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Recognition of some finite simple groups by the orders of vanishing elements

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Abstract. Let G be a finite group. A vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G. Denote by $\mathrm{Vo}(G)$ the set of the orders of vanishing elements of G. A finite group all of whose elements have prime power order is called a CP-group. Generally, a finite group G is called a VCP-group if every element in $\mathrm{Vo}(G)$ is a prime power. Here, we classify completely the non-solvable VCP-groups and show that, except for A_7, the non-solvable VCP-groups coincide with the non-solvable CP-groups. Moreover, as a consequence, we give a new characterization of the simple VCP-groups, namely, if G is a finite group and M is a finite non-abelian simple VCP-group except for $\mathrm{L}_2(9)$ such that $\mathrm{Vo}(G) = \mathrm{Vo}(M)$, then $G \cong M$. In addition, we also prove that if $\mathrm{Vo}(G) = \mathrm{Vo}(\mathrm{L}_2(9))$, then either $G \cong \mathrm{L}_2(9)$, or $G \cong NA$, where $A \cong \mathrm{SL}_2(4)$ and N is an elementary abelian 2-group and a direct sum of natural $\mathrm{SL}_2(4)$-modules.

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1. Introduction

Given a finite group G, a vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G. We will denote by $\mathrm{Van}(G)$ the set of vanishing elements of G. Denote by $\pi(G)$ the set of prime divisors of the orders of G and by $\pi_e(G)$ the set of the element orders of G. Our aim in this paper is to analyze a particular subset of $\pi_e(G)$, the set $\mathrm{Vo}(G)$ of the orders of elements in $\mathrm{Van}(G)$. We know that $\mathrm{Vo}(G)$ can encode some information about the structure of G (see [1,5,15,17]).

Following [9], we call groups all of whose elements have prime power order CP-groups. Generally, we say that a group G is a VCP-group if every element in $\mathrm{Vo}(G)$ is a prime power. Clearly, a CP-group is a VCP-group. However, the converse is not true. For the group A_7, $\mathrm{Vo}(\mathrm{A}_7) = \{3, 4, 5, 7\}$, but $\pi_e(\mathrm{A}_7) = \{1, 2, 3, 4, 5, 6, 7\}$.

The finite non-solvable VCP-groups were investigated in [16]. Since $\mathrm{Vo}(\mathrm{A}_7) = \{3, 4, 5, 7\}$, it is obvious that the simple group A_7 is missing in that theorem (the bug was pointed out in [8]). In this paper, we classify completely the non-solvable VCP-groups and

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show that, except for A_7, the non-solvable VCP-groups coincide with the non-solvable CP-groups. It is known that the quotient group of $\mathrm{Aut}(L_2(9))$ modulo $\mathrm{Inn}(L_2(9))$ is isomorphic to the elementary abelian group of order 4; in other words, there are 3 subgroups of index 2 in $\mathrm{Aut}(L_2(9))$. One of them is isomorphic to S_6 and is not a VCP-group; another one is isomorphic to PGL(2, 9) and possesses a cyclic subgroup of order 10. Close inspection shows that the remaining subgroup is in fact a VCP-group. Note that in the following theorem, M_{10} denotes the only VCP-subgroup of index 2 in $\mathrm{Aut}(L_2(9))$.

Theorem 1.1. *Let G be a finite non-solvable* VCP-*group and let N be the solvable radical of G. Then G satisfies one of the following statements*:

(1) *If $N = 1$, then G is isomorphic to one of the following groups*: $L_2(q)$ *for* $q = 5,7,8,9,17$, $L_3(4)$, $Sz(8)$, $Sz(32)$, A_7, *or* M_{10}.
(2) *If $N > 1$, then N is a 2-group, furthermore, one of the following holds*:

  (2.1) *N is a direct sum of minimal normal subgroup of G, each of order 2^4 and isomorphic to the natural* $SL_2(4)$-*module* (*recall that* $L_2(5) \cong SL_2(4) \cong A_5$) *and G/N is isomorphic to* $L_2(5)$.
  (2.2) *N is abelian and G/N is isomorphic to* $L_2(8)$
  (2.3) *N is nilpotent of class* 6 *and G/N is isomorphic to* $Sz(8)$, *or* $Sz(32)$.

