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Curvature, torsion and the quadrilateral gaps

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Abstract. For a manifold with an affine connection, we prove formulas which infinitesimally quantify the gap in a certain naturally defined open geodesic quadrilateral associated to a pair of tangent vectors u, v at a point of the manifold. We show that the first-order infinitesimal obstruction to the quadrilateral to close is always zero, the second-order infinitesimal obstruction to the quadrilateral to close is $-T(u, v)$, where T is the torsion tensor of the connection, and if $T = 0$, then the third-order infinitesimal obstruction to the quadrilateral to close is $(1/2)R(u, v)(u + v)$ in terms of the curvature tensor of the connection. Consequently, the torsion of the connection, and if the torsion is identically zero, then also the curvature of the connection can be recovered uniquely from knowing all the quadrilateral gaps. In particular, this answers a question of Rajaram Nityananda about the quadrilateral gaps on a curved Riemannian surface. The angles of $3\pi/4$ and $-\pi/4$ radians feature prominently in the answer, along with the value of the Gaussian curvature. This article is essentially self-contained, and written in an expository style.

Keywords. Affine connection; curvature; torsion; geodesics.

Mathematics Subject Classification. 53B05, 53B20.

1. Introduction

Let M be a surface with a Riemannian metric, and let P_0 be a point on M. Let $u \in T_{P_0}M$ be a unit tangent vector. Now consider the following journey on M. To begin with, choose the geodesic starting at P_0 in the direction given by u, and travel along it for a distance s to arrive at a point $P_1(s)$. Next, turn left in $\pi/2$ radians, and travel along the geodesic in that direction for a distance s, to arrive at a point $P_2(s)$. Repeat this twice more: turn left at $P_2(s)$ in $\pi/2$ radians and travel along the geodesic in that direction for a distance s to arrive at a point $P_3(s)$, and then turn left at $P_3(s)$ in $\pi/2$ radians and travel along the geodesic in that direction for a distance s to finally arrive at a point $P_4(s)$. This defines an open rectangle on M, with vertices $P_0, \dots, P_4$, whose legs are the above four geodesic segments. We assume that we have chosen an orientation for M around P_0, so that the left turns make unambiguous sense.

If the surface M is flat, then for small values of s, we will be back at the starting point, that is, our journey will be along a geodesic quadrilateral with $P_4(s) = P_0$. But if M is

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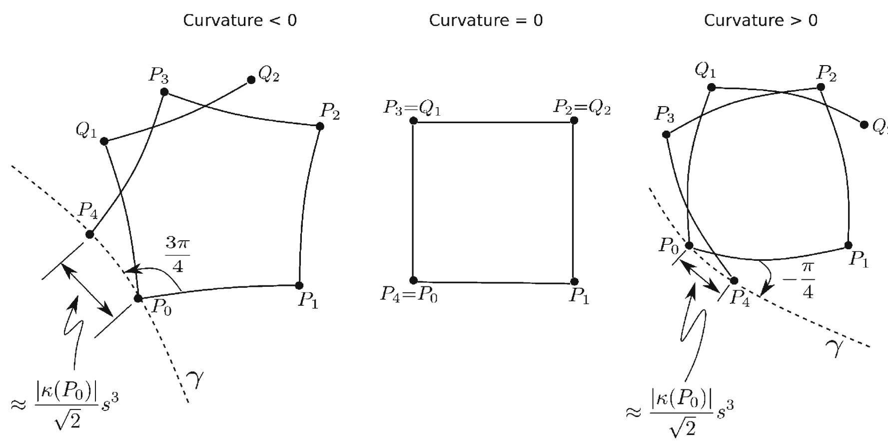
not flat, then we do not come back, that is, $P_4(s) \neq P_0$. One may say that the quadrilateral does not close, as it has a gap. Rajaram Nityananda asked for a quantification of this gap for a small s.

We have another geodesic starting at P_0 with initial direction $v \in T_{P_0}M$, where (u, v) is a right-handed orthonormal basis for $T_{P_0}M$. Travel for a distance s along it, to come to a point $Q_1(s)$. Turn right at $Q_1(s)$ in $\pi/2$ radians and travel along the geodesic in that direction for a distance s to arrive at a point $Q_2(s)$. Again, if M is flat, then $Q_2(s) = P_2(s)$ for a small s. Otherwise, we have an open rectangle with successive vertices Q_2, Q_1, P_0, P_1, P_2 joined by geodesic segments of length s. Again, one can ask what is the gap between $P_2(s)$ and $Q_2(s)$.

Here is our answer. We show that

$$\lim_{s\to 0} \frac{P_4(s) - P_0}{s^3} = \lim_{s\to 0} \frac{Q_2(s) - P_2(s)}{s^3} = \frac{\kappa(P_0)}{2}(u - v)$$

where $\kappa(P_0)$ denotes the Gaussian curvature of M at P_0.



The difference $P_4(s) - P_0$ or $Q_2(s) - P_2(s)$ between any pair of points of M can be understood in terms of a smooth embedding of a neighbourhood of P_0 into an affine space $\mathbb{R}^n$, which makes the difference a vector in the vector space $\mathbb{R}^n$. In particular, the points $P_4(s)$ trace a curve $\gamma(\lambda)$, parameterized by $\lambda = s^3$, and this has the tangent vector $(\kappa(P_0)/2)(u - v)$ at $\lambda = 0$.

Equivalently, for any smooth function f defined in a neighbourhood of P_0, we have

$$\lim_{s\to 0} \frac{f(P_4(s)) - f(P_0)}{s^3} = \lim_{s\to 0} \frac{f(Q_2(s)) - f(P_2(s))}{s^3} = \frac{\kappa(P_0)}{2}(u - v)(f).$$

In words, the sizes of both the gaps, that is, the distances from P_0 to $P_4(s)$ or from $P_2(s)$ to $Q_2(s)$ regarded as functions of s, are equal to $\frac{1}{\sqrt{2}}|\kappa(P_0)|s^3$ up to third order terms in s (that is, modulo s^4). Secondly, up to third order terms in s, the gaps make an angle of $-\pi/4$ radians with the first leg of the quadrilateral when $\kappa(P_0) > 0$, and the opposite angle of $3\pi/4$ radians when $\kappa(P_0) < 0$, regardless of the magnitude $|\kappa(P_0)|$ of the curvature.

The above result may be compared with the formula of Bertrand and Puiseux, which gives the circumference $C(r)$ of an infinitesimally small circle on M centered at $P_0 \in M$ with geodesic radius r. This formula says that the deviation of $C(r)$ from the Euclidean value $2\pi r$ is infinitesimally small of third order in r. More precisely,

$$\lim_{r\to 0} \frac{2\pi r - C(r)}{r^3} = \frac{\pi \kappa(P_0)}{3},$$

that is, $2\pi r - C(r) = \frac{1}{3}\pi\kappa(P_0)r^3$ up to third order terms in r.

We directly verify in Section 3 the result for the sphere $x^2 + y^2 + z^2 = r^2$ in the Euclidean space $\mathbb{R}^3$ (which has constant positive curvature $1/r^2$) and for the hyperboloid $t^2 - x^2 - y^2 = r^2$ in the Minkowski space $\mathbb{R}^{1,2}$ (which has constant negative curvature $-1/r^2$) by respectively using orthogonal or Lorentz transformations. We sketch a heuristic argument in Section 3, based on Riemann normal coordinates, to go from the case of constant curvature to that of arbitrary metrics on surfaces. We will not make this argument rigorous, as the formula $\lim_{s\to 0}(P_4(s) - P_0)/s^3 = (\kappa(P_0)/2)(u - v)$ is just the 2-dimensional case of the more general Theorem 1.1, which we now formulate.

Let M be a differential manifold of any dimension d, equipped with an affine connection ∇, that is, a connection on the tangent bundle TM. As a special case, M may be a (pseudo-)Riemannian manifold, and ∇ the connection induced by the given metric (which is the unique symmetric connection for which the covariant derivative of the metric is identically zero). We now generalize Nityananda's question to this general setup, where notions such as length and perpendicularity are not available. To any ordered triple (P, u, v) consisting of a point $P \in M$ and a pair of tangent vectors $u, v \in T_P M$, and any real number s such that $|s|$ is small enough (depending on (P, u, v)), we associate a new such triple $\mathbb{T}_s(P, u, v) = (P', u', v')$ constructed as follows: Let γ be the unique affinely parameterized geodesic starting at P with initial tangent vector u, so that $\gamma(0) = P$ and $\dot{\gamma}(0) = u$. We define

$$P' = \gamma(s), \; u' = v(s) \quad \text{and} \quad v' = -u(s),$$

where $u(s), v(s) \in T_{P'}M$ are the parallel transports of u, v along γ. We will sometimes write the triple (P', u', v') as $(P'(s), u'(s), v'(s))$ to make the dependence on s explicit. As an example, $\mathbb{T}_0(P, u, v) = (P'(0), u'(0), v'(0)) = (P, v, -u)$. The point P will be called as the *location* of the triple (P, u, v). We now begin with a point $P_0 \in M$ and $u, v \in T_{P_0}M$ and apply the operator $\mathbb{T}_s$ iteratively to define new points $P_1, P_2, \ldots$, where P_1 is the location of $\mathbb{T}_s(P, u, v)$, P_2 is the location of $\mathbb{T}_s(\mathbb{T}_s(P, u, v))$, and in general, P_n is the location of the triple $(\mathbb{T}_s)^n(P, u, v)$. Note that the operator $\mathbb{T}_s$ is invertible (but its two-sided inverse $\mathbb{T}_s^{-1}$ is not equal to $\mathbb{T}_{-s}$, in general). We apply the operator $\mathbb{T}_s^{-1}$ iteratively to define new points $Q_1, Q_2, \ldots$, where Q_1 is the location of $T_s^{-1}(P, u, v)$, Q_2 is the location of $\mathbb{T}_s^{-1}(\mathbb{T}_s^{-1}(P, u, v))$, and so on. In these terms, the quadrilateral gap of Nityananda is the gap between P_0 and $P_4(s)$ (or between $P_2(s)$ and $Q_2(s)$) when M is Riemannian of dimension 2 and (u, v) is an orthonormal basis for $T_{P_0}M$. The Theorem 1.1 describes these gap infinitesimally.

Theorem 1.1. *Let M be a smooth manifold equipped with an affine connection ∇. Then with notation as above, we have the following*:

(1) *If* $P_0 \in M$ *and* $u, v \in T_{P_0}M$, *then*

$$\lim_{s\to 0}\frac{P_4(s)-P_0}{s^2}=\lim_{s\to 0}\frac{P_2(s)-Q_2(s)}{s^2}=-T(u,v)\in T_{P_0}M,$$

where $T(u,v)=\nabla_u v-\nabla_v u-[u,v]$ *is the torsion tensor of* ∇.

(2) *If* $T\equiv 0$, *that is*, ∇ *is symmetric*, *then*

$$\lim_{s\to 0}\frac{P_4(s)-P_0}{s^3}=-\lim_{s\to 0}\frac{P_2(s)-Q_2(s)}{s^3}=\frac{1}{2}R(u,v)(u+v)\in T_{P_0}M,$$

where $R(u,v)=\nabla_u\nabla_v-\nabla_v\nabla_u-\nabla_{[u,v]}\in \mathrm{End}(T_{P_0}M)$ *is the curvature tensor of* ∇.

Our answer to Nityananda's question now follows by taking (u, v) to be an orthonormal basis for $T_{P_0}M$ when M is a surface with a Riemannian metric and ∇ is the Riemannian connection, and noting that in this case, $T(u, v) = 0$ and $R(u, v)$ is the skew-symmetric operator $\begin{pmatrix} 0 & \kappa_0 \\ -\kappa_0 & 0 \end{pmatrix}$ in terms of the basis (u, v), where κ_0 is the Gaussian curvature of M at P_0.

