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Structure of weakly one-sided duo Ore extensions

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Abstract. Marks (*J. Algebra* **280** (2004) 463–471) proved that if the skew polynomial ring $R[x;\sigma]$ is left or right duo, then $R[x;\sigma]$ is commutative. It is proved that if $R[x;\sigma]$ is weakly left (resp., right) duo over a reduced ring R with an endomorphism (resp., a monomorphism) σ, then $R[x;\sigma]$ is commutative. This concludes that a noncommutative skew polynomial ring is not weakly left duo when the base ring is reduced. It is also shown that if $R[x;\sigma]$ is weakly left duo then the polynomial ring $R[x]$ is weakly left duo. We next study the structure of the Ore extension $R[x;\sigma,\delta]$ when it is weakly left or right duo.

Keywords. Weakly left (right) duo ring; skew polynomial ring; ore extension; rigid endomorphism; commutative ring; radical.

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1. Weakly one-sided duo rings

Throughout this article, every ring is associative with identity unless otherwise stated. Let R be a ring and σ be an endomorphism of R. We follow [7,22] to use the definition of *Ore extension*. A map $\delta : R \to R$ is said to be a σ-derivation if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \sigma(a)\delta(b)+\delta(a)b$ for all $a, b \in R$. The Ore extension $R[x;\sigma,\delta]$ of R is the ring of polynomials with an indeterminate x over R, only subject to the relation $xa = \sigma(a)x+\delta(a)$ for all $a \in R$. Such a ring always exists and is unique up to isomorphism (see [7] for details). Note that the identity of $R[x;\sigma,\delta]$ is equal to $1 = 1_R$, the identity of R. So we obtain $\sigma(1) = 1$ and $\delta(1) = 0$ from the computation $1x + 0 = x = x1 = \sigma(1)x + \delta(1)$. If $\delta = 0$, then $R[x;\sigma,0]$ is written by $R[x;\sigma]$ and is said to be the skew polynomial ring with an indeterminate x over R. When σ is the identity map on R and $\delta = 0$, $R[x;\sigma,\delta]$ is written by $R[x]$ and is said to be the polynomial ring with an indeterminate x over R. $U(R)$

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denotes the group of all units in R. The Jacobson radical of R is denoted by $J(R)$. An ideal I of R is said to be a σ-ideal if $\sigma(I) \subseteq I$, it is said to be a δ-ideal if $\delta(I) \subseteq I$, and it is called a (σ, δ)-ideal if both inclusions hold. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n upper triangular matrix ring over R by $T_n(R)$, where $n \geq 2$. Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere.

Following Feller [5], a ring is said to be *right* (resp., *left*) *duo* if every right (resp., left) ideal is an ideal; and a ring is called *duo* if it is both right and left duo. One may see very useful results for duo rings in [4,20,23]. We next recall a generalization of right duo ring. Following Yao [24], a ring R is called *weakly right* (resp., *left*) *duo* if for each $a \in R$ there exists a positive integer $n = n(a)$, depending on a, such that $a^n R$ (resp., Ra^n) is a two-sided ideal; and R is called weakly duo if it is both weakly left and right duo. A ring R is called *right* (resp., *left*) *quasi-duo* if every maximal right (resp., left) ideal of R is two-sided. A ring is usually said to be *abelian* if every idempotent is central. Weakly right duo rings are both abelian and right quasi-duo by [24, Lemma 4] and [25, Proposition 2.2]. Recall that a ring is said to be *reduced* if it has no nonzero nilpotent elements. Reduced rings are easily shown to be abelian. A ring R is usually called *directly finite* if $ab = 1$ for $a, b \in R$ implies $ba = 1$. Abelian rings are clearly directly finite, and the class of abelian rings is closed under subrings. We use these facts freely.

The class of weakly one-sided duo rings is obviously closed under homomorphic images. In the following argument, we see that this class is not closed under subrings and that the weakly one-sided duo property is not left–right symmetric.

Let $R = W[K]$ be the first Weyl algebra over a field K of characteristic zero. For example, let $A = K\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over K, and $R = A/I$ where I is the ideal of A generated by $yx - xy - 1$. Then R is neither weakly left nor weakly right duo. But the ring $W(K)$ of all quotients of R is a division ring (hence duo), and $W[K] \subsetneq W(K)$.