From Theorem 1.1, we easily get the following result.

COROLLARY 1.2

If G is a non-solvable VCP-group, not isomorphic to A_7, then G is a CP-group.

For a set Ω of positive integers, let $h(\Omega)$ be the number of isomorphism classes of finite group G such that $\pi_e(G) = \Omega$. A group G is called characterizable (or recognizable) if $h(\pi_e(G)) = 1$. We define a similar concept for the set of orders of vanishing elements.

DEFINITION 1.3

For a set Ω of positive integers, let $v(\Omega)$ be the number of isomorphism classes of finite group G such that $\mathrm{Vo}(G) = \Omega$. A non-abelian group G is called V-characterizable (or V-recognizable) if $v(\mathrm{Vo}(G)) = 1$.

Notice that, applying Theorem 1.1, we easily prove the main theorem of [8] (the simple VCP-groups can be characterized by group order and the set of vanishing element orders). In this paper, we characterize the simple VCP-groups only by the orders of vanishing elements and obtain the following result.

Theorem 1.4. *Except for* $L_2(9)$, *the simple VCP-groups can be characterized by the orders of vanishing elements. In addition, if* $\mathrm{Vo}(G) = \mathrm{Vo}(L_2(9))$, *then either* $G \cong L_2(9)$, *or* $G \cong NA$, *where* $A \cong SL_2(4)$ *and N is an elementary abelian* 2-*group and a direct sum of natural* $SL_2(4)$-*modules.*

Notice that if $G \cong NA$, where $A \cong SL_2(4)$ and N is an elementary abelian 2-group and a direct sum of natural $SL_2(4)$-modules, then we easily conclude that $\mathrm{Vo}(G) = \mathrm{Vo}(L_2(9))$.

In particular, $\upsilon(\mathrm{Vo}(L_2(9)) = \infty$. Also, for the simple linear group $L_3(5)$, $\mathrm{Vo}(L_3(5)) = \mathrm{Vo}(\mathrm{Aut}(L_3(5))$, but $L_3(5) \ncong \mathrm{Aut}(L_3(5))$ because $|L_3(5)| \neq |\mathrm{Aut}(L_3(5))|$. On the other hand, we also know that both $L_2(9)$ and $L_3(5)$ are not recognizable (see [14]). So, it is natural to ask the following question:

Question. For a non-abelian simple group G, if $h(\pi_e(G)) = 1$, then is $\upsilon(\mathrm{Vo}(G)) = 1$? Conversely, if $\upsilon(\mathrm{Vo}(G)) = 1$, then is $h(\pi_e(G)) = 1$?

All further unexplained notation is standard, readers may refer to [4,10].

2. Preliminary results

Given a finite set of positive integers X, the *prime graph* $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p, and two distinct vertices p and q are adjacent if and only if there exists an element of X divisible by p and q. For a finite group G, the graph $\Pi(\pi_e(G))$, which we denote by $GK(G)$, is also known as the Gruenberg–Kegel graph of G. Denote the vertex set of $GK(G)$ by $\pi(G)$. The prime graph $\Pi(\mathrm{Vo}(G))$ which we denote by $\Gamma(G)$ in this paper, is called the vanishing prime graph of G. The vanishing prime graph was introduced in [6,7]. The following lemma provides some properties of the vanishing prime graph of a finite group and its relationship with the Gruenberg–Kegel graph. In what follows, we shall denote by $V(\mathcal{G})$ the vertex set of a graph $\mathcal{G}$, and by $n(\mathcal{G})$ the number of connected components of $\mathcal{G}$.

Lemma 2.1 [6,7]. *Let G be an finite group. Then the following hold*:

(1) *If G is solvable, then $\Gamma(G)$ has at most two connected components.*
(2) *If G is non-solvable and $\Gamma(G)$ is disconnected, then G has a unique non-abelian composition factor S, and $n(\Gamma(G)) \leq n(GK(S))$ unless G is isomorphic to A_7.*

Lemma 2.2 ([5, *Proposition* 2.1]). *Let G be a non-abelian simple group and p a prime number. If G is of Lie type, or if $p \geq 5$, then there exists $\chi \in \mathrm{Irr}(G)$ of p-defect zero.*

In the following lemma, we collect some basic remarks relating to the vanishing elements of a group G and the vanishing elements of the quotients of G. We shall freely use these results.