Thus, the torsion tensor T can be read off from the limit as $s \to 0$ of gaps$/s^2$. If the torsion is zero then the elementary Lemma 4.1 shows that the curvature tensor R at a point P_0 is uniquely determined by the function $(u, v) \mapsto R(u, v)(u + v)$, and so the curvature tensor R can be uniquely recovered from the limit as $s \to 0$ of gaps$/s^3$.

A heuristic argument. Without knowing Theorem 1.1, one could have argued as follows. We can expand the gap $Q_2(s) - P_2(s)$ as a power series in s, to write

$$Q_2(s)-P_2(s)=A_0(u,v)+sA_1(u,v)+s^2A_2(u,v)+s^3A_3(u,v) \bmod s^4.$$

The vectors $A_i(u, v)$ will have to be homogeneous polynomials of total degree i in the variables u, v, as $A_i(u, v)$ is the coefficient of s^i, and moreover, they should change sign when u and v are interchanged. As we know that $Q_2(0) = P_2(0) = P_0$, we must have $A_0 = 0$. The A_i's are to be made from ∇ alone. The obvious candidate for A_1 is $c(u - v)$ for some constant c. But for the Euclidean space itself, the gap is zero, so we must have $c = 0$. This means $A_1(u, v) = 0$. An obvious candidate for $A_2(u, v)$ is $T(u, v)$, the torsion. A degree 3 homogeneous candidate for $A_3(u, v)$ is $R(u, v)u$, but that is not skew-symmetric. Applying the idempotent projector for skew-symmetry in the group ring $\mathbb{R}[S_2]$ of the permutation group S_2 to $R(u, v)u$, we get the candidate $\frac{1}{2}(R(u, v)u - R(v, u)v)$ for A_3, which is homogeneous of degree 3 and skew-symmetric in u, v. But this equals $\frac{1}{2}R(u, v)(u + v)$, as $R(u, v)$ is skew-symmetric in u, v. The above argument (made in hindsight) is only suggestive: for example, there could be constant numerical coefficients c_i that multiply the candidates A_i, though the examples in Section 2 would show us that there is no further coefficient that multiplies $\frac{1}{2}R(u, v)(u + v)$. The actual proof of Theorem 1.1 is given in Section 5, and it does not refer to the above argument.

In our discussion so far, we had chosen a pair of vectors $u, v \in T_{P_0}M$, and we had parallel translated this pair. More generally, we can choose a basis $(u_1, \dots, u_d)$ for $T_{P_0}M$, where $d = \dim(M)$, and parallel transport it. This suggests that we should consider paths in the frame bundle $\pi : E \to M$ of M, which is a principal $GL(d)$-bundle. Recall that a connection ∇ on TM naturally gives rise to a vector field ξ on the total space of TM, whose flow is called as the geodesic flow of ∇. Similarly, ∇ naturally gives rise to d different horizontal vector fields $\xi_1, \dots, \xi_d$ on the total space of E. Here, recall that giving a connection ∇ on TM is equivalent to giving an invariant rank d vector subbundle D of TE, which is complementary to the kernel of the derivative map $\pi_* : TE \to \pi^*TM$

(the 'vertical' subbundle) at each point of E. For $y \in E$, an element of $T_y E$ is called a horizontal vector if it lies in the fiber $D_y \subset T_y E$. If $x \in M$ and if $(u_1, \ldots, u_d)$ is a basis for $T_x M$, so that we can regard $y = (u_1, \ldots, u_d)$ as a point of E lying over x, then these horizontal vector fields ξ_i are uniquely defined by the requirement that

$$\pi_*(y)\xi_i = u_i,$$

where $\pi_*(y) : T_y E \to T_x M$ is the derivative of π. These vector fields have been considered in the book of Nomizu (see Chapter 3, Section 1 of [1]) under the name 'basic vector fields'. With this definition, if we start at a point P_0 and parallel transport a basis $y = (u_1, \ldots, u_d)$ along the affinely parameterized geodesic γ through P_0 with $\gamma(0) = P_0$ and initial tangent vector $\dot{\gamma}(0) = u_i$ to the point $P_1 = \gamma(s)$, and if $y_1 = (u_1(s), \ldots, u_d(s))$ is the basis of $T_{P_1} M$ obtained by parallel translating the basis $y = (u_1, \ldots, u_d)$ along γ, then $y_1 \in E$ is exactly the point obtained by following the flow on E of the vector field ξ_i for parameter value s. This translates the question of geodesic quadrilateral gaps on M into a question about non-commutativity of flows of vector fields on the manifold E. The geodesic gap corresponding to (P_0, u_m, u_n) will just be the projection under $\pi : E \to M$ of the gap in E corresponding to the flows of vector fields ξ_m and ξ_n.

Let ξ and η be vector fields on a manifold M and let φ_s and ψ_s be the corresponding flows. Given $P_0 \in M$, for a small enough s, we can define the points

$$P_1 = \varphi_s(P_0), \quad P_2 = \psi_s \circ \varphi_s(P_0), \quad P_3 = \varphi_{-s} \circ \psi_s \circ \varphi_s(P_0),$$
$$P_4 = \psi_{-s} \circ \varphi_{-s} \circ \psi_s \circ \varphi_s(P_0)$$

and

$$Q_1 = \psi_s(P_0) \quad \text{and} \quad Q_2 = \varphi_s \circ \psi_s(P_0)$$

in M. The points P_0, P_1, P_2, P_3, P_4 and Q_{-2}, Q_{-1}, P_0, P_1, P_2 are the successive vertices of open quadrilaterals, whose sides are the flow lines of $\pm\xi$ and $\pm\eta$. The following is a well-known fundamental result, that interprets the bracket of two vector fields in terms of the gaps $P_4(0) - P_0$ and $P_2(s) - Q_2(s)$.

Lemma 1.2. *With notation as above*,

$$\lim_{s \to 0} \frac{P_4(s) - P_0}{s^2} = \lim_{s \to 0} \frac{P_2(s) - Q_2(s)}{s^2} = [\xi, \eta]_{P_0} \in T_{P_0} M.$$

For the sake of completeness, we give a proof of the above in Section 3. This proof introduces a certain trick which is important in the proof of Theorem 1.1.

We now apply the above lemma to the flows of ξ_m and ξ_n on E, starting at a point $y = (u_1, \ldots, u_d) \in E$ over $x = P_0 \in M$. The projection under $\pi : E \to M$ of the corresponding flow quadrilateral gaps in E are the geodesic quadrilateral gaps in M for the triple (P_0, u_m, u_n). Thus, we need to compute the bracket $[\xi_m, \xi_n]$ in E and its projection under $\pi_* : T_y E \to T_x M$. This is given by the following result proved in Sect. 4, which follows from the definitions by an easy computation. We expect this result – or some equivalent form – to be well known to experts.

Theorem 1.3. *Let $\pi : E \to M$ be the frame bundle associated to TM. Let $\xi_1, \ldots, \xi_d$ be the frame flow fields on the total space of E, associated to the affine connection ∇ on M.*

Let $x \in M$ and let $y = (u_1, \ldots, u_d) \in E$ with $\pi(y) = x$, so that $\pi_*(y)(\xi_i) = u_i$ for all i. Then the following holds:

(1) *The torsion of ∇ is given by $T(u_i, u_j) = -\pi_*(y)[\xi_i, \xi_j]$.*
(2) *In particular, if ∇ is symmetric, then $\pi_*[\xi_i, \xi_j] = 0$, so the bracket $[\xi_i, \xi_j]$ is a vertical vector field on E, hence can be naturally regarded as a section of the pullback to E of the bundle $\mathrm{Ad}(E) = \underline{\mathrm{End}}(TM)$ on M.*
(3) *If ∇ is symmetric, then the curvature of ∇ is given as follows. For any $x \in M$ and $y = (u_1, \ldots, u_d) \in E$ with $\pi(y) = x$, we have $R(u_i, u_j)_x = -[\xi_i, \xi_j]_y$ as elements of $\mathrm{End}(T_x M)$. As global sections of $\pi^*\underline{\mathrm{End}}(TM)$ over E, we have the equality $R(\pi_*\xi_i, \pi_*\xi_j) = -[\xi_i, \xi_j]$.*

The fact that the geodesic gap for (P_0, u_m, u_n) in M is the projection under $\pi : E \to M$ of the flow gap for ξ_m, ξ_n in E, together with Lemma 1.2, shows that the statement (1) of Theorem 1.3 gives a proof of the statement (1) of Theorem 1.1.

The verticality of $[\xi_i, \xi_j]$ for symmetric connections shows that up to second infinitesimal order in s, the geodesic quadrilateral gap on M is zero for symmetric connections. In fact, we proved Theorem 1.3 before proving Theorem 1.1, and this told us that the geodesic quadrilateral gap on M for a symmetric connection is a phenomenon of third or higher infinitesimal order in s. This is what motivated the third-order Taylor series calculation which is at the heart of the proof of Theorem 1.1.

In conclusion, we can therefore say that the first-order infinitesimal obstruction to the quadrilaterals to close is always zero, the second-order infinitesimal obstruction to the quadrilateral associated to $u, v \in T_{P_0}M$ to close is $T(u, v)$, and if ∇ is symmetric, then the third-order infinitesimal obstruction to this quadrilateral to close is $(1/2)R(u, v)(u + v)$. If $T \equiv 0$, then Lemma 4.1 shows how to uniquely recover the curvature tensor from the function $(u, v) \mapsto R(u, v)(u + v)$, showing that the torsion of a connection, and in case that is zero, the curvature of a connection, can be uniquely recovered from the knowledge of all the gaps.

If both the torsion tensor and the curvature tensor are identically zero, then by Theorem 1.3, the horizontal distribution on E defined by the connection is *involutive*, that is, closed under bracket. Hence by the Frobenius theorem, it follows that the connection is *integrable*, that is, E locally admits horizontal sections, and therefore there are no further obstructions for small-sized geodesic quadrilaterals to close, where the upper bound on $|s|$ depends on the starting triple (P, u, v). However, if such an (M, ∇) is not geodesically complete, then there can still be a gap in a geodesic quadrilateral, *even if* the quadrilateral exists. For example, if we take M to be the universal cover of $\mathbb{R}^2 - \{(0, 0)\}$, with Riemannian metric pulled up from the Euclidean metric $dx^2 + dy^2$ below, then the pull-back of any square in $\mathbb{R}^2 - \{(0, 0)\}$ which has winding number 1 around $(0, 0)$ is an open geodesic quadrilateral in M. But if ∇ has zero torsion, zero curvature and if moreover M is geodesically complete, then there is not even a global obstruction, that is, the geodesic quadrilaterals will close for all (P, u, v) and s. From this, it can be seen that such a pair (M, ∇) will be isomorphic to a quotient of the affine space $\mathbb{R}^d$ with its natural constant connection.

This article is essentially self-contained and written in a semi-expository style, with the hope that young mathematicians who are not necessarily specialists may find it readable and possibly instructive. Given its elementary nature, the result – in the Riemannian context – could well be a part of the 'folklore' of the subject.

2. Spheres and hyperboloids

In this section, we directly verify the Theorem 1.1 for (i) the sphere $x^2 + y^2 + z^2 = r^2$ in the Euclidean space $\mathbb{R}^3$ with metric $\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$, which has a constant curvature $1/r^2$, and (ii) the hyperboloid $t^2 - x^2 - z^2 = r^2, t > 0$ in the Minkowski space $\mathbb{R}^{1,2}$ with metric $-\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2$, which has a constant curvature $-1/r^2$.