Let S be the quotient field of the polynomial ring $F[t]$ with an indeterminate t over a field F of characteristic zero. Consider the field monomorphism $\sigma : S \to S$ defined by $\sigma\left(\frac{f(t)}{g(t)}\right) = \frac{f(t^2)}{g(t^2)}$. Next consider the skew power series ring $R = S[[x; \sigma]]$ by σ with an indeterminate x over S in which every element is of the form $\sum_{i=1}^{\infty} a_i x^i$, only subject to $xa = \sigma(a)x$ for all $a \in S$. Then R is left duo but not weakly right duo by [15, Example 1]. Next, consider the skew power series ring $R = S[[x; \sigma]]$ in which every element is of the form $\sum_{i=1}^{\infty} x^i a_i$, only subject to $ax = x\sigma(a)$ for all $a \in S$. Then R is right duo but not weakly left duo by a similar method in [15, Example 1].

2. When $R[x; \sigma]$ is weakly left or right duo

Marks in [21, Theorem 1] proved that if $R[x; \sigma]$ is left or right duo, then σ is the identity map on R. In what follows, we obtain a more general result when $R[x; \sigma]$ is weakly left or right duo over a reduced ring R. Note that $\sigma(U(R)) \subseteq U(R)$, i.e., $U(R) \subseteq \sigma^{-1}(U(R))$.

Lemma 2.1. *Let R be a ring and σ be an endomorphism of R.*

(1) *If $R[x; \sigma]$ is weakly left duo, then there exists $h \geq 1$ such that $\sigma^h(a) = a$ for all $a \in R$ (i.e., σ^h is the identity map on R); in particular, σ is bijective.*
(2) *If $R[x; \sigma]$ is weakly right duo, then σ is surjective.*
(3) *Suppose that $R[x; \sigma]$ is weakly right duo. Then the following are equivalent:*

(i) *σ is injective;*
(ii) *There exists $k \geq 1$ such that $\sigma^k(a) = a$ for all $a \in R$ (i.e., σ^k is the identity map on R).*

(4) *If $R[x;\sigma]$ is weakly left or right duo, then $\sigma^{-1}(U(R)) = U(R)$ and $\sigma(U(R)) = U(R)$.*

Proof. Write $E = R[x;\sigma]$.

(1) Suppose that E is weakly left duo. Then there exists $h \geq 1$ such that $(1+x)^h E \subseteq E(1+x)^h$. So, for every $0 \neq a \in R$, $(1+x)^h a = (a_0 + a_1x + \cdots + a_nx^n)(1+x)^h$ for some $a_0 + a_1x + \cdots + a_nx^n \in E$. But

$$(1+x)^h a = (1+\cdots+x^h)a = a + \cdots + \sigma^h(a)x^h$$

and

$$\begin{aligned}&(a_0 + a_1x + \cdots + a_nx^n)(1+x)^h\\ &\quad = (a_0 + a_1x + \cdots + a_nx^n)(1+\cdots+x^h) = a_0 + \cdots + a_nx^{n+h}.\end{aligned}$$

In this equality, we must have $n = 0$, obtaining $a_0 = a$ and $a_0 = \sigma^h(a)$. Therefore $\sigma^h(a) = a$, and this yields that σ is bijective.
(2) Suppose that E is weakly right duo. Assume on the contrary that $p \notin \sigma(R)$ for some $p \in R$. Then $px^k \notin x^kE$ for all $k \geq 1$ because

$$\begin{aligned}&x^k(a_0 + a_1x + \cdots + a_nx^n)\\ &\quad = \sigma^k(a_0)x^k + \sigma^k(a_1)x^{k+1} + \cdots + \sigma^k(a_n)x^{k+n} \neq px^k\end{aligned}$$

for all $a_0 + a_1x + \cdots + a_nx^n \in E$, contrary to E being weakly right duo. Thus σ is surjective.
(3) (ii) $\Rightarrow$ (i) is shown by the latter part of the proof of (1). Assuming (i), we know from (2) that σ is bijective, and then (ii) follows by applying (1) to the weakly left duo opposite ring $R[x;\sigma]^{op} \cong R^{op}[x;\sigma^{-1}]$.
(4) Let E be weakly left duo. Then σ is bijective by (1). So we have that $\sigma^{-1}(U(R)) = U(R)$ and $\sigma(U(R)) = U(R)$.

