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A characterization of totally disconnected compactly ruled groups

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Abstract. A locally compact group G is called compactly ruled if it is a directed union of compact open subgroups. We denote by $\mathcal{SUB}(G)$ the space of closed subgroups of G equipped with the *Chabauty topology*. In this paper, we show that the subspace $\mathcal{SUB}_{\mathrm{co}}(G)$ of $\mathcal{SUB}(G)$ consisting of compact open subgroups is dense in $\mathcal{SUB}(G)$ if and only if G is totally disconnected compactly ruled.

Keywords. Totally disconnected locally compact group; compactly ruled; Chabauty topology.

Mathematics Subject Classification. 22D05.

1. Introduction and main results

Given a locally compact group G with identity element e, we denote by $\mathcal{SUB}(G)$ the set of closed subgroups of G equipped with the Chabauty topology, which is compact (see [1, Chapitre VIII, §5, no. 3, Théorème 1]). Recall that a base of neighborhoods of $H \in \mathcal{SUB}(G)$ in the Chabauty topology is given by the sets

$$\mathcal{U}_G(H; K, W) \stackrel{\mathrm{def}}{=} \{L \in \mathcal{SUB}(G) \mid L \cap K \subseteq WH \text{ and } H \cap K \subseteq WL\}, \tag{1.1}$$

where K ranges through the set $\mathcal{K}(G)$ of all compact subsets of G and W through the filter $\mathcal{U}(e)$ of all neighborhoods of the identity (the statement "U is a neighborhood of a point x" is used in the Bourbaki sense, throughout this paper; i.e., U is any subset which contains an open subset containing x). In particular, the trivial subgroup $E = \{e\}$ has a neighborhood base consisting of sets

$$\mathcal{U}_G(E; K, W) = \{L \in \mathcal{SUB}(G) \mid L \cap K \subseteq W\}, \tag{1.2}$$

where $K \in \mathcal{K}(G)$ and $W \in \mathcal{U}(e)$.

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For a locally compact group G, the set of compact open subgroups of G is denoted by

$$\mathcal{SUB}_{co}(G) .$$

The identity component of G, denoted by G_0, is the connected component of the identity in G. We note that G_0 is compact if and only if $\mathcal{SUB}_{co}(G)$ is nonempty (see Corollary 2.E.7(2) in [2]). The following proposition is due to Gartside and Smith (Lemma 3.2 in [5]).

PROPOSITION

For a profinite group G, the subspace $\mathcal{SUB}_{co}(G)$ is dense in $\mathcal{SUB}(G)$.

The following question arises in an attempt to generalize the above proposition to other classes of locally compact topological groups.

Question A. For which locally compact groups G is the set $\mathcal{SUB}_{co}(G)$ dense in $\mathcal{SUB}(G)$?

Let $\mathfrak{X}$ be the class of locally compact groups G such that $\mathcal{SUB}_{co}(G)$ is dense in $\mathcal{SUB}(G)$. A group G belongs to the class $\mathfrak{X}$ is called an $\mathfrak{X}$-group. To state the main result of this paper, we need the following definition (see Proposition 1.3 and Definition 1.4 in [8]).

DEFINITION 1.1 (Compactly ruled group)

A topological group G is called compactly ruled if one of the following equivalent statements is satisfied:

(1) G is a directed union of compact open subgroups;
(2) G is locally compact and every compact subset of G is contained in a compact subgroup;
(3) G is locally compact and every finite subset of G is contained in a compact subgroup.

Some interesting properties and results concerning the Chabauty space of compactly ruled groups are given in Part 1, Subsection 1.2.1 of [8]. The affirmative answer to Question A is given in the following.

Theorem A (Theorem 3.12 and Theorem 3.13 below). *For a locally compact group G the following conditions are equivalent*:

(1) *G is compactly ruled.*
(2) *The closure $\overline{\mathcal{SUB}_{co}(G)}$ of $\mathcal{SUB}_{co}(G)$ coincides with $\{H \in \mathcal{SUB}(G) \mid G_0 \subseteq H\}$.*

In particular, $G \in \mathfrak{X}$ if and only if G is totally disconnected and compactly ruled.