2.1 *Sphere in Euclidean* $\mathbb{R}^3$

Let $\mathbb{R}^3$ be given the orientation in which the standard basis (e_1, e_2, e_3) is right-handed. We define $M \subset \mathbb{R}^3$ by the equation $x^2 + y^2 + z^2 = 1$. Let M be given the orientation under which (e_2, e_3) is a right-handed basis at $e_1 \in M$. The metric tensor on M is induced from the Euclidean metric tensor $\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$ on $\mathbb{R}^3$. The data (P, u, v) that consists of a point $P \in M$ and a right-handed orthonormal basis (u, v) for $T_P M$ is the same as a right-handed orthonormal basis (P, u, v) for $\mathbb{R}^3$, and so all such triples form the manifold N which is a principal homogeneous space under the action of $SO(3)$. Under this action, a group element $A \in SO(3)$ takes any triple (P, u, v) to the new triple (AP, Au, Av). We will identify N with $SO(3)$ by representing each triple (P, u, v) by the matrix $X \in SO(3)$ whose columns are P, u and v respectively, so that in the parlance of the Introduction, the location of X is the first column of X.

For $s \in \mathbb{R}$ and $X = (P, u, v) \in N$, recall that $\mathbb{T}_s(X)$ is the new triple (P', u', v') as defined in the Introduction. As the action of $SO(3)$ on M preserves the metric, it preserves the geodesics and parallel transport of tangent vectors. Hence $\mathbb{T}_s(X) = \mathbb{T}_s(XI) = X\mathbb{T}_s(I)$, which shows that the action of $\mathbb{T}_s$ on any X is given by right multiplication by $A(s) = \mathbb{T}_s(I)$.

We now evaluate the matrix $A(s) = \mathbb{T}_s(I)$. The triple (P_0, u, v) corresponding to I has $P_0 = (1, 0, 0) = e_1,\ u = (0, 1, 0) = e_2,\ v = (0, 0, 1) = e_3$. The geodesics on M are the great circles, and the distance along these is given by the angle subtended at the origin in $\mathbb{R}^3$. Hence it is immediate that

$$A(s) = \begin{pmatrix} \cos(s) & 0 & \sin(s) \\ \sin(s) & 0 & -\cos(s) \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence the vertices P_i of the geodesic quadrilateral are the first columns of the powers $A(s)^i$, and the vertices Q_i are the first columns of the negative powers $A(s)^{-i}$. This gives

$$P_2(s) = (\cos^2(s), \cos(s)\sin(s), \sin(s)),$$
$$Q_2(s) = (\cos^2(s), \sin(s), \cos(s)\sin(s)),$$

hence $Q_2(s) - P_2(s) = (0, \sin(s)(1 - \cos(s)), - \sin(s)(1 - \cos(s)))$. It follows that

$$\lim_{s\to 0} \frac{Q_2(s) - P_2(s)}{s^3} = (0, 1/2, -1/2) = \frac{1}{2}(u - v),$$

where $u, v \in T_P M$ are the vector e_2, e_3. This proves the result for the gap between $P_2(s)$ and $Q_2(s)$.

The first column of $A(s)^4$ is

$$P_4 = \begin{bmatrix} \cos^4(s) + 2\cos(s)\sin^2(s) \\ \cos^3(s)\sin(s) - \cos^2(s)\sin(s) + \sin^3(s) \\ \cos^2(s)\sin(s) - \cos(s)\sin(s) \end{bmatrix}.$$

This implies that

$$\lim_{s\to 0} \frac{P_4(s) - P_0}{s^3} = (0, 1/2, -1/2) = \frac{1}{2}(u - v).$$

Finally, to get the result for a sphere $x^2 + y^2 + z^2 = r^2$, which has constant Gaussian curvature $1/r^2$, we replace the variable s by s/r in $A(s)$, and multiply the first column of a power of $A(s/r)$ by r to get the vertices P_i, Q_i. This gives

$$\lim_{s\to 0} \frac{P_4(s) - P_0}{s^3} = \lim_{s\to 0} \frac{Q_2(s) - P_2(s)}{s^3} = (0, 1/2r^2, -1/2r^2) = \frac{\kappa}{2}(u - v).$$

2.2 *Hyperboloid in Minkowskian* $\mathbb{R}^{1,2}$

Let $\mathbb{R}^{1,2}$ be the Minkowski space with pseudo-Riemannian metric $-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$, and orientation chosen such that the basis $e_t = (1, 0, 0)$, $e_x = (0, 1, 0)$, $e_y = (0, 0, 1)$ is right-handed. Let $M \subset \mathbb{R}^{1,2}$ be defined by $t^2 - x^2 - y^2 = 1$ and $t > 0$. The induced metric tensor g on M is positive definite, and has constant Gaussian curvature -1. Let M be given the orientation under which the basis (e_x, e_y) of $T_{e_t}M$ is right-handed.

Note that M has a transitive free action of the proper orthochronous Lorentz group $L_+^\uparrow$ which is the connected component of $SO(1, 2)$. Analogous to the previous example, all triples (P, u, v), where $P \in M$ and $(u, v) \in T_P M$ is a right-handed orthonormal basis, form a principal $L_+^\uparrow$-space. Again, such a triple (P, u, v) can be identified with the matrix X in $L_+^\uparrow$ whose columns are P, u, v respectively. Thus, N can be identified with $L_+^\uparrow$, and the action of $L_+^\uparrow$ becomes left multiplication. The action of $L_+^\uparrow$ preserves the metric, the geodesics, and parallel transport of vectors, as before. So once again, we take I as our base triple, and find that if we put $B(s) = \mathbb{T}_s(I)$, then for any triple X, we must have $\mathbb{T}_s(X) = XB(s)$.

As the geodesics and parallel transport have a simple description starting from the triple I, we can see that

$$B(s) = \begin{pmatrix} \cosh(s) & 0 & -\sinh(s) \\ \sinh(s) & 0 & -\cosh(s) \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence starting from the triple $(P_0 = e_t, u = e_x, v = e_y)$, the vertices P_i of the geodesic quadrilateral are the first columns of the powers $B(s)^i$, and the vertices Q_i are the first columns of the negative powers $B(s)^{-i}$. This gives

$$P_2 = (\cosh^2(s), \cosh(s)\sinh(s), \sinh(s)),$$
$$Q_2 = (\cosh^2(s), \sinh(s), \cosh(s)\sinh(s))$$

and

$$P_4 = \begin{bmatrix} \cosh^4(s) - 2\cosh(s)\sinh^2(s) \\ \cosh^3(s)\sinh(s) - \cos^2(s)\sin(s) - \sin^3(s) \\ \cosh^2(s)\sinh(s) - \cosh(s)\sinh(s) \end{bmatrix}.$$

Now a direct calculation gives the results

$$\lim_{s\to 0} \frac{P_4(s) - P_0}{s^3} = \lim_{s\to 0} \frac{Q_2(s) - P_2(s)}{s^3} = -\frac{1}{2}(u - v).$$

A scaling by the factor r, where $B(s)$ is replaced by $B(s/r)$ and the points P_i and Q_i are replaced by r-times the first column of powers of $B(s/r)$ gives the result for the hyperboloid defined by $t^2 - x^2 - y^2 = r^2$ in $\mathbb{R}^{1,2}$, which has curvature $-1/r^2$.

2.3 *The case of an arbitrary surface metric g*

Let $P_0 \in M$, and let the Gaussian curvature of the given metric g on M take the value $\kappa_0 = \kappa(P_0)$ at P_0. There exists a coordinate neighbourhood U of P_0 with local coordinates x, y, known as Riemann normal coordinates centered at P_0, such that $P_0 = (0, 0)$, and the metric tensor takes the form

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 - \frac{\kappa_0}{3}(y^2\mathrm{d}x^2 - 2xy\mathrm{d}x\mathrm{d}y + x^2\mathrm{d}y^2) \\ +h_1(x, y)\mathrm{d}x^2 + h_2(x, y)\mathrm{d}x\mathrm{d}y + h_3(x, y)\mathrm{d}y^2,$$

where each $h_i(x, y)$ is of order ≥ 3 in x, y, that is, $h_i(x, y) \in \mathfrak{m}^3 \subset C^\infty(U)$ where $\mathfrak{m} \subset C^\infty(U)$ is the maximal ideal generated by x, y, (which is the set of all C^∞-functions on U that vanish at P_0). Hence, up to terms of second-order (means modulo $\mathfrak{m}^3$), the pair (U, g) is the same as the pair (U, g') where g' is a Riemannian metric on U with constant Gaussian curvature κ_0, as in Riemann normal coordinates, both g and g' have the common form

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 - \frac{\kappa_0}{3}(y^2\mathrm{d}x^2 - 2xy\mathrm{d}x\mathrm{d}y + x^2\mathrm{d}x^2) \bmod \mathfrak{m}^3\mathrm{Sym}^2(T^*M).$$

One can heuristically argue from this that the points $P_i(s)$, $Q_i(s)$ for the two metrics g and g' are congruent modulo s^4, and consequently the quadrilateral gaps $P_4(s) - P_0$ or $Q_2(s) - P_2(s)$ for the two metrics are congruent modulo s^4. Hence the validity of the gap formula $P_4(s) - P_0 = Q_2(s) - P_2(s) = s^3(\kappa_0/2)(u - v) \bmod s^4$ for constant curvature metrics, which we verified above, implies its validity for all Riemannian surfaces. We do not make this argument rigorous here, instead, we deduce the validity of the gap formula for arbitrary Riemannian surfaces from the Theorem 1.1, as explained in the Introduction.

3. Brackets, flows and gaps

Taylor expansion and equivalence modulo s^n. We recall some elementary facts about rings of real-valued smooth functions. All smooth functions on $(-a, a)$ that vanish at $s = 0$ form the principal ideal $(s) = sC^\infty(-a, a)$ in the ring $C^\infty(-a, a)$, and all smooth functions $f(s, t)$ on $D = (-a, a) \times (-a, a)$ that vanish at $(0, 0)$ form the ideal $(s, t) \subset C^\infty(D)$. If W is any open subset of some $\mathbb{R}^n$ (or more generally, a smooth manifold), then all smooth $f : (-a, a) \times W \to \mathbb{R}$ that vanish on $\{0\} \times W$ form the principal ideal $(s) = sC^\infty((-a, a) \times W)$ in the ring $C^\infty((-a, a) \times W)$, and all smooth functions f on $D \times W$ that vanish at $\{(0, 0)\} \times W$ form the ideal $(s, t) \subset C^\infty(D \times W)$. The following lemma lists some frequently used elementary facts.

Lemma 3.1.

(1) *Taylor expansion: For any smooth f on $(-a,a)\times W$ and any $n \geq 0$, there exist unique functions $g_0,\dots,g_n \in C^\infty(W)$ such that $f = g_0 + g_1 s + \cdots + g_n s^n \bmod (s^{n+1})$ in the ring $C^\infty((-a,a)\times W)$, where $g_0(w) = f(0,w)$ and $g_i(w) = \frac{1}{i!}\frac{\partial^i f}{\partial s^i}(0,w)$ for $i = 1,\dots,n$.*

(2) *In particular, if for each $w_0 \in W$ we have $f(s,w_0) \in s^n C^\infty(-a,a)$, then $f \in s^n C^\infty((-a,a)\times W)$.*

(3) *Both the above statements hold with $(-a,a)$ replaced by D, where we have the Taylor expansion*

$$f = \sum_{0\leq j+k\leq n} h_{jk}\, s^j t^k \bmod (s,t)^{n+1}$$

in the ring $C^\infty(D\times W)$, where

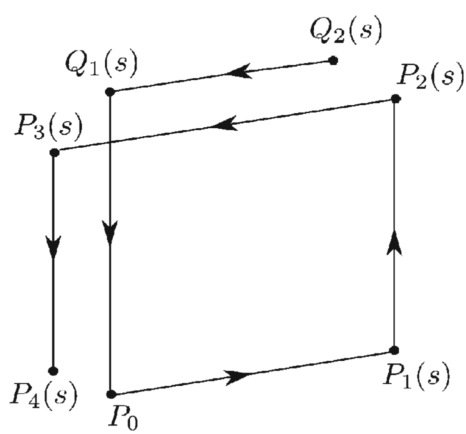
$$h_{jk}(w) = \frac{1}{j!k!}\frac{\partial^{j+k} f}{\partial s^j \partial t^k}(0,0,w) \in C^\infty(W).$$

(4) *For $a,b > 0$, all functions on $U = (-a,a)\times(-b,b)$ that vanish on the diagonal $\Delta \subset U$ (which is defined by $s = t$) form the principal ideal $(s-t)$ in $C^\infty(U)$. A similar statement holds for $\Delta\times W \subset U\times W$.*

Brackets and gaps in flows. For vector fields ξ and η on a differential manifold M, the bracket $[\xi,\eta] = \xi\circ\eta - \eta\circ\xi$ can be seen in terms of two different kinds of quadrilateral gaps that are related to their flows. Let φ_s and ψ_s be the respective flows, defined locally for small values of s. In particular, if $U \subset M$ is an open subset whose closure is compact, then there exists some $a > 0$ such that both φ_s and ψ_s are defined as smooth maps $(-a,a)\times U \to M$.