Next suppose that E is weakly right duo. Then σ is surjective by (2). We use this fact freely. To show $\sigma^{-1}(U(R)) \subseteq U(R)$ and $\sigma(U(R)) = U(R)$, we apply the method in the proof of [21, Theorem 1]. Let $b \in U(R)$. Then $b = \sigma(a)$ for some $a \in R$. Since E is weakly right duo, there exists $k \geq 1$ such that $xa^k \in a^kE$. Then $xa^k = a^kh(x)$ for some $h(x) = b_0 + b_1x + \cdots + b_lx^l \in E$. We obtain

$$\sigma(a)^kx = \sigma(a^k)x = xa^k = a^kh(x) = a^kb_0 + a^kb_1x + \cdots + a^kb_lx^l,$$

entailing $\sigma(a)^k = a^kb_1 \in a^kR$. But $\sigma(a)^k \in U(R)$, so $a^kR = R$. This implies $a \in U(R)$, noting that abelian rings are directly finite. Then $\sigma^{-1}(U(R)) \subseteq U(R)$ (hence $\sigma^{-1}(U(R)) = U(R)$) and $\sigma(U(R)) = U(R)$. □

Let R be a ring and σ be an endomorphism of R. Considering the fact that σ is the identity map on R and R is commutative when $R[x;\sigma]$ is left duo, one may conjecture that if $R[x;\sigma]$ is weakly left duo then R is also commutative and σ is the identity map on R, i.e., $h = 1$ in (1) of Lemma 2.1. However, the following examples exclude the possibility.

Example 2.2.

(1) Let $R = D_3(\mathbb{Z}_2)$. Then R is a weakly duo ring because every matrix in R is either invertible or nilpotent, but it is neither left nor right duo by [24, Example 1]. Note that $R[x]$ is isomorphic to $D_3(\mathbb{Z}_2[x])$. We claim that $R[x]$ is also a weakly duo ring. Letting $p(x) = \begin{pmatrix} f & g & h \\ 0 & f & k \\ 0 & 0 & f \end{pmatrix} \in R[x]$ (where $f, g, h, k \in \mathbb{Z}_2[x]$), we have $p(x)^4 = f^4 I_3$ that is central in $R[x]$, where I_3 is the identity matrix of R. Therefore, $R[x]p(x)^4 = p(x)^4 R[x]$ is an ideal of $R[x]$. However, $R[x]$ is neither left nor right duo. For $E_{12}E_{23} = E_{13}$, $E_{13} \neq CE_{12}$ and $E_{13} \neq E_{23}D$ for any $C, D \in R[x]$.

(2) Let T be the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and consider the automorphism σ of T defined by $\sigma((a, b)) = (b, a)$. Let

$$R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha \in \{(0,0), (1,1)\} \text{ and } \beta, \gamma, \delta \in T \right\}$$

be a subring of $D_3(T)$. The automorphism σ of T is extended to the automorphism $\sigma : R \to R$ which is defined by $\sigma((\upsilon_{ij})) = (\sigma(\upsilon_{ij}))$. Note that σ^2 is the identity map on R and that $R[x;\sigma]$ is isomorphic to $\left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in T[x;\sigma] \right\}$, where every coefficient of a is in $\{(0,0), (1,1)\}$.

Let $u(x) = \begin{pmatrix} f & g & h \\ 0 & f & k \\ 0 & 0 & f \end{pmatrix} \in R[x;\sigma]$. Then $f = 0$ or $f = (1,1)x^{n_1} + (1,1)x^{n_2} + \cdots + (1,1)x^{n_t}$ with $n_i \geq 0$ when $f \neq 0$. So $f^2 = 0$ or $f^2 = (1,1)x^{2n_1} + (1,1)x^{2n_2} + \cdots + (1,1)x^{2n_t}$ since the characteristic of $T[x;\sigma]$ is 2. Thus f^2 is central in $T[x;\sigma]$. Next we get $u(x)^4 = \begin{pmatrix} f^4 & p & q \\ 0 & f^4 & r \\ 0 & 0 & f^4 \end{pmatrix}$, where $p = f^2(fg+gf)+(fg+gf)f^2$, $q = f^2(fh+gk+hf)+(fg+gf)(fk+kf)+(fh+gk+hf)f^2$, and $r = f^2(fk+kf)+(fk+kf)f^2$. Since f^2 is central, $p = 0$, $q = (fg+gf)(fk+kf)$ and $r = 0$. Thus $u(x)^4 = \begin{pmatrix} f^4 & 0 & q \\ 0 & f^4 & 0 \\ 0 & 0 & f^4 \end{pmatrix}$, and so $u(x)^8 = \begin{pmatrix} f^8 & 0 & 0 \\ 0 & f^8 & 0 \\ 0 & 0 & f^8 \end{pmatrix}$ is central in $R[x;\sigma]$. Therefore, $R[x;\sigma]u(x)^8 = u(x)^8 R[x;\sigma]$ is an ideal of $R[x;\sigma]$ and hence $R[x;\sigma]$ is a weakly duo ring. Moreover, we can show that $R[x;\sigma]$ is neither left nor right duo by using the same method as in (1).