Example 1.19 in [8] shows that the subspace $\mathcal{SUB}_{c}(G)$ of $\mathcal{SUB}(G)$ consisting of compact subgroups need not be closed, even if G is compactly ruled.

PROPOSITION A (Proposition 3.14 below)

For a compacly ruled group G the following two conditions are equivalent:

(1) *$\mathcal{SUB}_{c}(G)$ is closed in $\mathcal{SUB}(G)$.*
(2) *G is compact.*

2. Chabauty topology

Let G be a locally compact group and $\mathcal{SUB}(G)$ the hyperspace of all closed subgroups of G. The *Chabauty topology* on $\mathcal{SUB}(G)$ has the sets

$$\mathcal{O}_1(K) = \{H \in \mathcal{SUB}(G) \mid H \cap K = \emptyset\},$$
$$\mathcal{O}_2(V) = \{H \in \mathcal{SUB}(G) \mid H \cap V \neq \emptyset\},$$

as an open subbase, where V and K run, respectively, over all open and compact subsets of G. The hyperspace $\mathcal{SUB}(G)$ endowed with the Chabauty topology is called the *Chabauty space* of the group G.

The following facts are well-known (see Theorems 1 and 2 of [13]).

PROPOSITION 2.1

Let G be a locally compact group and $(H_i)_{i\in I}$ be a net in $\mathcal{SUB}(G)$.

(1) *If $(H_i)_{i\in I}$ is directed by inclusion, then $(H_i)_{i\in I}$ converges to $\overline{\cup_{i\in I} H_i}$.*
(2) *If $(H_i)_{i\in I}$ is directed by inverse inclusion, then $(H_i)_{i\in I}$ converges to $\cap_{i\in I} H_i$.*

We say that a topological group G is *topologically finitely generated* if there is a finite subset F of G such that the subgroup $\langle F\rangle$ generated by F is dense in G. As an immediate consequence of Proposition 2.1(1), we get the following well-known result.

PROPOSITION 2.2

The subspace of topologically finitely generated subgroups is dense in $\mathcal{SUB}(G)$.

A continuous mapping $f : X \longrightarrow Y$ is proper if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X (See [3]).

PROPOSITION 2.3

Let G_1 and G_2 be two locally compact topological groups.

(1) *If $\phi\colon G_1 \to G_2$ is a proper morphism between locally compact topological groups, then the following mapping*

$$\mathcal{SUB}(\phi) : \mathcal{SUB}(G_1) \to \mathcal{SUB}(G_2), \quad H \mapsto \phi(H)$$

is continuous. Moreover, if ϕ is surjective, then $\mathcal{SUB}(\phi)$ is also surjective.

(2) *If $\phi\colon G_1 \to G_2$ be a continuous open morphism. Then the following mapping*

$$\mathcal{SUB}^*(\phi) : \mathcal{SUB}(G_2) \to \mathcal{SUB}(G_1), \quad H \mapsto \phi^{-1}(H)$$

is continuous.

Proof. See [7]. □

Remark 2.4. Since the identity component G_0 of a compactly ruled group G is compact, the canonical projection $\pi : G \to G/G_0$ is proper and so $\mathcal{SUB}(\pi) : \mathcal{SUB}(G) \to \mathcal{SUB}(G/G_0)$ is continuous.

Since every continuous injective mapping of a compact space onto a Hausdorff space is a homeomorphism (Theorem 3.1.13 in [3]), we obtain the following.

PROPOSITION 2.5 ([13], Proposition 1)

Let A be a closed subgroup of a locally compact group G. Then $\mathcal{SUB}(A)$ is naturally homeomorphic to the subspace

$$\{H \in \mathcal{SUB}(G) \mid H \subseteq A\}$$

of $\mathcal{SUB}(G)$.

We close this section with the following elementary lemma (Lemma 7.3 in [6]).

Lemma 2.6. *Let G be a locally compact group and $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ two nets converging to A and B respectively in $\mathcal{SUB}(G)$. If $A_i \subseteq B_i$ holds eventually, then $A \subseteq B$.*

Remark 2.7. In particular, the above lemma implies that if a net $(A_i)_{i\in I}$ of closed subgroups converges to A then $\bigcap_{i\in I} A_i \subseteq A$.