If $P_0 \in M$ and if s is small enough, the following construction makes sense. Let $P_1(s) = \varphi_s(P_0)$, $P_2(s) = \psi_s(P_1(s))$, $P_3(s) = \varphi_{-s}(P_2(s))$ and $P_4(s) = \psi_{-s}(P_3(s))$. Let $Q_1(s) = \psi_s(P_0)$ and $Q_2(s) = \varphi_s(Q_1(s))$. These points form two open quadrilaterals.

The first open quadrilateral has the five vertices $P_0, P_1(s),\dots,P_4(s)$, with successive vertices joined by integral curves of ξ, η, $-\xi$ and $-\eta$, respectively. The second open quadrilateral has the five successive vertices $Q_2(s)$, $Q_1(s)$, P_0, $P_1(s)$, $P_2(s)$, with successive vertices joined by integral curves of $-\xi$, $-\eta$, ξ and η, respectively. As in general the two flows may not commute, $P_4(s) \neq P_0$ and $P_2(s) \neq Q_2(s)$, in general.



Given any point $x \in M$, there exists a real number $a > 0$ and a neighbourhood W of x in M, such that for any $s \in (-a, a)$ and $P_0 \in W$, the corresponding points $P_1(s)$, $P_2(s)$, $P_3(s)$, $P_4(s)$, $Q_1(s)$, $Q_2(s)$ are defined and are smooth functions from $(-a, a) \times W$ to M. This is a consequence of the basic existence and uniqueness theorem for ordinary differential equations.

Given any smooth function f in a neighbourhood of $x \in M$, we now define the following two functions on $(-a, a) \times W$ where $a > 0$ and W is a neighbourhood of x in M, where both a and W are chosen to be sufficiently small. The *flow gap functions* $F_{\mathrm{I}}(P_0, \xi, \eta, f, s)$ and $F_{\mathrm{II}}(P_0, \xi, \eta, f, s)$ are the functions from $(-a, a) \times W$ to M defined by

$$F_{\mathrm{I}}(P_0, \xi, \eta, f, s) = f(P_4(s)) - f(P_0), \text{ and}$$
$$F_{\mathrm{II}}(P_0, \xi, \eta, f, s) = f(P_2(s)) - f(Q_2(s)).$$

We now rephrase the Lemma 1.2 in terms that will be more useful to us.

Lemma 3.2. *With notation as above, we have the following relations in the ring* $C^\infty((-a, a) \times W)$ *of all smooth real functions on* $(-a, a) \times W$, *where* (s^3) *is the ideal consisting of all multiples of* s^3.

(*i*) $F_{\mathrm{I}}(P_0, \xi, \eta, f, s) = s^2[\xi, \eta]_{P_0}(f) \bmod (s^3)$, *and*
(*ii*) $F_{\mathrm{II}}(P_0, \xi, \eta, f, s) = s^2[\xi, \eta]_{P_0}(f) \bmod (s^3)$.

Strategy of the proof of the Lemma 3.2. To begin with, note that as a consequence of the Lemma 3.1(2), it is enough to prove the above lemma for a fixed P_0, instead of allowing P_0 to vary over an open set W. Though the limits corresponding to them are equal, each of the two gap functions $F_{\mathrm{I}}(P_0, \xi, \eta, f, s)$ and $F_{\mathrm{II}}(P_0, \xi, \eta, f, s)$ has a different advantage. Using F_{I}, we see that as s varies the point $P_4(s)$ moves on a smooth curve through P_0, whose tangent at P_0, when calculated with respect to the new parameter s^2, is $[\xi, \eta]_{P_0}$. On the other hand, the definition of F_{II} has more symmetry and has fewer compositions of flows, so F_{II} is much easier to calculate. Given this, we will first prove the statement about F_{II} in Lemma 3.2. Though a direct proof of the statement about F_{I} is possible along the same lines by working a bit harder, we will instead use a different trick. This involves deducing the statement (i) about F_{I} for a fixed P_0 from the full form of the statement (ii) about F_{II} in which P_0 is not fixed but varies over an open set $W \subset M$. A similar trick will be deployed later in the proof of the Theorem 1.1, where we first prove the statement about the gap function G_{II} related to the geodesic quadrilateral Q_2, Q_1, P_0, P_1, P_2, and then use this trick to deduce the statement about the gap function G_I related to the geodesic quadrilateral P_0, P_1, P_2, P_3, P_4, which is considerably more difficult (being much longer and computationally messy) to approach directly.

Proof of Lemma 3.2(ii). As explained above, it is enough to prove the statement for a fixed P_0, instead of allowing it to vary over W. For small enough s, t let

$$\mathcal{P}(s, t) = \psi_t \circ \varphi_s(P_0) \in M,$$

in particular, $P_0 = \mathcal{P}(0, 0)$, $P_1(s) = \mathcal{P}(s, 0)$ and $P_2(s) = \mathcal{P}(s, s)$. Let f be a smooth function on a neighbourhood of P_0. Then regarded as a function on $(-a, a) \times (-a, a)$, we have

$$\frac{\partial}{\partial s} f(\mathcal{P}(s, 0)) = \xi_{\mathcal{P}(s,0)}(f) \quad \text{and} \quad \frac{\partial}{\partial t} f(\mathcal{P}(s, t)) = \eta_{\mathcal{P}(s,t)}(f)$$

By expansions in s and t, we have the following equalities.

$$
\begin{aligned}
f(\mathcal{P}(s,0)) &= f(P_0) + s\xi_{P_0}(f) + \frac{s^2}{2}\xi_{P_0}\xi(f) + s^3 h_1(s),\\
f(\mathcal{P}(s,t)) &= f(\mathcal{P}(s,0)) + t\eta_{\mathcal{P}(s,0)}(f) + \frac{t^2}{2}\eta_{\mathcal{P}(s,0)}(\eta(f)) + t^3 h_2(s,t),\\
\eta_{\mathcal{P}(s,0)}(f) &= \eta_{P_0}(f) + s\xi_{P_0}(\eta(f)) + s^2 h_3(s),\\
\eta_{\mathcal{P}(s,0)}(\eta(f)) &= \eta_{P_0}(\eta(f)) + sh_4(s)
\end{aligned}
$$

where $h_1(s)$, $h_3(s)$ and $h_4(s)$ are smooth real functions on $(-a,a)$, while $h_2(s,t)$ is a smooth real function on $(-a,a)\times(-a,a)$. Hence we get

$$
\begin{aligned}
f(\mathcal{P}(s,t)) - f(P_0) &= s\xi_{P_0}(f) + t\eta_{P_0}(f) + st(\xi_{P_0}\eta(f))\\
&\quad + \frac{s^2}{2}\xi_{P_0}\xi(f) + \frac{t^2}{2}\eta_{P_0}\eta(f) + H(s,t)
\end{aligned}
$$

where the error term $H(s,t)$ is given by

$$
H(s,t) = s^3 h_1(s) + t^3 h_2(s,t) + s^2 t h_3(s) + \frac{st^2}{2}h_4(s).
$$

Now put $t = s$ in the above, that is, restrict the above function to the diagonal $\Delta \subset (-a,a)\times(-a,a)$. As $P_1(s) = \mathcal{P}(s,0)$ and $P_2(s) = \mathcal{P}(s,s)$, we get

$$
\begin{aligned}
f(P_2(s)) - f(P_0) &= s(\xi_{P_0}(f) + \eta_{P_0}(f))\\
&\quad + s^2(\xi_{P_0}\eta(f)) + \frac{s^2}{2}(\xi_{P_0}\xi(f) + \eta_{P_0}\eta(f)) + s^3 h(s)
\end{aligned}
$$

where $h(s)$ is a smooth function on $(-a,a)$. Similarly, by first travelling along the flow of η to reach Q_1 and then along the flow of ξ to reach Q_2 (that is, by just interchanging ξ and η in the above expression), we get

$$
\begin{aligned}
f(Q_2(s)) - f(P_0) &= s(\eta_{P_0}(f) + \xi_{P_0}(f))\\
&\quad + s^2(\eta_{P_0}\xi(f)) + \frac{s^2}{2}(\eta_{P_0}\eta(f) + \xi_{P_0}\xi(f)) + s^3 k(s)
\end{aligned}
$$

for some smooth function $k(s)$ on $(-a,a)$. Taking the difference,

$$
\begin{aligned}
f(P_2(s)) - f(Q_2(s)) &= s^2(\xi_{P_0}\eta(f) - \eta_{P_0}\xi(f)) \bmod (s^3)\\
&= s^2[\xi,\eta]_{P_0}(f) \bmod (s^3),
\end{aligned}
$$

which proves the statement of Lemma 3.2.(ii). □

The trick which gets F_{I} from F_{II}. We have proved that for any $x \in M$, there is a neighbourhood $x \in W \subset M$ and an $a > 0$ such that both the quadrilaterals are defined for any $P_0 \in W$ and $|s| < a$, and we have

$$
F_{\mathrm{II}}(P_0,\xi,\eta,f,s) = s^2[\xi,\eta]_{P_0}(f) \bmod s^3 C^\infty((-a,a)\times W),
$$

which is the statement (ii) of the Lemma. To deduce from this the statement (i) of the Lemma, we begin by noting the identity

$$F_{\rm I}(P_0, \xi, \eta, f, s) = F_{\rm II}(P_2(s), -\xi, -\eta, f, s)$$

which is immediate from the definitions. Now for a fixed P_0, consider the map $(-a, a) \to M$ that sends $\tau \mapsto P_2(\tau) = \psi_\tau \circ \varphi_\tau(P_0)$. As $P_2(0) = P_0$, by continuity there exists $0 < b < a$ such that $P_2(\tau) \in W$ whenever $\tau \in (-b, b)$. The smooth map

$$(-b, b) \times (-b, b) \to (-a, a) \times W : (s, \tau) \mapsto (s, P_2(\tau))$$

induces a ring homomorphism $C^\infty((-a, a) \times W) \to C^\infty((-b, b) \times (b, b))$ under which the relationship in the ring $C^\infty((-a, a) \times W)$ in the statement of the Lemma 3.2(ii) gives the following relationship in the ring $C^\infty((-b, b) \times (-b, b))$:

$$F_{\rm II}(P_2(\tau), \xi, \eta, f, s) = s^2[\xi, \eta]_{P_2(\tau)}(f) \bmod s^3 C^\infty((-b, b) \times (-b, b)).$$

It should be noted that the above step used the full form of Lemma 3.2(ii), in which P_0 varies over a neighbourhood W of $x \in M$, as we needed to apply it to the variable base point $P_2(\tau) \in W$.