The weakly left and right duo properties have in fact different impact on an Ore extension $R[x;\sigma]$ of endomorphism types as seen in Lemma 2.1. These properties do an essential role in the following theorem. Recall that for an endomorphism σ of a ring R, σ is called a *rigid* endomorphism if $a\sigma(a) = 0$ for $a \in R$ implies $a = 0$, and the ring R is called σ-*rigid* if there exists a rigid endomorphism σ of R, as in [8,16]. It is well-known that every σ-rigid ring is reduced and σ is injective.

Let R be a ring and σ be an endomorphism of R. Recall that an element $a \in R$ is called σ-*nilpotent* if for every $m \geq 1$, there exists $n \geq 1$ such that $a\sigma^m(a)\cdots\sigma^{mn}(a) = 0$, and a subset S of R is called σ-*nil* if every element of S is σ-nilpotent.

Remark 2.3. For a ring R with an endomorphism σ, if R is σ-rigid then σ^k is also rigid for all $k \geq 1$ and R does not have nonzero σ-nil subsets.

In what follows, we use the reducedness of R freely, recalling that σ-rigid rings are reduced. For the proof of the first affirmation, let $a\sigma^k(a) = 0$ for all $k \geq 1$ and $a \in R$, then $a^2 = 0$ by [8, Lemma 4]. Hence $a = 0$ because R is reduced.

For the proof of the second affirmation, let a be any element of a σ-nil subset S of R. There exists $n \geq 1$ such that $a\sigma^m(a)\cdots\sigma^{mn}(a) = 0$ for every $m \geq 1$. Then $0 = a\sigma^m(a)\cdots\sigma^{mn}(a) = a\sigma^m(a\sigma^m(a)\cdots\sigma^{m(n-1)}(a))$ implies

$$a^2\sigma^m(a)\cdots\sigma^{m(n-1)}(a) = 0,$$

by [8, Lemma 4]. From $0 = a^2\sigma^m(a)\cdots\sigma^{m(n-1)}(a) = a^2\sigma^m(a\sigma^m(a)\cdots\sigma^{m(n-2)}(a))$, we get

$$a^3\sigma^m(a)\cdots\sigma^{m(n-2)}(a) = 0,$$

by [8, Lemma 4]. Continuing this process, we finally obtain $a^n\sigma^m(a) = 0$. This yields $a^{n+1} = 0$, by [8, Lemma 4]. But R is reduced, and $a = 0$ follows. Consequently $S = \{0\}$, and therefore R does not have nonzero σ-nil subsets.

Let R be a ring and σ be an endomorphism of R. Due to Leroy *et al.* [19], let

$$N_\sigma(R) = \{a \in R \mid a\sigma(a)\cdots\sigma^n(a) = 0 \text{ for some integer } n \geq 1\}.$$

The following is stated prior to [19, Lemma 8]:

(1) $N_\sigma(R) = \{a \in R \mid (ax)^n = 0 \text{ for some positive integer } n\}$, where $ax \in R[x;\sigma]$; and
(2) since $\sigma(N_\sigma(R)) \subseteq N_\sigma(R)$, we have that if $N_\sigma(R)$ is an ideal of R then $N_\sigma(R)[x;\sigma]$ is also an ideal of $R[x;\sigma]$. So σ induces an endomorphism, also denoted by σ, on $R/N_\sigma(R)$, obtaining $(R/N_\sigma(R))[x;\sigma] \cong R[x;\sigma]/N_\sigma(R)[x;\sigma]$.

Some results in the following theorem are based on this argument.