Lemma 2.3 [6,8]. *Let N be a normal subgroup of G.*

(1) *Any character of G/N can be viewed, by inflation, as a character of G. In particular, if $xN \in \mathrm{Van}(G/N)$, then $xN \subseteq \mathrm{Van}(G)$.*
(2) *If $p \in \pi(N)$ and N has an irreducible character of p-defect zero, then every element of N of order divisible by p is a vanishing element of G.*
(3) *If $m \in \mathrm{Vo}(G/N)$, then there exists an integer n such that $mn \in \mathrm{Vo}(G)$.*

Remark 2.4. Let G be a simple group of Lie type. By Lemma 2.2, G has characters of p-defect zero for every prime p and hence by [10, Theorem 8.17] every non-identity element of G is a vanishing element. Hence $\mathrm{Vo}(G) = \pi_e(G) - \{1\}$.

3. On non-solvable VCP-groups

The following results come from Corollary A and Corollary C of [5], which will be used to show that the non-solvable VCP-groups have at least three connected components.

Lemma 3.1. *Let G be a finite group, and let p and q be prime numbers. Then the following hold*:

(1) *If p does not divide any element in* $\mathrm{Vo}(G)$, *then G has a normal Sylow p-subgroup.*
(2) *If every element in* $\mathrm{Van}(G)$ *is a* $\{p, q\}$*-element, then G is solvable.*

Lemma 3.2 *Assume that G is a non-solvable* VCP-*group. Then* $\Gamma(G)$ *has at least three connected components.*

Proof. We assume that G is a non-solvable VCP-group. Using Feit–Thompson's theorem, we conclude that 2 divides some element in $\mathrm{Vo}(G)$. Therefore, it follows by the hypothesis and part (2) of Lemma 3.1 that $\Gamma(G)$ has at least three connected components, and we are done. □

Lemma 3.3 ([6, *Proposition* 2.10]). *Let S be a sporadic simple group, or an alternating group on n letters with* $n \geq 8$. *Then S has an irreducible character* ϕ *which extends to* $\mathrm{Aut}(S)$ *and an element g of order* 6 *such that* $\phi(g) = 0$.

In the following lemma, the method of proof has appeared in Lemma 5.1 of [6]. For the reader's convenience, we include this proof.

Lemma 3.4 *Let G be a group, and N a non-trivial normal subgroup of G. If N is a p-group and* $G/N \cong \mathrm{A}_7$, *then the vertex* 2 *in* $\Gamma(G)$ *is not an isolated point.*

Proof. Recalling Lemma 2.3(1) and Lemma 2.3(3), the vanishing prime graph of a factor group of G is a subgraph of $\Gamma(G)$, hence we may assume that N is minimal normal in G. The graph $\Gamma(\mathrm{A}_7)$ has four vertices (namely 2, 3, 5 and 7).

In order to prove that a prime $q \neq p$ is adjacent to p in $\Gamma(G)$, we shall argue that there exists a p'-element $x = gN \in \mathrm{Van}(G/N)$ of order divisible by q and such that $C_N(x) \neq 1$. Then, taking a non-trivial $y \in C_N(x)$, the element gy has order divisible by pq and $gy \in gN \subseteq \mathrm{Van}(G)$, by Lemma 2.3(1).

Since the dimension of the eigenspace of the eigenvalue 1 is invariant by field extensions, considering also that the dimensions of eigenspaces add up in a direct sum of modules, we can reduce to the case that N is an absolutely irreducible A_7-module. So, we refer to the character tables of A_7 in [11].