3. The class $\mathfrak{X}$ and its closure properties

The following fundamental result due to Van Dantzig (1931) is the starting point for the structure theory of totally disconnected locally compact groups (Theorem 7.7 in [9]).

Theorem 3.1 (van Dantzig). *Let G be a totally disconnected locally compact group. Then $\mathcal{SUB}_{\text{co}}(G)$, the set of all compact open subgroups of G, is a basis of identity neighborhoods.*

We begin with the following result.

Lemma 3.2. *For a locally compact group G, the following two conditions are equivalent*:

(1) *G is totally disconnected.*
(2) *The trivial subgroup E is an adherent point of $\mathcal{SUB}_{\text{co}}(G)$.*

In particular, any $\mathfrak{X}$-group is totally disconnected.

Proof.

(1) $\Rightarrow$ (2). By van Dantzig's theorem (Theorem 3.1), any identity neighborhood $V \in \mathcal{U}(e)$ contains a compact open subgroup H_V of G. It is clear that the net $(H_V)_{V\in\mathcal{U}(e)}$ is directed by inverse inclusion and so, by Proposition 2.1(2), $(H_V)_{V\in\mathcal{U}(e)}$ converges to $\bigcap_{V\in\mathcal{U}(e)} H_V = E$.
(2) $\Rightarrow$ (1). Follows immediately from Lemma 2.6. □

In a locally compact topological group G, an element $g \in G$ is called a compact element if the monothetic subgroup $\overline{\langle g \rangle}$ is compact. We shall denote the set of all compact elements in G by $\mathtt{comp}\,(G)$. Then Weil's lemma says that for any element $g \in G$, either $g \in \mathtt{comp}\,(G)$ or else $\langle g \rangle$ is isomorphic to $\mathbb{Z}$ and therefore is, in particular, discrete (see [10], Proposition 7.43).

We introduce the following definition ([8], Definition 2).

DEFINITION 3.3 (Periodic group)

A topological group G is called *periodic* if

(1) G is locally compact and totally disconnected.
(2) $\mathtt{comp}\,(G) = G$.

PROPOSITION 3.4

Any $\mathfrak{X}$-group is periodic.

Proof. Let G be a locally compact group such that $\mathcal{SUB}_{\mathrm{co}}(G)$ is dense in $\mathcal{SUB}(G)$. By Lemma 3.2, the group G is totally disconnected. It remains to prove that $\mathtt{comp}\,(G) = G$. We proceed to prove this by contradiction, and suppose that $\mathtt{comp}\,(G) \neq G$. By Theorem 2 in [14], the set $\mathtt{comp}\,(G)$ is closed in G and therefore $G \setminus \mathtt{comp}\,(G)$ is a nonempty open subset of G. Then $\mathcal{O}_2(G \setminus \mathtt{comp}\,(G))$ is a nonempty open subset of $\mathcal{SUB}(G)$ and so $\mathcal{O}_2(G \setminus \mathtt{comp}\,(G)) \cap \mathcal{SUB}_{\mathrm{co}}(G) \neq \varnothing$. Consequently, there exists $H \in \mathcal{SUB}_{\mathrm{co}}(G)$ such that $H \cap (G \setminus \mathtt{comp}\,(G)) \neq \varnothing$, a contradiction. Thus $G = \mathtt{comp}\,(G)$ and so G is periodic. □

Remark 3.5. The converse of the above proposition is not valid, in general. For example, if G is an infinite discrete torsion finitely generated group, G is isolated in $\mathcal{SUB}(G)$ and so $\mathcal{SUB}_{\mathrm{co}}(G)$ is not dense in $\mathcal{SUB}(G)$.

Since a subset A of a topological space X is dense in X if and only if every neighborhood U of X meets A, we get the following:

PROPOSITION 3.6

Let G be a totally disconnected locally compact group. Then $\mathcal{SUB}_{\mathrm{co}}(G)$ is dense in $\mathcal{SUB}(G)$ if and only if, for any compact subset K of G, $H \in \mathcal{SUB}(G)$ and $U \in \mathcal{SUB}_{\mathrm{co}}(G)$, the following system

$$\mathtt{Sys}\,(K, U, H) = \begin{cases} X \cap K & \subseteq UH \\ H \cap K & \subseteq UX \end{cases}$$

has a solution X in $\mathcal{SUB}_{\mathrm{co}}(G)$.