Under the ring homomorphism $C^\infty((-b, b) \times (-b, b)) \to C^\infty(-b, b)$ induced by the inclusion of the diagonal, under which $s \mapsto s$ and $\tau \mapsto s$, the extension of the ideal $s^3 C^\infty(-b, b) \times (-b, b))$ is the ideal $s^3 C^\infty(-b, b)$. Applying this homomorphism to the above equality (means, putting $s = \tau$), we get

$$F_{\rm II}(P_2(s), \xi, \eta, f, s) = s^2[\xi, \eta]_{P_2(s)}(f) \bmod s^3 C^\infty(-b, b).$$

Now, $P_2(s)$ is a smooth function of s, and for $s = 0$ it takes the value P_0. Hence by Lemma 3.1(1), we have

$$[\xi, \eta]_{P_2(s)}(f) = [\xi, \eta]_{P_0}(f) \bmod s C^\infty(-b, b).$$

Hence substitution gives us

$$F_{\rm II}(P_2(s), \xi, \eta, f, s) = s^2[\xi, \eta]_{P_0}(f) \bmod s^3 C^\infty(-b, b).$$

Hence we finally have

$$\begin{aligned} F_{\rm I}(P_0, \xi, \eta, f, s) &= F_{\rm II}(P_2(s), -\xi, -\eta, f, s) \\ &= s^2[-\xi, -\eta,]_{P_0}(f) \bmod s^3 C^\infty(-b, b) \\ &= s^2[\xi, \eta,]_{P_0}(f) \bmod s^3 C^\infty(-b, b) \text{ as desired.} \end{aligned}$$

4. Connections: Basic notions

In this section, we first recall some well known basic notions related to affine connections (see e.g. [1] for a more comprehensive introduction), and then formulate the Lemma 4.1.

An *affine connection* on a smooth manifold M associates to any tangent vector fields ξ and η on an open subset $U \subset M$ a new tangent vector field $\nabla_\xi \eta$ on U. This is $C^\infty(U)$-linear in the variable ξ, while it is $\mathbb{R}$-linear in η and satisfies the Leibniz rule: $\nabla_\xi(f\eta) = \xi(f)\eta + f\nabla_\xi(\eta)$ for all $f \in C^\infty(U)$. In particular, if $P \in U$ and $u \in T_P M$, then the vector $\nabla_u \eta \in T_P M$, which is defined to be $(\nabla_\xi \eta)_P$, where ξ is any tangent vector field in an open neighbourhood of P such that $\xi_P = u$, is well defined.

Let M be a manifold with an affine connection ∇. Let $\gamma(s)$ be a parameterized curve in M defined on an interval $s \in (-a, a) \subset \mathbb{R}$ (this means $\gamma : (-a, a) \to M$ is smooth). A smooth vector field $v(s)$ along γ consists of giving a vector $v(s) \in T_{\gamma(s)}M$ for all $s \in (-a, a)$, which is smooth as a function of s. For example, the *tangent vector field of* γ, denoted by $\dot{\gamma}$ or $\mathrm{d}\gamma/\mathrm{ds}$, is a smooth vector field along γ. If ∇ is an affine connection on M, and γ and v are as above, then we can define the *covariant derivative* of v along γ to be a certain smooth tangent vector field along γ, denoted by $\nabla_{\dot{\gamma}}v$ or $\nabla_{\mathrm{d}\gamma/\mathrm{ds}}v$. While this can be naturally defined in a coordinate-free manner by working over the graph of γ in $(-a, a) \times M$, what we will find more useful is its local coordinate expression, which we next describe.

For this, let U be a coordinate chart around P with coordinates $(x^1, \ldots, x^d)$. The *connection coefficients* of ∇ are the real-valued smooth functions Γ^i_{jk} on U defined by

$$\nabla_{\frac{\partial}{\partial x^k}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma^i_{jk}\frac{\partial}{\partial x^i}.$$

where we have used the *summation convention* under which there is understood to be a summation over an index which is repeated with one occurence as a subscript and another as a superscript. The superscript i on x^i is to be regarded as a subscript in the expression $\partial/\partial x^i$. The summation signs are suppressed.

Let the point $\gamma(s)$ have coordinates $(x^1(s), \ldots, x^d(s))$, so that the tangent vector is $\dot{\gamma} = \frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\partial}{\partial x^i}$. Let $v(s) = v^i(s)\frac{\partial}{\partial x^i}$. Let $\Gamma^i_{jk}(s) = \Gamma^i_{jk}(\gamma(s))$. Then

$$(\nabla_{\dot{\gamma}}v)(s) = \left(\frac{\mathrm{d}v^i}{\mathrm{d}s}(s) + \Gamma^i_{jk}(s)v^j(s)\frac{\mathrm{d}x^k}{\mathrm{d}s}(s)\right)\frac{\partial}{\partial x^i}\bigg|_{\gamma(s)},$$

which we write more briefly as $(\nabla_{\dot{\gamma}}v)^i = \frac{\mathrm{d}v^i}{\mathrm{d}s} + \Gamma^i_{jk}v^j\dot{\gamma}^k$. We say that the vector field v along γ is *parallel transported* along γ if $\nabla_{\dot{\gamma}}v = 0$, that is, $\frac{\mathrm{d}v^i}{\mathrm{d}s} = -\Gamma^i_{jk}v^j\dot{\gamma}^k$. The parameterized curve $\gamma(s)$ is called a *geodesic* if its tangent vector field $\dot{\gamma}$ is parallel transported along γ, that is, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. In coordinate terms, this is the equation

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} = -\Gamma^i_{jk}\frac{\mathrm{dx^j}}{\mathrm{ds}}\frac{\mathrm{d}x^k}{ds}.$$

Given any $P \in M$ and $u \in T_PM$, for a small enough $a > 0$ there is a unique geodesic $\gamma : (-a, a) \to M$ such that $\gamma(0) = P$ and $\mathrm{d}\gamma/\mathrm{d}s(0) = u$.

Co-ordinates and vector fields on the frame bundle. We will denote by $\pi : E \to M$ the frame bundle associated to TM, which is a principal $GL(d)$-bundle where $d = \dim M$, whose fiber over $x \in M$ is the set of all linear bases for T_xM. If $(x^1, \ldots, x^d)$ are local coordinates on M defined on an open subset $U \subset M$, then on the open subset $\pi^{-1}(U) \subset E$, we get coordinates $(x^1, \ldots, x^d, \dot{x}^1_1, \ldots, \dot{x}^i_j, \ldots, \dot{x}^d_d)$. These are defined as follows. If $x \in U$ has coordinates $(a^1, \ldots, a^d)$, and if $y = (u_i, \ldots, u_d)$ is a basis for T_xM, with $u_j = b^i_j\partial_i$ where $\partial_i = \partial/\partial x^i$, then $\pi(y) = x$, and the coordinates of y are $x^i = a^i$ and $\dot{x}^i_j = b^i_j$.

The group $GL_d(\mathbb{R})$ acts on the right on E, under which a matrix $A = (A^p_q) \in GL_d(\mathbb{R})$ moves the point $y = (u_1, \ldots, u_d) \in E$ to the point $(v_1, \ldots, v_d) = (u_1, \ldots, u_d)A \in E$ where $v_i = u_jA^j_i$. A connection ∇ on TM is the same as a distribution (vector subbundle) $D \subset TE$ which is (i) preserved by the action of $GL_d(\mathbb{R})$, and (ii) is 'horizontal', that is, supplementary to the kernel of $\pi_* : TE \to \pi^*TM$. As π is a submersion, the kernel of π_* is a vector subbundle of rank d^2. This is called the 'vertical subbundle', and it is naturally ismorphic to $\pi^*\underline{\mathrm{End}}(TM)$.

We now define d global vector fields $\xi_1, \ldots, \xi_d$ on E, which are everywhere linearly independent and span D. At any point $y = (u_1, \ldots, u_d) \in E$ over $x \in M$, the linear map $\pi_*(y) : D_y \to T_x M$ is an isomorphism, so there is a unique $\xi_i \in D_y$ such that

$$\pi_*(y)\xi_i = u_i$$

for all $i = 1, \ldots, d$. In coordinate terms, if Γ^i_{jk} are the connection coefficients for the connection ∇ on TM in the local coordinates (x^i), then D_y has the basis $\xi_1, \ldots, \xi_d$, where

$$\xi_m = \dot{x}^i_m \frac{\partial}{\partial x^i} - \Gamma^i_{jk} \dot{x}^j_\ell \dot{x}^k_m \frac{\partial}{\partial \dot{x}^i_\ell}.$$

In the above, there is a summation over the indices i, j, k, ℓ, which are repeated with one occurence as a subscript and another as a superscript. For this purpose, the index ℓ in $\frac{\partial}{\partial \dot{x}^i_\ell}$ is regarded as a superscript. The summation sign is suppressed.

How to recover the Riemann curvature tensor from the data $R(u, v)(u + v)$? The *torsion tensor* of an affine connection ∇ associates to any pair (ξ, η) of tangent vector fields on an open subset $U \subset M$ the tangent vector field $T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$ on U. This turns out to be $C^\infty(U)$-linear in both ξ and η, justifying the name 'tensor'. In particular, if $P \in M$ and $u, v \in T_P M$, we get a well-defined vector $T(u, v) \in T_P M$, which is $\mathbb{R}$-linear in each of u, v. An affine connection is said to be *symmetric* if the tensor T is identically zero on M.

The *Riemann curvature tensor* of ∇ is the operator that associates to any triple (ξ, η, ζ) of tangent vector fields on an open subset $U \subset M$ the tangent vector field $R(\xi, \eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta$ on U. This turns out to be $C^\infty(U)$-linear in each of the three variables, justifying the name 'tensor'. In particular, if $P \in M$ and $u, v, w \in T_P M$, then we get a well-defined vector $R(u, v)w \in T_P M$, which is $\mathbb{R}$-linear in each of u, v, w.

The curvature tensor satisfies $R(u, v)w = -R(v, u)w$ (skew-symmetry) and moreover if ∇ is symmetric, then $R(u, v)w + R(v, w)u + R(w, u)v = 0$ (the algebraic Bianchi identity). From these, the following lemma is immediate.

Lemma 4.1. *If* ∇ *is symmetric*, *then for any three vectors* $u, v, w \in T_P M$, *the following identities hold*:

$$\begin{aligned} R(u, v)u &= \tfrac{1}{2} R(2u, v)(2u + v) - R(u, v)(u + v), \ \textit{and} \\ R(u, v)w &= \tfrac{1}{3}(R(u, v + w)(v + w) - R(u, v)v - R(u, w)w \\ &\quad + R(u + w, v)(u + w) - R(u, v)u - R(w, v)w). \end{aligned}$$

Hence the curvature operator $TM \times_M TM \times_M TM \to TM : (u, v, w) \mapsto R(u, v)w$ *can be recovered uniquely from the map* $TM \times_M TM \to TM : (u, v) \mapsto R(u, v)(u + v)$.

5. Frame flow, torsion and curvature

The vector fields $\xi_1, \ldots, \xi_d$ are so defined that the integral curve of ξ_m on E through a point $y = (u_1, \ldots, u_d) \in E$ projects under $\pi : E \to M$ to the geodesic through $x = \pi(y)$

with tangent $u_m \in T_x M$, and the integral curve in E parallel translates each of $u_1, \ldots, u_d$ along this geodesic through x. Hence we call ξ_m as the *m-th frame flow field*, and its flow on E as the *m-th frame flow*. This is a natural generalization of the notion of the geodesic flow on TM, under which only the tangent vector is parallel translated, instead of a full frame.