Theorem 2.4. *Let R be a reduced ring and σ be an endomorphism* (*resp., a monomorphism*) *of R. Suppose that $R[x;\sigma]$ is a weakly left* (*resp., right*) *duo ring. Then we have the following results*:

(1) *σ is rigid* (*i.e., $R[x;\sigma]$ is reduced*).
(2) *$N_\sigma(R) = 0$ and hence $R[x;\sigma]$ is commutative* (*i.e., R is commutative and σ is the identity map on R*).
(3) *$J(R[x;\sigma]) = 0$.*

Proof. First, note that σ is an automorphism by Lemma 2.1.

(1) Write $E = R[x;\sigma]$. Let $a\sigma(a) = 0$ for $a \in R$. Suppose that E is weakly left duo. Then there exists $k \geq 1$ such that $a^k x = (b_0 + b_1x + \cdots + b_mx^m)a^k$ for some $b_0 + b_1x + \cdots + b_mx^m \in E$. This implies $a^k = b_1\sigma(a)^k$ since $a^kx = b_1\sigma(a^k)x$. From $a\sigma(a) = 0$, we obtain

$$0 = a\sigma(a) = ab_1\sigma(a)^k = ab_1\sigma(a^k) = a^{k+1}$$

and hence $a = 0$ using the assumption that R is reduced. Therefore σ is rigid.

Next suppose that E is weakly right duo. Then there exists $k \geq 1$ such that $xa^k = a^k(b_0 + b_1x + \cdots + b_mx^m)$ for some $b_0 + b_1x + \cdots + b_mx^m \in E$. This yields

$\sigma(a^k) = a^k b_1$ because $xa^k = \sigma(a^k)x$. From $a\sigma(a) = 0$, we obtain $0 = a\sigma(a) = a\sigma(a)^k b_1 = a\sigma(a^k)b_1 = a^{k+1}b_1$. Since R is reduced, $(ab_1)^{k+1} = 0$, forcing $ab_1 = 0$; and $\sigma(a)^k = \sigma(a^k) = a^k b_1 = 0$ follows. This entails $\sigma(a) = 0$ and so $a = 0$ because σ is injective. This concludes that σ is rigid and so $R[x;\sigma]$ is reduced by [9, Proposition 3].
(2) By [19, Corollary 9], $(R/N_\sigma(R))[x;\sigma]$ is a commutative ring. Moreover, it can be shown that $N_\sigma(R) = 0$ by the same computation as the described in the previous sentence of this theorem, since σ is rigid by (1). These facts imply that $R[x;\sigma]$ is commutative.
(3) This follows immediately from Amitsur's theorem [1], since σ is the identity map by (2). □

By Theorem 2.4, one can say that a noncommutative skew polynomial ring is never weakly one-sided duo when the base ring is reduced. The weakly left (right) duo property does not pass to polynomial rings as follows: Let R be a noncommutative division ring. Then R is weakly left (right) duo. However, $R[x]$ is not weakly left(right) duo by Theorem 2.4(2), since $R[x]$ is not commutative.

For a ring R, the upper nilradical (i.e., the sum of all nil ideals) and the set of all nilpotent elements of a ring R are denoted by $\mathrm{Nil}^*(R)$ and $N(R)$, respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N^*(R) \subseteq N(R)$. We also denote the sum of all σ-nil σ-ideals of R by $\mathrm{Nil}^*_\sigma(R)$ for an endomorphism σ of a ring R. When σ is an automorphism, $\mathrm{Nil}^*_\sigma(R)$ is the unique maximal σ-nil σ-ideal of R, called the *σ-upper nilradical* of R, by [17, Theorem 3.5]. The automorphism σ induces an automorphism on $R/\mathrm{Nil}^*_\sigma(R)$ (also written by σ) and $R/\mathrm{Nil}^*_\sigma(R)$ has no nonzero σ-nil σ-ideals, i.e., $\mathrm{Nil}^*_\sigma(R/\mathrm{Nil}^*_\sigma(R)) = \{0\}$, also by [17, Theorem 3.5].

Theorem 2.5. *Let σ be an endomorphism (resp., a monomorphism) of R. Suppose that $R[x;\sigma]$ is a weakly left (resp., right) duo ring. Then $N(R) = \mathrm{Nil}^*(R) = \mathrm{Nil}^*_\sigma(R) = N_\sigma(R)$, and $R/N(R)$ is commutative.*

Proof. Assume that $R[x;\sigma]$ be a weakly left duo. Then σ is bijective by Lemma 2.1(1), and so we have $\mathrm{Nil}^*(R) = \mathrm{Nil}^*_\sigma(R)$ by [11, Proposition 3.17]. Let $a \in N(R)$. Then $a^n = 0$ for some $n \geq 1$. Since $R[x;\sigma]/J(R[x;\sigma])$ is reduced and $J(R[x;\sigma]) = (N_\sigma(R) \cap J(R)) + N_\sigma(R)[x;\sigma]x$ by [18, Proposition 2.3(1)], $a \in J(R[x;\sigma])$ and this implies $a \in N_\sigma(R)$. Thus $N(R) \subseteq N_\sigma(R)$.