DEFINITION 3.7 (Directed family)

A *directed family* of subsets of a set X is a family $(X_i)_{i\in I}$ of subsets of X such that, for every $i, j \in I$, there is some $k \in I$ such that $X_i \cup X_j \subseteq X_k$.

When $(X_i)_{i\in I}$ is a directed family, we write

$$\biguplus_{i\in I} X_i$$

for the union $\bigcup_{i\in I} X_i$.

PROPOSITION 3.8 (Stability properties of the class $\mathfrak{X}$)

The class of groups $\mathfrak{X}$ has the following properties:

(1) *The class $\mathfrak{X}$ is stable under taking open subgroups.*
(2) *Let G be a locally compact group such that $G = \biguplus_{i\in I} O_i$, where $(O_i)_{i\in I}$ is a directed family of open subgroups of G. If $O_i \in \mathfrak{X}$ for each i, then $G \in \mathfrak{X}$.*

Proof.

(1) Let $G \in \mathfrak{X}$ and let U be an open subgroup of G. As the injection map $i : U \to G$ is open, it induces a surjective continuous map $\mathcal{SUB}^*(i) : \mathcal{SUB}(G) \to \mathcal{SUB}(U)$ (see Proposition 2.3(2)). Then $\mathcal{SUB}_{\rm co}(U) = \mathcal{SUB}^*(i)\,(\mathcal{SUB}_{\rm co}(G))$ is dense in $\mathcal{SUB}(U)$. □
(2) Let K be a compact subset of G, H a closed subgroup of G and $U \in \mathcal{SUB}_{\rm co}(G)$. By hypothesis the family $(O_i)_{i\in I}$ is directed and so there is $i \in I$ such that $K \subseteq O_i$. As $O_i \in \mathfrak{X}$, then the system $\mathtt{Sys}(K, U \cap O_i, H \cap O_i)$ has a solution X in $\mathcal{SUB}_{\rm co}(O_i)$. Then we have

$$\begin{cases} X \cap K & \subseteq\ (U \cap O_i)(H \cap O_i) \subseteq UH \\ H \cap O_i \cap K & =\ H \cap K \subseteq (U \cap O_i)X \subseteq UX \end{cases}$$

and so, since $\mathcal{SUB}_{\rm co}(O_i) \subseteq \mathcal{SUB}_{\rm co}(G)$, X is a solution in $\mathcal{SUB}_{\rm co}(G)$ of the system $\mathtt{Sys}(K, U, H)$. Thus $G \in \mathfrak{X}$. □

We have the following result due to Gartside and Smith (see Lemma 3.2 in [5]); a proof is included for completeness.

PROPOSITION 3.9

The class $\mathfrak{X}$ contains the class of profinite groups.

Proof. Let G be a profinite group. Let K be a compact subset of G, $U \in \mathcal{SUB}_{\rm co}(G)$ and $H \in \mathcal{SUB}(G)$. In view of Proposition 12.1.9 and Theorem 12.3.26 of [12] (see also Theorem 7.7 in [9]), there exists a compact open normal subgroup V of G such that $V \subseteq U$. It is easy to see that $X = VH$ is a solution of $\mathtt{Sys}(K, U, H)$ in $\mathcal{SUB}_{\rm co}(G)$. □

PROPOSITION 3.10

Let G be a locally compact totally disconnected group. If G is compactly ruled then $G \in \mathfrak{X}$.

We present two proofs of this proposition.

First proof. Since G is compactly ruled, G is a directed union of compact open subgroups (see Proposition 1.3 in [8]). From Proposition 3.8(3) and Proposition 3.9 we conclude that $G \in \mathfrak{X}$.