If $p \neq 2, 3, 5, 7$, then for every $\phi \in \mathrm{IBr_p}(\mathrm{A}_7) = \mathrm{Irr}(\mathrm{A}_7)$ we have $\langle \phi, 1_{\langle x \rangle} \rangle \neq 0$, for $x = (1, 2)(3, 4, 5, 6)$.
If $p = 3$, then $\langle \phi, 1_{\langle x \rangle} \rangle \neq 0$, for every $\phi \in \mathrm{IBr_p}(\mathrm{A}_7)$ when $x = (1, 2)(3, 4, 5, 6)$.
If $p = 5$, then for $x = (1, 2)(3, 4, 5, 6)$, there exists $\phi \in \mathrm{IBr_p}(\mathrm{A}_7)$ such that $\langle \phi, 1_{\langle x \rangle} \rangle \neq 0$.
If $p = 7$, then for $x = (1, 2)(3, 4, 5, 6)$, there also exists $\phi \in \mathrm{IBr_p}(\mathrm{A}_7)$ such that $\langle \phi, 1_{\langle x \rangle} \rangle \neq 0$.

If $p = 2$, then $\langle \phi, 1_{\langle x \rangle} \rangle \neq 0$, when $\phi(1) \neq 6$ and $x = (1, 2, 3)(4, 5, 6)$ or $x = (1, 2, 3, 4, 5, 6, 7)$ or when $\phi(1) = 6$ and $x = (1, 2, 3)(4, 5, 6)$ or $x = (1, 2, 3, 4, 5)$. □

Recall that a Frobenius group is a product $G = FH$ with $F \trianglelefteq G$, $1 < H < G$, $F \cap H = 1$ and such that two elements $x \in F$ and $y \in H$ commute only if $x = 1$ or $y = 1$ (i.e. H acts fixed-point freely on F by conjugation). In this setting, F is called the kernel and H a complement of G. A classical result, Thompson's nilpotency criterion, implies that F is nilpotent. Moreover, every subgroup of H of order the product of two (possibly equal) primes is cyclic and every Sylow subgroup of H is either cyclic or a generalized quaternion group.

Lemma 3.5. *Let G be a* VCP-*group. Assume that G contains a proper normal subgroup N of odd order and that $G \backslash N \subseteq \mathrm{Van}(G)$. Then G is a solvable group.*

Proof. We may assume that G is of even order. Then there is a Sylow 2-subgroup S of G such that $S \cap N = 1$. It follows by the hypothesis $G \backslash N \subseteq \mathrm{Van}(G)$ that NS is a Frobenius group with kernel N and a complement S. Hence S is a cyclic group or a generalized quaternion group.

If S is cyclic, then G is solvable. Assume that S is a generalized quaternion group. We may assume that N is the normal subgroup of G with the greatest possible odd order. Then the group G/N contains a central involution by [3, Theorem 2]. Hence the hypothesis implies that G/N is a 2-group, and so G is solvable. □

In the following, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We assume that G is a non-solvable VCP-group. Then by Lemma 3.2, $\Gamma(G)$ has at least three connected components. Let N be the solvable radical of G. Then by part (2) of Lemma 2.1, G has a normal series

$$1 \leq N < M \leq G,$$

where G/M is a solvable group, and M/N is a non-cyclic simple group. Now we consider the group $\bar{G}{:=}G/N$. As N is the solvable radical of G, $\bar{G} \leq \mathrm{Aut}(\bar{M})$ and $G/M \leq \mathrm{Out}(\bar{M})$.

Let $\bar{M}$ be a sporadic simple group, or an alternating group on n letters with $n \geq 8$. Then, by Lemma 3.3, $\bar{M}$ has an irreducible character ϕ which extends to $\mathrm{Aut}(\bar{M})$ and an element g of order 6 such that $\phi(g) = 0$. So $g \in \mathrm{Van}(\bar{G})$, and thus $\mathrm{Van}(G)$ contains an element of order divisible by 6, a contradiction.