Also, by [18, Proposition 2.3(1)], $N_\sigma(R)$ is a σ-nil ideal of R, and $N_\sigma(R)$ is clearly a σ-ideal of R. Hence, $N_\sigma(R) \subseteq \mathrm{Nil}^*_\sigma(R)$. This implies that $N_\sigma(R) = \mathrm{Nil}^*_\sigma(R)$ since $\mathrm{Nil}^*_\sigma(R) \subseteq N_\sigma(R)$.

Consequently, we have

$$N(R) \subseteq N_\sigma(R) = Nil^*_\sigma(R) = Nil^*(R) \subseteq N(R).$$

Note $\sigma(N(R)) = N(R)$. Then the automorphism σ induces an automorphism on $R/N(R)$ (also written by σ). So we obtain that $(R/N(R))[x;\sigma]$ is weakly left duo from the assumption that $R[x;\sigma]$ is weakly left duo. Thus $R/N(R)$ is commutative, by Theorem 2.4(2).

The proof for the right case is similar. □

By Theorem 2.5, Köthe's conjecture (i.e., the upper nilradical contains all nil left ideals) holds for weakly left (resp., right) duo rings $R[x;\sigma]$, where σ is an endomorphism (resp., a monomorphism) of a ring R.

The following is an immediate consequence of Theorem 2.5.

COROLLARY 2.6

(1) *Let R be a ring and suppose that $R[x]$ is a weakly left or right duo ring. Then $R/\mathrm{Nil}^*(R)$ is a commutative reduced ring.*
(2) *Let R be a reduced ring and suppose that $R[x]$ is a weakly left or right duo ring. Then R is a commutative ring.*

Let σ be an endomorphism of a ring R and $h \geq 1$. In the following, we see an information about the skew polynomial ring $R[x;\sigma^h]$ when $R[x;\sigma]$ is weakly left (right) duo. Observe that $R[x;\sigma]$ is naturally graded by the Abelian groups $R[x^i;\sigma]$ with $i \geq 1$, i.e., $R[x;\sigma] = \bigoplus_{i=0}^{\infty} R[x^i;\sigma]$ and $R[x^i;\sigma]R[x^j;\sigma] \subseteq R[x^{i+j};\sigma]$. Note that $R[x^i;\sigma]$ is isomorphic to $R[x;\sigma^i]$.

PROPOSITION 2.7

Let R be a ring and σ be an endomorphism of R. If $R[x;\sigma]$ is weakly left (*resp.*, *right*) *duo, then $R[x;\sigma^h]$ is weakly left* (*resp.*, *right*) *duo for every $h \geq 1$.*

Proof. Suppose that $R[x;\sigma]$ is weakly left duo and let $h \geq 1$. By the argument prior to this proposition, $R[x^h;\sigma]$ is isomorphic to $R[x;\sigma^h]$. So it suffices to prove that $R[x^h;\sigma]$ is weakly left duo. Take $f(x^h) = \sum_{i=0}^{n} a_i x^{hi}$, $g(x^h) = \sum_{j=0}^{m} b_j x^{hj}$ in $R[x^h;\sigma]$. Since $R[x;\sigma]$ is weakly left duo, there exists $k \geq 1$ such that $f(x^h)^k g(x^h) = g'(x) f(x^h)^k$ for some $g'(x) = c_0 + c_1 x + \cdots + c_t x^t \in R[x;\sigma]$.