Second proof. Let H be a closed subgroup of G. We shall prove that there exists a net of compact open subgroups of G which converges to H. As G is compactly ruled, by Proposition 1.3 in [8], there exists a directed family $(H_\alpha)_{\alpha\in I}$ of compact open subgroups of G such that

$$G = \biguplus_{\alpha\in I} H_\alpha$$

and so

$$H = \biguplus_{\alpha\in I} (H_\alpha \cap H)$$

By Proposition 2.1, we have

$$\lim_{\alpha\in I} (H_\alpha \cap H) = H$$

in $\mathcal{SUB}(H)$ and so in $\mathcal{SUB}(G)$. On the other hand, for every $\alpha \in I$, $H_\alpha \cap H$ is a closed subgroup of the profinite group H_α and therefore, by Proposition 3.9, we have $H_\alpha \cap H = \lim_{j\in S_\alpha} Z_{(j,\alpha)}$ for a net $Z_{(j,\alpha)}$ of compact open subgroups of H_α (and so of G). Therefore,

$$H = \lim_{\alpha\in I} \lim_{j\in S_\alpha} Z_{(j,\alpha)}.$$

Then there exists a subnet $(Z_p)_{p\in P}$ of the net $(Z_{(j,\alpha)})$ such that

$$H = \lim_{p\in P} Z_p,$$

by the Theorem on Iterated Limits of [11], p. 69.

Let G be a locally compact group. For $n \geq 1$, we put

$$\mathtt{comp}_n(G) \overset{\text{def}}{=} \left\{(g_1, \ldots, g_n) \in G^n \;\middle|\; \overline{\langle g_1, \ldots, g_n\rangle} \text{ is compact}\right\}.$$

It is clear that $\mathtt{comp}_1(G) = \mathtt{comp}\,(G)$.

Lemma 3.11. *Let G be a totally disconnected locally compact group. If for any $n \geq 1$, $\mathtt{comp}_n(G)$ is dense in G^n, then for every $U \in \mathcal{SUB}_{\text{co}}(G)$ and $g \in G$, the subgroup $\langle U, g\rangle$ is compact and open in G.*

Proof. Since $\mathtt{comp}_1(G) = \mathtt{comp}\,(G)$ and since $\mathtt{comp}\,(G)$ is closed in G (Theorem 2 in [14]) then $\mathtt{comp}\,(G) = G$. Let V be a compact open subgroup of G such that $g \in V$ (see Lemma 1.2 in [8]). Since VU is compact, there exists a finite symmetric subset $B = \{b_1 = e, b_2, \ldots, b_m\}$ such that $B \subseteq UVU$ and $UV \subseteq BU$. Thus $UBU = BU$. On the other hand, since $\mathtt{comp}_m(G)$ is dense in G^m, then

$$\mathtt{comp}_m(G) \cap \prod_{i=1}^{m} b_i U \neq \varnothing$$

Let $(\gamma_1, \ldots, \gamma_m) \in \mathtt{comp}_m(G) \cap \prod_{i=1}^{m} b_i U$ and let

$$C = \{\gamma_1, \ldots, \gamma_m\} \cup \{\gamma_1, \ldots, \gamma_m\}^{-1} \cup \{e\}$$

We have $CU = BU$. Then $UCU = CU$ and therefore for any $k \geq 1$, $(CU)^k = C^k U$. Then $VU \subseteq \langle BU \rangle = \langle CU \rangle = \langle C \rangle U$. The subgroup $H \stackrel{\text{def}}{=} \langle C \rangle U$ is a product of a relatively compact and a compact open subgroup and so it is compact. As $\langle U, g \rangle \subseteq H$, $\langle U, g \rangle$ is compact. □

Theorem 3.12. *Let G be a locally compact group. The following conditions are equivalent*:

(1) *G is totally disconnected compactly ruled.*
(2) *$G \in \mathfrak{X}$.*
(3) *G is totally disconnected and for any $n \geq 1$, $\mathtt{comp}_n(G)$ is dense in G^n.*