Proof of Theorem 1.3. We will denote $\partial f/\partial x^i$ by $f_{,i}$. We will use the notation

$$\partial_i = \frac{\partial}{\partial x^i} \quad \text{and} \quad D_q^p = \frac{\partial}{\partial \dot{x}_p^q}$$

With this, we have

$$\xi_m = \dot{x}_m^i \partial_i - \Gamma^i_{jk} \dot{x}_a^j \dot{x}_m^k D_i^a \quad \text{and} \quad \xi_n = \dot{x}_n^p \partial_p - \Gamma^p_{qr} \dot{x}_s^q \dot{x}_n^r D_p^s$$

Note that $\partial_i D_q^p = D_q^p \partial_i$, $\partial_i (\dot{x}_q^p) = 0$ and $D_j^i (\dot{x}_q^p) = \delta_q^i \delta_j^p$ in terms of Kronecker symbols. Hence we have

$$\begin{aligned}\xi_m \circ \xi_n &= (\dot{x}_m^i \partial_i - \Gamma^i_{jk} \dot{x}_a^j \dot{x}_m^k D_i^a)(\dot{x}_n^p \partial_p - \Gamma^p_{qr} \dot{x}_s^q \dot{x}_n^r D_p^s) \\ &= -\Gamma^i_{cb} \dot{x}_m^b \dot{x}_n^c \partial_i + (-\Gamma^p_{qc,b} + \Gamma^i_{qb}\Gamma^p_{ic} + \Gamma^i_{cb}\Gamma^p_{qi}) \dot{x}_s^q \dot{x}_m^b \dot{x}_n^c D_p^s \\ &\quad + \text{ second order partial derivative terms.}\end{aligned}$$

Similarly, by exchanging m and n and renaming the dummy variables,

$$\begin{aligned}\xi_n \circ \xi_m &= -\Gamma^i_{bc} \dot{x}_m^b \dot{x}_n^c \partial_i + (-\Gamma^p_{qb,c} + \Gamma^i_{qc}\Gamma^p_{ib} + \Gamma^i_{bc}\Gamma^p_{qi}) \dot{x}_s^q \dot{x}_m^b \dot{x}_n^c D_p^s \\ &\quad + \text{ second order partial derivative terms.}\end{aligned}$$

As the bracket is a differential operator of first order (which is because the second order terms exactly cancel as partial derivatives commute), we get

$$\begin{aligned}[\xi_m, \xi_n] &= \xi_m \circ \xi_n - \xi_n \circ \xi_m \\ &= x(-\Gamma^i_{cb} + \Gamma^i_{bc}) \dot{x}_m^b \dot{x}_n^c \partial_i + \text{ linear combinations of the } D_p^s.\end{aligned}$$

Recall that at any point $y = (u_1, \ldots, u_n) \in E$ over $x \in M$, we have $\pi_*(y)(\xi_r(y)) = u_r$, where $\pi_*(y) : T_y E \to T_x M$ is the derivative of $\pi : E \to M$. $T(\partial_m, \partial_n) = (\Gamma^i_{nm} - \Gamma^i_{mn})\partial_i$, that is, $T^i_{mn} = \Gamma^i_{nm} - \Gamma^i_{mn}$. So taking $u_r = \partial_r$, which corresponds to $\dot{x}_r^s = \delta_r^s$, the above gives

$$\pi_*[\xi_m, \xi_n] = -T^i_{mn} \partial_i = -T(\partial_m, \partial_n)$$

As the ∂_r form a basis for $T_x M$, this shows that the torsion of ∇ is given by $T(u_i, u_j) = -\pi_*(y)[\xi_i, \xi_j]$ as claimed. This proves the statement (1) in the Theorem. □

Now assuming ∇ is symmetric, that is, $\Gamma^i_{jk} = \Gamma^i_{kj}$, we get

$$[\xi_m, \xi_n] = \xi_m \circ \xi_n - \xi_n \circ \xi_m$$

$$
\begin{aligned}
&= ((-\Gamma^p_{qc,b} + \Gamma^i_{qb}\Gamma^p_{ic} + \Gamma^i_{cb}\Gamma^p_{qi}) \\
&\quad - (-\Gamma^p_{qb,c} + \Gamma^i_{qc}\Gamma^p_{ib} + \Gamma^i_{bc}\Gamma^p_{qi}))\dot{x}^q_s \dot{x}^b_m \dot{x}^c_n D^s_p \\
&= (\Gamma^p_{qb,c} - \Gamma^p_{qc,b} + \Gamma^i_{qb}\Gamma^p_{ic} - \Gamma^i_{qc}\Gamma^p_{ib})\dot{x}^q_s \dot{x}^b_m \dot{x}^c_n D^s_p \\
&= -R^p_{qbc}\dot{x}^q_s \dot{x}^b_m \dot{x}^c_n D^s_p,
\end{aligned}
$$

where

$$R^p_{qbc} = \Gamma^p_{qc,b} - \Gamma^p_{qb,c} + \Gamma^i_{qc}\Gamma^p_{ib} - \Gamma^i_{qb}\Gamma^p_{ic}$$

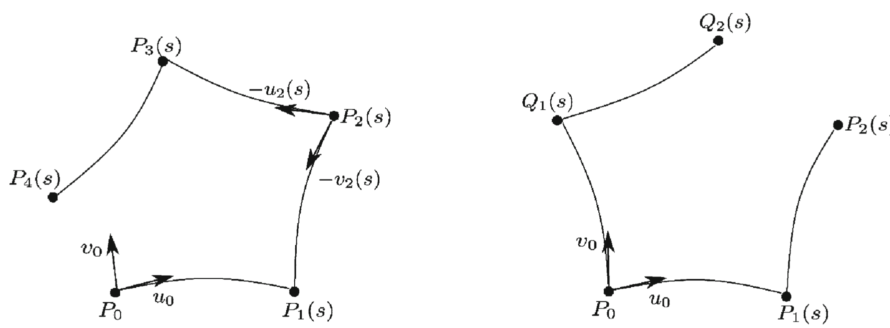
are the coordinates of the Riemann curvature tensor when ∇ is symmetric. Taking $\dot{x}^j_i = u^j_i$, it follows that

$$R(u_i, u_j)_x = -[\xi_i, \xi_j]_y$$

as elements of $\mathrm{End}(T_x M)$. This completes the proof of (3), and hence that of the Theorem 1.3.

6. Quadrilateral gap for an affine connection

Let there be given tangent vectors $u_0, v_0 \in T_{P_0}M$ at some point $P_0 \in M$. For small values of s, recall from the Introduction that by repeated application of the operator $\mathbb{T}_s$ or its inverse to the triple (P_0, u_0, v_0) we defined two open geodesic quadrilaterals $P_0, P_1(s), P_2(s), P_3(s), P_4(s)$ and $Q_2(s), Q_1(s), P_0, P_1(s), P_2(s)$. The successive vertices in each quadrilateral are joined by geodesic segments as described earlier.



By the fundamental existence and uniqueness theorem for ODEs with smooth parameters, applied to the geodesic equation and the parallel transport equation, for $(x, u_x, v_x) \in TM \times_M TM$ there exists an open neighbourhood $W \subset TM \times_M TM$ and a real number $a > 0$ such that for any triple $(P_0, u_0, v_0) \in W$ and $s \in (-a, a)$, the triples $\mathbb{T}^i_s(P_0, u_0, v_0)$ are defined for $i = -2, -1, \ldots, 3, 4$ and depend smoothly on the tuple (s, P_0, u_0, v_0), giving smooth morphisms $(-a, a) \times W \to TM \times_M TM$. Given any open neighbourhood U of $x \in M$, if we choose $a > 0$ and W to be sufficiently small then the above quadrilaterals lie entirely in U.

In terms of the notation introduced above, we now restate the Theorem 1.1 in the following somewhat stronger form, which allows (P_0, u_0, v_0) to vary.

Theorem 6.1. *Let M be a smooth manifold with an affine connection ∇. Then for any C^∞ function f defined on a neighbourhood U of a point $x \in M$, and for any $u_x, v_x \in T_x M$, the following relations hold in the ring $C^\infty((-a, a) \times W)$ for sufficiently small $a > 0$ and open neighbourhood W of (x, u_x, v_x) in $TM \times_M TM$, where $(s^n) \subset C^\infty((-a, a) \times W)$ denotes the principal ideal generated by s^n.*

(1) $f(P_2(s)) - f(Q_2(s)) = -s^2 T(u_0, v_0)(f) \bmod (s^3)$, *where T is the torsion tensor of ∇.*
(2) $f(P_4(s)) - f(P_0) = -s^2 T(u_0, v_0)(f) \bmod (s^3)$.
If moreover, ∇ is symmetric, then we have
(3) $f(P_2(s)) - f(Q_2(s)) = -\frac{1}{2}s^3(R(u_0, v_0)(u_0 + v_0))(f) \bmod (s^4)$, *where R is the curvature tensor of ∇.*
(4) $f(P_4(s)) - f(P_0) = \frac{1}{2}s^3(R(u_0, v_0)(u_0 + v_0))(f) \bmod (s^4)$.

Strategy of the proof. It is enough to take the function f above to be a coordinate function x^i w.r.t a smooth local chart $(U, x^1, \ldots, x^d)$ around x. We will fix (P_0, u_0, v_0) to begin with, and make a laborious but straight-forward computation of the Taylor series in s for the function $x^i(Q_2(s)) - x^i(P_2(s))$, which will yield the statements (1) and (3). By Lemma 3.1(2), these imply the full forms of (1) and (3) (with variable (P_0, u_0, v_0)). Note that we have already proved (1) as a consequence of Lemma 1.2 and Theorem 1.3, so this gives another proof of the same.

Finally, we employ a certain trick involving an identity between two kinds of gaps and a Taylor series expansion (which is a somewhat more complicated version of the trick that we use in the proof of Lemma 3.2), to show that the full form of (1) (with variable (P_0, u_0, v_0)) implies the form of (2) where (P_0, u_0, v_0) is fixed, and the full form of (3) implies the form of (4) where (P_0, u_0, v_0) is fixed. Again, the full form of (2) and (4) follows by Lemma 3.1(2). The reverse implications also follow from similar arguments. The Theorem 1.1 stated in the Introduction is an immediate consequence of the Theorem 6.1.

Proofs of 6.1(1) and 6.1(3). To begin with, we fix the triple (P_0, u_0, v_0). Let γ_1 be the geodesic on M with affine parameter s, normalized by $\gamma_1(0) = P_0$ and $\frac{d\gamma_1}{ds}(0) = u_0$. Let $u_1(s), v_1(s) \in T_{\gamma_1(s)}M$ be the parallel transports of $u_0, v_0 \in T_{P_0}M$ along γ_1. Let $\gamma_{2,s}$ be the geodesic on M with affine parameter t, normalized by $\gamma_{2,s}(0) = \gamma_1(s)$ and $\frac{d\gamma_{2,s}}{dt}(0) = v_1(s)$. For s, t in a small domain $D = (-a, a) \times (-a, a)$ in $\mathbb{R}^2$, let

$$\mathcal{P}(s, t) = \gamma_{2,s}(t) \in M$$

which defines a smooth function $\mathcal{P} : D \to M$. In particular, $\mathcal{P}(s, 0) = \gamma_1(s)$.

Let U be a coordinate chart around P_0 with coordinates $(x^1, \ldots, x^d)$, with $P_0 = (x_0^1, \ldots, x_0^d)$. We choose $a > 0$ to be small enough such that the image of $\mathcal{P} : D \to M$ lies in U. Let $x^i(s, t)$ denote the coordinates of $\mathcal{P}(s, t)$. We use the notation $\Gamma^i_{jk}(0) = \Gamma^i_{jk}(P_0)$ and $\Gamma^i_{jk}(s, t) = \Gamma^i_{jk}(\mathcal{P}(s, t))$. The point $\gamma_1(s)$ has coordinates

$$x_1^i(s) = x^i(s, 0).$$

In terms of the summation convention, we have $u_0 = u_0^i \frac{\partial}{\partial x^i}$, $v_0 = v_0^i \frac{\partial}{\partial x^i}$, $u_1(s) = u_1^i \frac{\partial}{\partial x^i}$ and $v_1(s) = v_1^i \frac{\partial}{\partial x^i}$, where u_1^i and v_1^i are the functions of s that are the coefficients of $u_1(s)$ and $v_1(s)$.