Now, we assume that $\bar{M} \cong \mathrm{A}_7$. Then $M = G$, as otherwise $\bar{G} \cong \mathrm{S}_7$, and hence G has vanishing elements of order divisible by 6, a contradiction. By the classification theorem of the finite simple groups, we can now suppose that $\bar{M}$ is a simple group of Lie type (note that $\mathrm{A}_5 \cong \mathrm{L}_2(5)$ and $\mathrm{A}_6 \cong \mathrm{L}_2(9)$). Then by Lemma 2.2, for any prime divisor p of $|\bar{M}|$, there exists $\chi_p \in \mathrm{Irr}(\bar{M})$ such that χ_p is of p-defect zero, and so according to Lemma 2.3(2), every element of $\bar{M}$ of order divisible by p is a vanishing element of $\bar{G}$. Hence, $\bar{M}$ is a simple CP-group. Then by [9, Proposition 3], $\bar{M}$ is isomorphic to one of the following groups: $\mathrm{L}_2(q)$, $q \in \{5, 7, 8, 9, 17\}$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$, or $\mathrm{Sz}(32)$. Recall that $\bar{G} \leq \mathrm{Aut}(\bar{M})$, by direct inspection (see [4]), $\bar{G}$ is isomorphic to one of the following groups:

$\mathrm{L}_2(q)$, $q \in \{5, 7, 8, 9, 17\}$, $\mathrm{L}_3(4)$, Sz(8), Sz(32), or M_{10}. Therefore, if $N = 1$, then G satisfies the type (1) of the theorem.

Next, we show that if $G/N \cong \mathrm{A}_7$, then $N = 1$. Otherwise, set $H := O^p(N)$ for some prime p such that $N/H > 1$. Now we consider the group G/H. Recall that $(G/H)/(N/H) \cong G/N \cong \mathrm{A}_7$. Then by Lemma 3.4, the vertex 2 in $\Gamma(G/H)$ is not an isolated point. Since the graph $\Gamma(G/H)$ is a subgraph of $\Gamma(G)$, the vertex 2 in $\Gamma(G)$ is also not an isolated point. Hence we obtain a contradiction, which implies that if $G/N \cong \mathrm{A}_7$, then $N = 1$.

In the following, we assume that $N > 1$. Now, we prove that if K is a normal subgroup of G, $K \leq N$, and N/K is nilpotent, then N/K is a 2-group. Without loss of generality, we may assume that N/K is a r-group for some prime r. We consider the group $\tilde{G} := G/K$. Recall that $\tilde{G}/\tilde{N} \cong G/N$ is isomorphic to one of the following groups: $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, Sz(8), Sz(32), or M_{10}. Therefore, it follows by Remark 2.4 and [4], $\tilde{G}\backslash\tilde{N} \subseteq \mathrm{Van}(\tilde{G})$, and so by Lemma 3.5, we get that $r = 2$, and we are done.

Next, we claim that N is a 2-group. Otherwise, G has a normal series

$$1 \leq T < K < N \leq G,$$

where K/T is a s-group with s an odd prime.

Now we consider the group $\hat{G} := G/T$. As $\hat{N}$ is a $\{2, s\}$-group, $\hat{G}\backslash\hat{N}$ has an element $\hat{g}$ of order t, where t is an odd prime and $t \neq s$. Since $\hat{G}\backslash\hat{N} \subseteq \mathrm{Van}(\hat{G})$, it follows by the hypothesis that the group $\langle\hat{g}\rangle$ acts fixed-point freely on $\hat{N}$, and so $\hat{N}$ is nilpotent. Then, arguing as in the fourth paragraph above, we obtain a contradiction, which implies that N is a 2-group. Since $G\backslash N \subseteq \mathrm{Van}(G)$ and G is a VCP-group. This also implies that G is a CP-group. Now, applying [9, Theorem 6] and [9, p. 198], we complete the proof. □

4. Recognition of simple VCP-groups

Next, we first give a case which forces that $\mathrm{Vo}(G)$ and $\pi_e(G)$ are almost the same.

Lemma 4.1 ([17, *Lemma* 3.2]). *Let G be a finite group, and let N be a normal subgroup of G. Assume that every non-identity element of G/N is a vanishing element of G/N. If N is an elementary abelian p-group for some prime p, then*

$$\pi_e(G) = \mathrm{Vo}(G) \cup \{1, p\}.$$

Moreover, if $\Gamma(G)$ is disconnected, then $p \in V(\Gamma(G))$.