Proof. The implication (1) ⇒ (2) follows from Proposition 3.10. (2) ⇒ (3) By Proposition 3.4 the group G is totally disconnected. Let $n \geq 1$ and let $g_1, \ldots, g_n \in G$ and $U \in \mathcal{SUB}_{\text{co}}(G)$. Let $K = \{g_1, \ldots, g_n\}$ and $L = \overline{\langle K \rangle}$. As $G \in \mathfrak{X}$, the system $\mathtt{Sys}(K, U, L)$ has a solution $H \in \mathcal{SUB}_{\text{co}}(G)$. Then $K \subseteq UH$ and therefore $(g_1 U \times g_2 U \times \cdots \times g_n U) \cap \mathtt{comp}_n(G) \neq \varnothing$. Consequently, $\mathtt{comp}_n(G)$ is dense in G^n. (3) ⇒ (1) Let $U \in \mathcal{SUB}_{\text{co}}(G)$. For a finite subset F of G, let U_F the subgroup generated by $U \cup F$. It is clear that

$$G = \biguplus_{F \in \mathcal{P}_f(G)} U_F,$$

where $\mathcal{P}_f(G)$ denotes the set of all finite subsets of G. By Lemma 3.11 any $U_F \in \mathcal{SUB}_{\text{co}}(G)$ and so G is compactly ruled. □

Theorem 3.13. *For a locally compact group G the following conditions are equivalent*:

(1) *G is compactly ruled.*
(2) *The closure $\overline{\mathcal{SUB}_{\text{co}}(G)}$ of $\mathcal{SUB}_{\text{co}}(G)$ coincides with $\{H \in \mathcal{SUB}(G) \mid G_0 \subseteq H\}$.*

Proof.

(1) ⇒ (2). By hypothesis G is compactly ruled and so G/G_0 is compactly ruled. As $\mathcal{SUB}^*(\pi)$ is continuous,

$$\begin{aligned}\{H \in \mathcal{SUB}(G) \mid G_0 \subseteq H\} &= \mathcal{SUB}^*(\pi)(\mathcal{SUB}(G/G_0)) \\ &= \mathcal{SUB}^*(\pi)(\overline{\mathcal{SUB}_{\text{co}}(G/G_0)}) \\ &\subseteq \overline{\mathcal{SUB}^*(\pi)(\mathcal{SUB}_{\text{co}}(G/G_0))} \\ &= \overline{\mathcal{SUB}_{\text{co}}(G)}.\end{aligned}$$

Consequently $\{H \in \mathcal{SUB}(G) \mid G_0 \subseteq H\} = \overline{\mathcal{SUB}_{\text{co}}(G)}$.
(2) ⇒ (1). By hypothesis $\mathcal{SUB}_{\text{co}}(G)$ is nonempty and so G_0 is compact. Then the map $\mathcal{SUB}(\pi)$ is continuous. We have

$$\begin{aligned}&\mathcal{SUB}(\pi)(\{H \in \mathcal{SUB}(G) \mid G_0 \subseteq H\}) \\ &\quad = \mathcal{SUB}(\pi)(\overline{\mathcal{SUB}_{\text{co}}(G)}) \subseteq \overline{\mathcal{SUB}(\pi)(\mathcal{SUB}_{\text{co}}(G))}.\end{aligned}$$

Consequently, $\mathcal{SUB}(G/G_0) = \overline{\mathcal{SUB}_{co}(G/G_0)}$. By Lemma 3.11, G/G_0 is compactly ruled and, as G_0 is compact, then G is compactly ruled. □

PROPOSITION 3.14

For a compactly ruled group G the following two conditions are equivalent:

(1) $\mathcal{SUB}_c(G)$ *is closed in* $\mathcal{SUB}(G)$.
(2) G *is compact.*

Proof. The implication (2) ⇒ (1) is trivial. The implication (1) ⇒ (2) follows immediately from Proposition 2.1(1) □

In accordance with Gartside-Smith [4, Definition 5.1], we adopt the following:

DEFINITION 3.15 (Isolated subgroup)

If G is a locally compact group and H a closed subgroup of G then H is said to be an *isolated subgroup* of G if H is an isolated point of $\mathcal{SUB}(G)$.

Combining Proposition 2.2 with Proposition 3.10 we obtain (see Theorem 5.6 and Lemma 5.4(ii) in [4])

PROPOSITION 3.16

Let G be a totally disconnected compactly ruled group. Any isolated subgroup of G is topologically finitely generated compact open.

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