The curve γ_1 satisfies the geodesic equation, hence

$$\frac{\mathrm{d}^2 x_1^i}{\mathrm{d}s} = \frac{\mathrm{d}u_1^i}{\mathrm{d}s} = -\Gamma^i_{jk}(s)u_1^j(s)u_1^k(s)$$

along γ_1. Also, as $v_1(s)$ is parallel transported along γ_1, we have

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}s} = -\Gamma^i_{jk}(s)v_1^j(s)u_1^k(s).$$

The steps in the following calculation are obtained by taking Taylor expansions in s and making various substitutions.

$$\begin{aligned}
x_1^i(s) &= x_0^i + s\frac{\mathrm{d}x_1^i}{\mathrm{d}s}(0) + \frac{s^2}{2}\frac{\mathrm{d}^2x_1^i}{\mathrm{d}s^2}(0) + \frac{s^3}{3!}\frac{\mathrm{d}^3x_1^i}{ds^3}(0) \bmod (s^4),\\
\frac{\mathrm{d}x_1^i}{\mathrm{d}s} &= u_1^i \text{ as } \gamma \text{ is a geodesic with tangent } u_1,\\
\frac{\mathrm{d}^2x_1^i}{\mathrm{d}s^2} = \frac{\mathrm{d}u_1^i}{\mathrm{d}s} &= -\Gamma^i_{jk}u_1^j u_1^k \text{ by the geodesic equation},\\
\frac{\mathrm{d}^3x_1^i}{\mathrm{d}s^3} &= -\Gamma^i_{jk,\ell}u_1^j u_1^k u_1^\ell + \Gamma^i_{jk}\Gamma^j_{ab}u_1^a u_1^b u_1^k + \Gamma^i_{jk}\Gamma^k_{ab}u_1^a u_1^b u_1^j.
\end{aligned}$$

It follows that in the ring $C^\infty(-a, a)$ we have the relation

$$\begin{aligned}
x_1^i(s) = x_0^i + su_0^i - \frac{s^2}{2}\Gamma^i_{jk}(0)u_0^j u_0^k\\
+ \frac{s^3}{3!}(-\Gamma^i_{jk,\ell}(0)u_0^j u_0^k u_0^\ell + \Gamma^i_{jk}(0)\Gamma^j_{ab}(0)u_0^a u_0^b u_0^k\\
+ \Gamma^i_{jk}(0)\Gamma^k_{ab}(0)u_0^a u_0^b u_0^j) \bmod (s^4).
\end{aligned}$$

By Taylor expansion and substitutions, we have the following:

$$\begin{aligned}
v_1^i(s) &= v_0^i + s\frac{\mathrm{d}v_1^i}{\mathrm{d}s}(0) + \frac{s^2}{2}\frac{\mathrm{d}^2v_1^i}{\mathrm{d}s^2}(0) \bmod (s^3).\\
\frac{\mathrm{d}v_1^i}{\mathrm{d}s} &= -\Gamma^i_{jk}v_1^j u_1^k \text{ as } v_1(s) \text{ is parallel transported along } \gamma_1.\\
\frac{\mathrm{d}^2v_1^i}{\mathrm{d}s^2} &= -\Gamma^i_{jk,l}v_1^j u_1^k u_1^l + \Gamma^i_{jk}\Gamma^j_{mn}v_1^m u_1^n u_1^k + \Gamma^i_{jk}\Gamma^k_{mn}v_1^j u_1^m u_1^n.
\end{aligned}$$

Hence we get

$$\begin{aligned}
v_1^i(s) = v_0^i + s(-\Gamma^i_{jk}(0)v_0^j u_0^k)\\
+ \frac{s^2}{2}(-\Gamma^i_{jk,l}(0)v_0^j u_0^k u_0^l + \Gamma^i_{jk}(0)\Gamma^j_{mn}(0)v_0^m u_0^n u_0^k\\
+ \Gamma^i_{jk}(0)\Gamma^k_{mn}(0)v_0^j u_0^m u_0^n) \mathrm{mod}(s^3) \text{ in the ring } C^\infty(-a, a).
\end{aligned}$$

Next, we consider the geodesic $\gamma_{2,s}(t)$, which we write as $\gamma_2(t)$ or just γ_2 when s is suppressed from the notation. We will denote the coordinates of $\gamma_2(t)$ by $x_2^i(t)$, leaving out the mention of s. The curve $\gamma_{2,s}(t)$ is defined by $\gamma_{2,s}(0) = P_1(s)$, and

$$\frac{\partial \gamma_{2,s}^i}{\partial t}(s,0) = v_1^i(s)$$

which we will simply write as $\frac{\mathrm{d}x_2^i}{\mathrm{d}t}(0) = v_1^i$ by suppressing s from the notation.

The steps in the following calculation are obtained by Taylor expansion in t and making substitutions. The coefficients of t^n terms are functions of s, belonging to $C^\infty(-a, a)$. The various relations are in the ring $C^\infty(D)$ modulo certain ideals $J \subset C^\infty(D)$. We suppress the mention of s and t in some places for brevity of notation, when these are understood.

$$\begin{aligned}
x_2^i(t) &= x_1^i + t\frac{\mathrm{d}x_2^i}{\mathrm{d}t}(0) + \frac{t^2}{2}\frac{\mathrm{d}^2x_2^i}{\mathrm{d}t^2}(0) + \frac{t^3}{3!}\frac{\mathrm{d}^3x_2^i}{\mathrm{d}t^3}(0) \bmod t^4,\\
\frac{\mathrm{d}x_2^i}{\mathrm{d}t}(0) &= v_1^i(s)\\
&= v_0^i + s(-\Gamma^i_{jk}(0)v_0^j u_0^k) + \frac{s^2}{2}(-\Gamma^i_{jk,l}(0)v_0^j u_0^k u_0^l\\
&\quad + \Gamma^i_{jk}(0)\Gamma^j_{mn}(0)v_0^m u_0^n u_0^k + \Gamma^i_{jk}(0)\Gamma^k_{mn}(0)v_0^j u_0^m u_0^n) \bmod s^3,\\
\frac{\mathrm{d}^2x_2^i}{\mathrm{d}t^2} &= -\Gamma^i_{jk}\frac{\mathrm{d}x_2^j}{\mathrm{d}t}\frac{\mathrm{d}x_2^k}{\mathrm{d}t} \text{ by the geodesic equation,}\\
\frac{\mathrm{d}^2x_2^i}{\mathrm{d}t^2}(0) &= -\Gamma^i_{jk}(s,0)v_1^j(s)v_1^k(s),\\
\Gamma^i_{jk}(s,0) &= \Gamma^i_{jk}(0) + s\Gamma^i_{jk,\ell}(0)u_0^\ell \bmod s^2,\\
\frac{\mathrm{d}^2x_2^i}{\mathrm{d}t^2}(0) &= -(\Gamma^i_{jk}(0) + s\Gamma^i_{jk,\ell}(0)u_0^\ell)(v_0^j\\
&\quad - s\Gamma^j_{ab}(0)v_0^a u_0^b)(v_0^k - s\Gamma^k_{ab}(0)v_0^a u_0^b) \bmod s^2\\
&= -\Gamma^i_{jk}(0)v_0^j v_0^k - s(\Gamma^i_{jk,\ell}(0)u_0^\ell v_0^j v_0^k - \Gamma^i_{jk}(0)\Gamma^j_{ab}(0)v_0^a u_0^b v_0^k\\
&\quad - \Gamma^i_{jk}(0)\Gamma^k_{ab}(0)v_0^a u_0^b v_0^j) \bmod s^2,\\
\frac{\mathrm{d}^3x_2^i}{\mathrm{d}t^3} &= -\Gamma^i_{jk,\ell}\frac{\mathrm{d}x_2^\ell}{\mathrm{d}t}\frac{\mathrm{d}x_2^j}{\mathrm{d}t}\frac{\mathrm{d}x_2^k}{\mathrm{d}t} + \Gamma^i_{jk}\Gamma^j_{ab}\frac{\mathrm{d}x_2^a}{\mathrm{d}t}\frac{\mathrm{d}x_2^b}{\mathrm{d}t}\frac{\mathrm{d}x_2^k}{\mathrm{d}t}\\
&\quad + \Gamma^i_{jk}\Gamma^k_{ab}\frac{\mathrm{d}x_2^a}{\mathrm{d}t}\frac{\mathrm{d}x_2^b}{\mathrm{d}t}\frac{\mathrm{d}x_2^j}{\mathrm{d}t}.
\end{aligned}$$

We now evaluate the last equation at $t = 0$, that is, at $(s, 0) \in D$, modulo the principal ideal (s). Note that $\frac{\mathrm{d}x_2^j}{\mathrm{d}t}(s,0) = v_1^j(s)$, so $\frac{\mathrm{d}x_2^j}{\mathrm{d}t}(s,0) = v_0^j \bmod s$. Similarly, $\Gamma^i_{jk}(s,0) = \Gamma^i_{jk}(0) \bmod s$, and $\Gamma^i_{jk,\ell}(s,0) = \Gamma^i_{jk,\ell}(0) \bmod s$. Hence,

$$\begin{aligned}
\frac{\mathrm{d}^3x_2^i}{\mathrm{d}t^3}(0) &= -\Gamma^i_{jk,\ell}(0)v_0^\ell v_0^j v_0^k + \Gamma^i_{jk}(0)\Gamma^j_{ab}(0)v_0^a v_0^b v_0^k\\
&\quad + \Gamma^i_{jk}(0)\Gamma^k_{ab}(0)v_0^a v_0^b v_0^j \bmod s.
\end{aligned}$$

Hence by substitutions, we get the following equations in the ring $C^\infty(D)$ modulo the ideal $(s,t)^4 = (s^4, s^3t, s^2t^2, st^3, t^4) \subset C^\infty(D)$.

$$\begin{aligned}
x_2^i(t) &= x_1^i + t\frac{\mathrm{d}x_2^i}{\mathrm{d}t}(0) + \frac{t^2}{2}\frac{\mathrm{d}^2x_2^i}{\mathrm{d}t^2}(0) + \frac{t^3}{3!}\frac{\mathrm{d}^3x_2^i}{\mathrm{d}t^3}(0)\,\mathrm{mod}(t^4)\\
&= x_0^i + su_0^i + tv_0^i - \frac{s^2}{2}\Gamma^i_{jk}(0)u_0^ju_0^k - \frac{t^2}{2}(\Gamma^i_{jk}(0)v_0^jv_0^k)\\
&\quad + ts(-\Gamma^i_{jk}(0)v_0^ju_0^k)\\
&\quad + \frac{s^3}{3!}(-\Gamma^i_{jk,\ell}(0)u_0^ju_0^ku_0^\ell + \Gamma^i_{jk}(0)\Gamma^j_{ab}(0)u_0^au_0^bu_0^k\\
&\quad + \Gamma^i_{jk}(0)\Gamma^k_{ab}(0)u_0^au_0^bu_0^j)\\
&\quad + \frac{s^2t}{2}(-\Gamma^i_{jk,l}(0)v_0^ju_0^ku_0^l + \Gamma^i_{jk}(0)\Gamma^j_{mn}(0)v_0^mu_0^nu_0^k\\
&\quad + \Gamma^i_{jk}(0)\Gamma^k_{mn}(0)v_0^ju_0^mu_0^n)\\
&\quad - \frac{st^2}{2}((\Gamma^i_{jk,\ell}(0)u_0^\ell v_0^jv_0^k - \Gamma^i_{jk}(0)\Gamma^j_{ab}(0)v_0^au_0^bv_0^k\\
&\quad - \Gamma^i_{jk}(0)\Gamma^k_{ab}(0)v_0^au_0^bv_0^j)\\
&\quad + \frac{t^3}{3!}(-\Gamma^i_{jk,\ell}(0)v_0^\ell v_0^jv_0^k + \Gamma^i_{jk}(0)\Gamma^j_{ab}(0)v_0^av_0^bv_0^k\\
&\quad + \Gamma^i_{jk}(0)\Gamma^k_{ab}(0)v_0^av_0^bv_0^j)\mathrm{mod}(s,t)^4.
\end{aligned}$$