Let $g'(x) = \sum_{l=0}^{p} c_l x^{hl} + \sum_{s=1}^{q} c_s x^s$, where s is not a multiple of h for all s. Then, by comparing the terms of both sides of

$$f(x^h)^k g(x^h) = \left(\sum_{l=0}^{p} c_l x^{hl}\right) f(x^h)^k + \left(\sum_{s=1}^{v} c_s x^s\right) f(x^h)^k,$$

we obtain $(\sum_{s=1}^{v} c_s x^s) f(x^h)^k = 0$. Consequently we now have

$$f(x^h)^k g(x^h) = g'(x) f(x^h)^k = (c_0 + c_h x^h + \cdots + c_{hp} x^{hp}) f(x^h)^k,$$

noting $c_0 + c_h x^h + \cdots + c_{hp} x^{hp} \in R[x^h, \sigma]$. This completes the proof. The proof for the right case is similar. □

For an endomorphism (resp., a monomorphism) σ of R, if $R[x;\sigma]$ is weakly left (resp., right) duo, $R[x]$ is weakly left (resp., right) duo, by Lemma 2.1(1) and Proposition 2.7. But the converse need not hold: Let $R_i = \mathbb{Z}$ for all integer i, and set $R = \prod_{i=-\infty}^{\infty} R_i$ be the direct product of R_i's. Then $R[x]$ is commutative (hence (weakly) duo). Consider the isomorphism σ of R defined by $\sigma((a_i)) = (b_i)$ with $b_{i+1} = a_i$ for all i, where $(a_i), (b_i) \in R$. Then $R[x;\sigma]$ is neither weakly left nor weakly right duo by Lemma 2.1

or Theorem 2.4. In fact, letting $(a_i) \in R$ with $a_1 = 1$ and $a_i = 0$ for all $i \neq 1$, we have $(a_i)^k R[x;\sigma] \nsubseteq R[x;\sigma](a_i)^k$ and $(a_i)^k R[x;\sigma] \nsupseteq R[x;\sigma](a_i)^k$ for all $k \geq 1$. Note that σ^h is not the identity map on R for all $h \geq 1$.

We next observe the form of units in $R[x;\sigma]$ when $R[x;\sigma]$ is weakly left or right duo.

Theorem 2.8. *Let σ be an endomorphism (resp., a monomorphism) of R. If $R[x;\sigma]$ is weakly left (resp., right) duo, then $U(R[x;\sigma]) = U(R) + N(R)[x;\sigma]x$.*

Proof. Suppose that $R[x;\sigma]$ is weakly left duo. By Theorem 2.5, $N(R)$ is an ideal such that $\sigma(N(R)) = N(R)$. Then we have $(R/N(R))[x;\sigma] \cong R[x;\sigma]/N(R)[x;\sigma]$ is reduced weakly left duo and so $(R/N(R))[x;\sigma] = (R/N(R))[x]$ by Theorem 2.4(2). Put $R/N(R) = \bar{R}$.

Let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in U(R[x;\sigma])$. Then $f(x)g(x) = g(x)f(x) = 1$ for some $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma]$. Note that $a_0 \in U(R)$. Since $\bar{R}$ is reduced, we get that $\bar{a}_i\bar{b}_j = \bar{0}$ for all i, j with $i + j \geq 1$, by applying the proof of [2, Lemma 1]. In particular, $\bar{a}_i\bar{b}_0 = \bar{0}$ for all $i \geq 1$, and so $\bar{a}_i = \bar{0}$ because $\bar{b}_0\bar{a}_0 = \bar{1}$. This leads us to the conclusion that $a_i \in N(R)$ for all $i \geq 1$, as desired.

Conversely, let $h(x) = c_0 + c_1x + \cdots + c_kx^k \in U(R) + N(R)[x;\sigma]x$. That is, $c_0 \in U(R)$ and $c_i \in N(R)$ for any $1 \leq i \leq k$. Then, by Theorem 2.5, $c_i \in N_\sigma(R)$ for any $i \geq 1$. Moreover, $J(R[x;\sigma]) = (N_\sigma(R) \cap J(R)) + N_\sigma(R)[x;\sigma]x$ by [18, Proposition 2.3(1)], and thus $c_1x + \cdots + c_kx^k \in J(R[x;\sigma])$. Letting $c_0d = dc_0 = 1$ for some $d \in R$, we have $1 + (c_1x + \cdots + c_kx^k)d = (c_0 + c_1x + \cdots + c_kx^k)d$ is right invertible in $R[x;\sigma]$. This implies that $h(x) = c_0 + c_1x + \cdots + c_kx^k$ is right invertible in $R[x;\sigma]$; hence $h(x) \in U(R[x;\sigma])$ because $R[x;\sigma]$ is directly finite.