In the following lemma, we will use the following elementary fact: let G and H be two groups, if $\mathrm{Vo}(G) = \pi_e(H) - \{1\}$, then, for any $m \in \mathrm{Vo}(G)$ and any non-identity factor n of m, we have that $n \in \mathrm{Vo}(G)$.

Lemma 4.2. *Let G be a finite group, and let N be a normal subgroup of G. Assume that N is solvable, that every non-identity element of G/N is a vanishing element of G/N and* $\mathrm{Vo}(G) = \mathrm{Vo}(G/N)$. *If $\Gamma(G)$ is disconnected and G/N is recognizable, then $N = 1$.*

Proof. Assume that $N > 1$. Let $1 \leq V < N$ such that N/V is a chief factor of G. Since N is solvable, N/V is an elementary abelian p-group, for some prime p. Now, we consider the group $\tilde{G} := G/V$. Applying Lemma 4.1 in the group $\tilde{G}$, we get

$$\pi_e(\tilde{G}) = \mathrm{Vo}(\tilde{G}) \cup \{1, p\}.$$

Recall that every non-identity element of G/N is a vanishing element of G/N, thus $\mathrm{Vo}(G/N) = \pi_e(G/N) - \{1\}$. Moreover, $\mathrm{Vo}(G) = \mathrm{Vo}(G/N)$, hence we conclude

$$\mathrm{Vo}(G) = \pi_e(G/N) - \{1\} = \mathrm{Vo}(G/N).$$

By the first equation above, we have that all non-identity factors of each element of $\mathrm{Vo}(G)$ are also included in $\mathrm{Vo}(G)$. Notice that any element in $\mathrm{Vo}(\tilde{G})$ is a factor of some element in $\mathrm{Vo}(G)$. Hence we obtain

$$\mathrm{Vo}(\tilde{G}) \subseteq \mathrm{Vo}(G) = \mathrm{Vo}(G/N) = \mathrm{Vo}(\tilde{G}/\tilde{N}).$$

Therefore, we conclude

$$\pi_e(\tilde{G}) = \mathrm{Vo}(\tilde{G}) \cup \{1, p\} \subseteq \mathrm{Vo}(\tilde{G}/\tilde{N}) \cup \{1, p\} \subseteq \pi_e(\tilde{G}).$$

So, we have

$$\pi_e(\tilde{G}) = \mathrm{Vo}(\tilde{G}/\tilde{N}) \cup \{1, p\}.$$

Recall that $\mathrm{Vo}(G/N) = \pi_e(G/N) - \{1\}$ and that $\mathrm{Vo}(G/N) = \mathrm{Vo}(\tilde{G}/\tilde{N})$, we obtain

$$\pi_e(\tilde{G}) = \pi_e(G/N) \cup \{p\}.$$

Since $\Gamma(G)$ is disconnected, it follows by [6, Proposition 4.2] that $V(\Gamma(G)) = \pi(G)$, which implies that $p \in V(\Gamma(G))$. Notice that all non-identity factors of each element of $\mathrm{Vo}(G)$ are included in $\mathrm{Vo}(G)$ and that $\mathrm{Vo}(G) = \pi_e(G/N) - \{1\}$. Thus we have that $p \in \pi_e(G/N)$. Hence we get

$$\pi_e(\tilde{G}) = \pi_e(G/N).$$

As $h(\pi_e(G/N)) = 1$, $\tilde{G} \cong G/N$, a contradiction. Hence $N = 1$, and we are done. □

In the following, we begin to prove Theorem 1.4.

Proof of Theorem 1.4. When M is one of the simple group $\mathrm{L}_2(q)$ for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, or A_7, using [4], we collect $\mathrm{Vo}(M)$ and $V(\Gamma(M))$ in Table 1.

Since $\mathrm{Vo}(G) = \mathrm{Vo}(M)$, according to Table 1, we have $n(\Gamma(G)) \geq 3$ and part (1) of Lemma 2.1 implies that G is a non-solvable VCP-group. Let N be the solvable radical of G. If $N = 1$, then, by Theorem 1.1 and Table 1, we get $G \cong M$. So, we may assume $N > 1$. By Theorem 1.1, N is a 2-group and $G/N \cong \mathrm{L}_2(5)$, $\mathrm{L}_2(8)$, $\mathrm{Sz}(8)$, or $\mathrm{Sz}(32)$. Moreover, as N is a 2-group, we get that $|G| = 2^t \cdot |G/N|$, where t is a positive integer.