The point $P_2(s)$ is the point $\mathcal{P}(s,s)$. Hence putting $t = s$ in the above expression, renaming dummy indices and collecting terms, we see that the coordinates of $P_2(s)$ as a function of $s \in (-a,a)$ are given in the ring $C^\infty(-a,a)$ modulo the ideal (s^4) by

$$\begin{aligned}
P_2(s)^i &= x_0^i + s(u_0^i + v_0^i) - \frac{s^2}{2}\Gamma^i_{jk}(0)(u_0^ju_0^k + v_0^jv_0^k) - s^2\Gamma^i_{jk}(0)v_0^ju_0^k\\
&\quad + \frac{s^3}{3!}(-\Gamma^i_{pq,r}(0) + \Gamma^i_{jr}(0)\Gamma^j_{pq}(0)\\
&\quad + \Gamma^i_{rk}(0)\Gamma^k_{pq}(0))(u_0^pu_0^qu_0^r + v_0^pv_0^qv_0^r)\\
&\quad + \frac{s^3}{2}(-\Gamma^i_{bc,a}(0) + \Gamma^i_{jb}(0)\Gamma^j_{ca}(0)\\
&\quad + \Gamma^i_{ck}(0)\Gamma^k_{ba}(0))(u_0^au_0^bv_0^c + u_0^av_0^bv_0^c)\mathrm{mod}(s^4).
\end{aligned}$$

A similar calculation, in which the roles of u_0 and v_0 are interchanged, gives the coordinates $Q_2(s)^i$ of the point $Q_2(s)$. Subtracting the two expressions, we get in the ring $C^\infty(-a,a)$ the following relations modulo the ideal (s^3):

$$\begin{aligned}
P_2(s)^i - Q_2(s)^i &= s^2(-\Gamma^i_{jk}(0)v_0^ju_0^k + \Gamma^i_{jk}(0)u_0^jv_0^k)\\
&= s^2(\Gamma^i_{pq}(0) - \Gamma^i_{qp}(0))u_0^pv_0^q\\
&= -s^2T_{pq}(0)u_0^pv_0^q, \text{ where } T_{pq} \text{ is the torsion tensor}\\
&= -s^2T(u_0,v_0)^i\,\mathrm{mod}(s^3).
\end{aligned}$$

This proves part (1) of Theorem 6.1. Now suppose that $T(u_0, v_0) = 0$, or in coordinate terms, $(-\Gamma^i_{jk} + \Gamma^i_{kj})u_0^j v_0^k = 0$ for all i. Then in the calculation of $P_2(s)^i - Q_2(s)^i \bmod(s^4)$, a large number of terms cancel and we get

$$P_2(s)^i - Q_2(s)^i = -\frac{s^3}{2}(\Gamma^i_{pr,q} - \Gamma^i_{pq,r} + \Gamma^i_{qj}\Gamma^j_{pr} - \Gamma^i_{rj}\Gamma^j_{pq})u_0^q v_0^r (u_0^p + v_0^p) \bmod(s^4)$$

Note that

$$\Gamma^i_{pr,q} - \Gamma^i_{pq,r} + \Gamma^i_{qj}\Gamma^j_{pr} - \Gamma^i_{rj}\Gamma^j_{pq} = R^i_{pqr}$$

is just the coordinate form of the Riemann curvature tensor. Hence the above reads

$$P_2(s)^i - Q_2(s)^i = -\frac{s^3}{2} R^i_{pqr}(0) u_0^q v_0^r (u_0^p + v_0^p) \bmod(s^4).$$

Hence we have proved the statement (3) of the Theorem 6.1.

Proof that 6.1(1) $\Rightarrow$ 6.1(2) and 6.1(3)$\Rightarrow$ 6.1(4). As the first step, we define two kinds of geodesic gap functions G_{I} and G_{II}. With hypothesis and notation as in the statement of Theorem 6.1, we put

$$G_{\mathrm{I}}(P_0, u_0, v_0, f, s) = f(P_4(s)) - f(P_0) \quad \text{and}$$
$$G_{\mathrm{II}}(P_0, u_0, v_0, f, s) = f(P_2(s)) - f(Q_2(s)).$$

For a given f, we regard both G_{I} and G_{II} as smooth functions $(-a, a) \times W \to \mathbb{R}$.

By a scaling argument, in order to prove the result for a triple (P, u, v), it is enough to verify it for $(P, \lambda u, \lambda v)$ for all λ in a neighbourhood of 0. Hence it is enough to take the point $(x, u_x, v_x) \in TM \times_M TM$ to be of the form $(x, 0, 0)$. Then note that $\mathbb{T}_0(x, 0, 0) = \mathbb{T}_0^{-1}(x, 0, 0) = (x, 0, 0)$. Hence by continuity, $(x, 0, 0)$ has an open neighbourhood V, and there is some $0 < b < a$ such that $\mathbb{T}_s^i(P_0, u_0, v_0) \in W$ for all $-2 \le i \le 4$, whenever $s \in (-b, b)$ and $(P_0, u_0, v_0) \in V$. For a given (P_0, u_0, v_0), recall that

$$\mathbb{T}_s^2(P_0, u_0, v_0) = (P_2(s), -u_2(s), -v_2(s)),$$

where $u_2(s)$ and $v_2(s)$ are respectively the parallel transports of u_0 and v_0 along the leg $P_0P_1(s)$ followed by the leg $P_1(s)P_2(s)$ of the geodesic quadrilaterals. Now consider the map $(-b, b) \to W$ that sends

$$\tau \mapsto (P_{0,\tau}, u_{0,\tau}, v_{0,\tau}) = (P_2(\tau), -u_2(\tau), -v_2(\tau)).$$

Composing the gap function G_{II} with this function $(-b, b) \to W$, we get a *parametric form* of G_{II}, defined by

$$(s, \tau) \mapsto G_{\mathrm{II}}(P_0, u_0, v_0, f, s, \tau) = G_{\mathrm{II}}(P_{0,\tau}, u_{0,\tau}, v_{0,\tau}, f, s)$$

which for a given f defines a smooth function $(-a, a) \times (-b, b) \to \mathbb{R}$. Hence applying the homomorphism $C^\infty((-a, a) \times W) \to C^\infty((-a, a) \times (-b, b))$ to the relation of Theorem 6.1(1) and (3), we get the following relations in $C^\infty((-a, a) \times (-b, b))$.

$$G_{\mathrm{II}}(P_{0,\tau}, u_{0,\tau}, v_{0,\tau}, f, s) = -s^2 T(u_{0,\tau}, v_{0,\tau})(f) \bmod s^3 C^\infty$$
$$((-a, a) \times (-b, b)), \text{ and when } T \equiv 0,$$

$$G_{\rm II}(P_{0,\tau}, u_{0,\tau}, v_{0,\tau}, f, s) = -\frac{s^3}{2} R(u_{0,\tau}, v_{0,\tau})(u_{0,\tau} + v_{0,\tau})(f) \bmod s^4 C^\infty((-a, a) \times (-b, b)).$$

Note that at $s = 0$, the smooth function $T(-u_2(s), -v_2(s))(f)$ takes the value $T(u_0, v_0)(f)$, so we have

$$T(-u_2(s), -v_2(s))(f) = T(u_0, v_0)(f) \bmod(s).$$

Similarly,

$$R(-u_2(s), -v_2(s))(-u_2(s) - v_2(s))(f) = -R(u_0, v_0)(u_0 + v_0)(f) \bmod(s).$$

The diagonal $\Delta \subset (-a, a) \times (-b, b)$ is defined by the equation $s = \tau$. This gives a closed imbedding

$$(-b, b) \hookrightarrow (-a, a) \times (-b, b) : \tau \mapsto (\tau, \tau).$$

Restricting functions on $(-a, a) \times (-b, b)$ to $(-b, b)$ under the above diagonal imbedding defines a homomorphism

$$C^\infty((-a, a) \times (-b, b)) \to C^\infty(-b, b)$$

under which $\tau \mapsto s$ and $s \mapsto s$. Under this homomorphism, the extension of the principal ideal $sC^\infty((-a, a) \times (-b, b))$ is the principal ideal $sC^\infty(-b, b)$. Hence putting $\tau = s$ in the relation

$$G_{\rm II}(P_{0,\tau}, u_{0,\tau}, v_{0,\tau}, f, s) = -s^2 T(u_{0,\tau}, v_{0,\tau})(f) \bmod s^3 C^\infty((-a, a) \times (-b, b)),$$

we get

$$\begin{aligned} &G_{\rm II}(P_2(s), -u_2(s), v_2(s), f, s)\\ &= -s^2 T(-u_2(s), -v_2(s))(f) \bmod s^3 C^\infty(-b, b)\\ &= -s^2 T(-u_0, -v_0)(f) \bmod s^3 C^\infty(-b, b)\\ &= -s^2 T(u_0, v_0)(f) \bmod s^3 C^\infty(-b, b) \qquad (*) \end{aligned}$$

Now suppose that $T \equiv 0$, so that we have

$$G_{\rm II}(P_{0,\tau}, u_{0,\tau}, v_{0,\tau}, f, s) = -\frac{s^3}{2} R(u_{0,\tau}, v_{0,\tau})(u_{0,\tau} + v_{0,\tau})(f) \bmod s^4 C^\infty((-a, a) \times (-b, b))$$

Restricting under the diagonal imbedding $(-b, b) \hookrightarrow (-a, a) \times (-b, b)$, we get the following relations in the ring $C^\infty(-b, b)$:

$$\begin{aligned} &G_{\rm II}(P_2(s), -u_2(s), -v_2(s), f, s)\\ &= -\frac{s^3}{2} R(-u_2(s), -v_2(s))(-u_2(s) - v_2(s))(f) \bmod(s^4),\\ &= -\frac{s^3}{2} R(-u_0, -v_0)(-u_0 - v_0)(f) \bmod(s^4),\\ &= \frac{s^3}{2} R(u_0, v_0)(u_0 + v_0)(f) \bmod(s^4) \qquad (**) \end{aligned}$$

It now remains to deduce the statements about $f(P_4(s)) - f(P_0)$. For this, recall that we have defined the other gap function G_{I} by $G_{\mathrm{I}}(P_0, u_0, v_0, f, s) = f(P_4(s)) - f(P_0)$. As G_{II} was defined by $G_{\mathrm{II}}(P_0, u_0, v_0, f, s) = f(P_2(s)) - f(Q_2(s))$, we have the obvious identity

$$G_{\mathrm{I}}(P_0, u_0, v_0, f, s) = G_{\mathrm{II}}(P_2(s), -u_2(s), -v_2(s), f, s).$$

We have already proved (see equation $(*)$ above) that the right-hand side is congruent modulo (s^3) to $-s^2 T(u_0, v_0)$. Therefore we have

$$G_{\mathrm{I}}(P_0, u_0, v_0, f, s) = -s^2 T(u_0, v_0) \bmod (s^3)$$

in $C^\infty(-b, b)$ which completes the proof of Theorem 6.1(2) for a fixed (P_0, u_0, v_0), and hence also for the general case of a variable (P_0, u_0, v_0) by applying Lemma 3.1 as already explained.

By equation $(**)$, if $T \equiv 0$, then $G_{\mathrm{II}}(P_2(s), -u_2(s), -v_2(s), f, s)$ is congruent modulo (s^4) to $\frac{s^3}{2} R(u_0, v_0)(u_0 + v_0)(f)$. Therefore we have

$$G_{\mathrm{I}}(P_0, u_0, v_0, f, s) = \frac{s^3}{2} R(u_0, v_0)(u_0 + v_0)(f) \bmod (s^4)$$

in $C^\infty(-b, b)$. This completes the proof of Theorem 6.1(4) for a fixed (P_0, u_0, v_0), and hence also for the general case of a variable (P_0, u_0, v_0) as already explained.

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