examples:

(1) For $n = p = 13 \equiv 1 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$ and H is cyclic of order 2. For $n = p = 17 \equiv 1 \pmod 4$ and $\ell = 29 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$ and H is cyclic of order 2.
(2) For $n = 2p = 2.41 \equiv 2 \pmod 4$ and $\ell = 13 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$ and H is cyclic of order 2. For $n = 2p = 2.51 \equiv 2 \pmod 4$ and $\ell = 61 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$ and H is cyclic of order 2.
(3) For $n = 2q = 2.19 \equiv -2 \pmod 4$ and $\ell = 53 \equiv 5 \pmod 8$, we have H is cyclic of order 2.
(4) For $n = pq = 5.7 \equiv 3 \pmod 4$ and $\ell = 53 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$, $(\frac{p}{q}) = -1$ and H is cyclic of order 2. For $n = pq = 5.11 \equiv 3 \pmod 4$ and $\ell = 13 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$, $(\frac{p}{q}) = 1$ and H is cyclic of order 2;

Theorem 4.3. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 5 \pmod 8$ is a prime, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. The rank $r_2(H)$ equals 2 if and only if n takes one of the following forms:*

(1) $n = \delta p$ *and* $(\frac{p}{\ell}) = 1$.
(2) $n = \delta p_1 p_2$ *and* $(\frac{p_1}{\ell}) = (\frac{p_2}{\ell}) = -1$.
(3) $n = pq$ *and* $(\frac{p}{\ell}) = 1$.
(4) $n = 2pq$ *and* $(\frac{p}{\ell}) = -1$.
(5) $n = p_1 p_2 q$ *and* $(\frac{p_1}{\ell}) = (\frac{p_2}{\ell}) = -1$.
(6) $n = \delta q_1 q_2$ *and at least one of the two symbols* $(\frac{q_1}{\ell})$, $(\frac{q_2}{\ell})$ *equals* -1.
(7) $n = q_1 q_2 q_3$ *and at most one of the symbols* $(\frac{q_1}{\ell})$, $(\frac{q_2}{\ell})$, $(\frac{q_3}{\ell})$ *equals* 1.

Numerical Examples 4.3. Here are some examples:

(1) For $n = p = 13 \equiv 1 \pmod 4$ and $\ell = 101 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = 1$ and H is of type $(2, 2)$. For $n = 2p = 2.73 \equiv 2 \pmod 4$ and $\ell = 109 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = 1$ and H is of type $(2, 4)$.
(2) For $n = p_1 p_2 = 17.37 \equiv 1 \pmod 4$ and $\ell = 29 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = -1$ and H is of type $(2, 2)$.
(3) For $n = pq = 13.11 \equiv 3 \pmod 4$ and $\ell = 53 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = 1$ and H is of type $(2, 2)$.
(4) For $n = 2pq = 2.17.11 \equiv -2 \pmod 4$ and $\ell = 29 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$ and H is of type $(2, 2)$.
(5) For $n = p_1 p_2 q = 17.37.23 \equiv 3 \pmod 4$ and $\ell = 61 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = -1$ and H is of type $(2, 2)$.

(6) For $n = q_1q_2 = 79.83 \equiv 1 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = -1$, $(\frac{q_2}{\ell}) = 1$ and H is of type $(2, 2)$. For $n = 2q_1q_2 = 2.47.59 \equiv 2 \pmod 4$ and $\ell = 13 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = -1$, $(\frac{q_2}{\ell}) = -1$ and H is of type $(2, 2)$.
(7) For $n = q_1q_2q_3 = 23.71.83 \equiv 3 \pmod 4$ and $\ell = 61 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = -1$, $(\frac{q_2}{\ell}) = -1$, $(\frac{q_3}{\ell}) = 1$ and H is of type $(2, 2)$.

Theorem 4.4. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{n\epsilon_0\sqrt{\ell}})$ be a real cyclic quartic number field, where $\ell \equiv 5 \pmod 8$ is a prime, n a square-free positive integer relatively prime to ℓ and ϵ_0 the fundamental unit of the quadratic subfield $k = \mathbb{Q}(\sqrt{\ell})$. The rank $r_2(H)$ equals 3 if and only if n takes one of the following forms*:

(1) $n = \delta p_1p_2$ *and* $(\frac{p_1}{\ell}) = -(\frac{p_2}{\ell}) = 1$.
(2) $n = \delta p_1p_2p_3$ *and* $(\frac{p_i}{\ell}) = -1$ *for all* $i \in \{1, 2, 3\}$.
(3) $n = 2pq$ *and* $(\frac{p}{\ell}) = 1$.
(4) $n = \delta q_1q_2$ *and* $(\frac{q_i}{\ell}) = 1$ *for all* $i \in \{1, 2\}$.
(5) $n = q_1q_2q_3$ *and only one of the symbols* $(\frac{q_i}{\ell})$, $i \in \{1, 2, 3\}$ *is* -1.
(6) $n = 2q_1q_2q_3$ *and at most one of the symbols* $(\frac{q_i}{\ell})$, $i \in \{1, 2, 3\}$ *is* 1.
(7) $n = p_1p_2q$ *and* $(\frac{p_1}{\ell}) = -(\frac{p_2}{\ell}) = 1$.
(8) $n = 2p_1p_2q$ *and* $(\frac{p_1}{\ell}) = (\frac{p_2}{\ell}) = -1$.
(9) $n = \delta pq_1q_2$ *and* $(\frac{p}{\ell}) = -1$ *and at least one of the symbols* $(\frac{q_i}{\ell})$, $i \in \{1, 2\}$ *is* -1.
(10) $n = pq_1q_2q_3$ *and* $(\frac{p}{\ell}) = -1$ *and at most one of the symbols* $(\frac{q_i}{\ell})$, $i \in \{1, 2, 3\}$ *is* 1.
(11) $n = p_1p_2p_3q$ *and* $(\frac{p_i}{\ell}) = -1$ *for all* $i \in \{1, 2, 3\}$.

Numerical Examples 4.4. Here are some examples:

(1) For $n = p_1p_2 = 37.89 \equiv 1 \pmod 4$ and $\ell = 5 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = 1$ and H is of type $(2, 4, 4)$. For $n = 2p_1p_2 = 2.41.53.89 \equiv 2 \pmod 4$ and $\ell = 29 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = 1$ and H is of type $(2, 2, 4)$.
(2) For $n = p_1p_2p_3 = 17.53.89 \equiv 1 \pmod 4$ and $\ell = 61 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = -1$, $(\frac{p_3}{\ell}) = -1$ and H is of type $(2, 2, 2)$. For $n = 2p_1p_2p_3 = 2.17.61.89 \equiv 2 \pmod 4$ and $\ell = 29 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = -1$, $(\frac{p_3}{\ell}) = -1$ and H is of type $(2, 2, 2)$.
(3) For $n = 2pq = 2.53.79 \equiv -2 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = 1$ and H is of type $(2, 2, 4)$.
(4) For $n = 2q_1q_2 = 2.59.83 \equiv 2 \pmod 4$ and $\ell = 29 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = (\frac{q_2}{\ell}) = -1$ and H is of type $(2, 2, 4)$. For $n = q_1q_2 = 67.71 \equiv 1 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = (\frac{q_2}{\ell}) = -1$ and H is of type $(2, 2, 2)$.
(5) For $n = q_1q_2q_3 = 19.47.71 \equiv -1 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = -1$, $(\frac{q_2}{\ell}) = (\frac{q_3}{\ell}) = 1$ and H is of type $(2, 2, 2)$.
(6) For $n = 2q_1q_2q_3 = 2.7.67.71 \equiv -2 \pmod 4$ and $\ell = 53 \equiv 5 \pmod 8$, we have $(\frac{q_1}{\ell}) = 1$, $(\frac{q_2}{\ell}) = (\frac{q_3}{\ell}) = -1$ and H is of type $(2, 2, 2)$.
(7) For $n = p_1p_2q = 13.17.43 \equiv 3 \pmod 4$ and $\ell = 61 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = -1$, $(\frac{p_2}{\ell}) = 1$ and H is of type $(2, 4, 4)$.
(8) For $n = 2p_1p_2q = 2.29.53.79 \equiv 2 \pmod 4$ and $\ell = 61 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = (\frac{p_2}{\ell}) = -1$ and H is of type $(2, 2, 2)$.
(9) For $n = pq_1q_2 = 37.47.71 \equiv 1 \pmod 4$ and $\ell = 5 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$, $(\frac{q_1}{\ell}) = -1$, $(\frac{q_2}{\ell}) = 1$ and H is of type $(2, 2, 2)$. For $n = 2pq_1q_2 = 2.17.31.83 \equiv 2 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$, $(\frac{q_1}{\ell}) = -1$, $(\frac{q_2}{\ell}) = 1$ and H is of type $(2, 2, 2)$.

(10) For $n = pq_1q_2q_3 = 5.43.31.71 \equiv 3 \pmod 4$ and $\ell = 13 \equiv 5 \pmod 8$, we have $(\frac{p}{\ell}) = -1$, $(\frac{q_1}{\ell}) = 1$, $(\frac{q_2}{\ell}) = (\frac{q_3}{\ell}) = -1$ and H is of type $(2, 2, 2)$.
(11) For $n = p_1p_2p_3q = 13.17.29.83 \equiv 3 \pmod 4$ and $\ell = 37 \equiv 5 \pmod 8$, we have $(\frac{p_1}{\ell}) = (\frac{p_2}{\ell}) = (\frac{p_3}{\ell}) = -1$ and H is of type $(2, 2, 2)$.

Remark 4.1. If the integer n does not take any value in Theorems 4.1, 4.2, 4.3, 4.4, then $r_2(H) \geq 4$.

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