By [16,17], the groups $\mathrm{L}_2(5)$, $\mathrm{Sz}(8)$ and $\mathrm{Sz}(32)$ are V-recognizable. Notice that any element in $\mathrm{Vo}(G/N)$ is a factor of some element in $\mathrm{Vo}(G)$. For the other cases of M, considering the fact that $|G| = 2^t \cdot |G/N|$, we get Table 2 from part (2) of Lemma 2.1 and Table 1.

Then, by Table 2, we get that the groups $\mathrm{L}_2(7)$, $\mathrm{L}_3(4)$, A_7 and $\mathrm{L}_2(17)$ are V-recognizable.

Table 1. $\mathrm{Vo}(M)$ and $V(\Gamma(M))$.

M	$\mathrm{Vo}(M)$	$V(\Gamma(M))$
$L_2(5)$	2, 3, 5	2, 3, 5
$L_2(7)$	2, 3, 4, 7	2, 3, 7
$L_2(8)$	2, 3, 7, 9	2, 3, 7
$L_2(9)$	2, 3, 4, 5	2, 3, 5
$L_2(17)$	2, 3, 4, 8, 9, 17	2, 3, 17
$L_3(4)$	2, 3, 4, 5, 7	2, 3, 5, 7
A_7	3, 4, 5, 7	2, 3, 5, 7
$\mathrm{Sz}(8)$	2, 4, 5, 7, 13	2, 5, 7, 13
$\mathrm{Sz}(32)$	2, 4, 5, 25, 31, 41	2, 5, 31, 41

Table 2. Information of G corresponding to M.

M	$\mathrm{Vo}(G)$	$V(\Gamma(G))$	$n(\Gamma(G))$	$G/N (N > 1)$
$L_2(7)$	2, 3, 4, 7	2, 3, 7	3	Impossible
$L_3(4)$	2, 3, 4, 5, 7	2, 3, 5, 7	4	Impossible
A_7	3, 4, 5, 7	2, 3, 5, 7	4	Impossible
$L_2(17)$	2, 3, 4, 8, 9, 17	2, 3, 17	3	Impossible
$L_2(8)$	2, 3, 7, 9	2, 3, 7	3	$L_2(8)$
$L_2(9)$	2, 3, 4, 5	2, 3, 5	3	$L_2(5)$

On the other hand, when M is isomorphic to $L_2(8)$, if $\mathrm{Vo}(G) = \mathrm{Vo}(M)$, then $G/N \cong M$; when M is isomorphic to $L_2(9)$, if $\mathrm{Vo}(G) = \mathrm{Vo}(M)$, then $G/N \cong L_2(5)$.

Assume that M is isomorphic to $L_2(8)$. Then we have that $G/N \cong M$. Recall that $\mathrm{Vo}(G) = \mathrm{Vo}(M)$; thus $\mathrm{Vo}(G) = \mathrm{Vo}(G/N)$. By Remark 2.4, every non-identity element of G/N is a vanishing element of G/N. By [2,13], the groups $L_2(8)$ and $L_2(17)$ are recognizable. Then by Lemma 4.2, we get that $N = 1$, a contradiction.

Now, we assume that M is isomorphic to $L_2(9)$. Note that N is a 2-group and that $G/N \cong L_2(5)$; thus $|G| = 2^t \cdot 3 \cdot 5$. Considering the fact that $G \backslash N \subseteq \mathrm{Van}(G)$, observe that there exists an element a in $G \backslash N$ with $o(a) = 3$. So we see that $C_G(a)$ is of order 3 (note that $|G|_3 = 3$). It follows from the main theorem of [12] that $G \cong NA$, where $A \cong SL_2(4)$ and N is an elementary abelian 2-group and a direct sum of natural $SL_2(4)$-modules. Then the proof is complete. □

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