The proof for the case of weakly right duo is similar. □

Let R be a ring and σ be an endomorphism of R. Suppose that σ is injective. Then there exists a universal over-ring $A(R,\sigma)$ of R, called the *σ-Cohn–Jordan extension* of R, such that σ extends to an automorphism of $A(R,\sigma)$, also denoted by σ, and $A(R,\sigma) = \bigcup_{i=0}^{\infty} \sigma^{-i}(R)$ (see [13] and [18] for details).

PROPOSITION 2.9

Let R be a ring and σ be a monomorphism of R.

(1) *If R is weakly left (resp., right) duo, then $A(R,\sigma)$ is weakly left (resp., right) duo.*
(2) *If $R[x;\sigma]$ is weakly left (resp., right) duo, then $A(R,\sigma)[x;\sigma]$ is weakly left (resp., right) duo.*

Proof.

(1) Suppose that R is weakly left duo. Let $a, b \in A = A(R,\sigma)$. Then there exists an integer $n \geq 1$ such that $\sigma^n(a), \sigma^n(b) \in R$. Since R is weakly left duo, $R\sigma^n(a)^k = R\sigma^n(a^k)$ is an ideal of R for some $k \geq 1$.
Here we claim that Aa^k is an ideal of A. From the arguments above, we obtain

$$\sigma^n(a^kb) = \sigma^n(a^k)\sigma^n(b) = c\sigma^n(a^k) = \sigma^n[\sigma^{-n}(c)a^k],$$

where $\sigma^n(a^k)\sigma^n(b) = c\sigma^n(a^k)$ for some $c \in R$ because $R\sigma^n(a^k)$ is an ideal of R. Then $a^kb = \sigma^{-n}(c)a^k$ is an equality in A because σ is an automorphism of A. This implies that

$a^k b \in Aa^k$, and so Aa^k is an ideal of A. Therefore, A is weakly left duo. The proof for the right case is similar.
(2) We apply the proof of [18, Proposition 1.8(2)]. Note that $A(R[x;\sigma],\sigma) = A(R,\sigma)[x;\sigma]$ since σ can be extended to a monomorphism of $R[x;\sigma]$ by setting $\sigma(x) = x$. Applying (1) to the ring $R[x;\sigma]$, we can obtain (2). □

The converse of Proposition 2.9(1) does not hold in general, by [18, Example 1.9].

The following argument is a sort of dual one of the construction of σ-Cohn–Jordan extension. Let R be a ring with an endomorphism σ, and set $K = \bigcup_{i=1}^{\infty} \ker(\sigma^i)$. Then K is a σ-ideal of R, and thus σ induces an injective endomorphism, also denoted by σ, of the ring $\bar{R} = R/K$. The following argument is based on this simple fact.

COROLLARY 2.10

Let R be a ring with an endomorphism σ. Suppose that $R[x;\sigma]$ is a weakly left (right) duo ring. Then

(1) *R and $\bar{R}[x;\sigma]$ are weakly left (right) duo.*
(2) *$A(\bar{R},\sigma)[x;\sigma]$ is weakly left (right) duo.*

Proof.

(1) It is obvious because R and $\bar{R}[x;\sigma]$ are homomorphic images of $R[x;\sigma]$.
(2) This is obtained from Proposition 2.9(2) applied to the ring $\bar{R}$. □

3. When $R[x;\sigma,\delta]$ is weakly left or right duo

In this section, we study the structure of the Ore extension $R[x;\sigma,\delta]$ when it is weakly left or right duo. First, note that Example 2.2(2) shows the existence of a weakly duo ring $R[x;\sigma,\delta]$ which is neither left nor right duo, defining $\delta : R \to R$ by $\delta(r) = dr - \sigma(r)d$, where $d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (1,0) \\ 0 & 0 & 0 \end{pmatrix} \in R$. Then δ is a nonzero inner σ-derivation of R and so $R[x;\sigma,\delta]$ is isomorphic to $R[y;\sigma]$, where $y = x - d$ (see [7, Exercise 2Y, page 44]). Thus, by Example 2.2(2), $R[x;\sigma,\delta]$ is weakly duo.

Let R be a reduced ring, σ be an endomorphism of R, and δ be a σ-derivation of R. If $R[x;\sigma]$ is weakly left (resp., right) duo where σ is an endomorphism (resp., a monomorphism) σ of R, then $R[x;\sigma]$ is reduced by Theorem 2.4. In the case of $R[x;\sigma,\delta]$, we have the same result as follows.

Theorem 3.1. *Let R be a ring, σ be an endomorphism of R, and δ be a σ-derivation of R. Suppose that $R[x;\sigma,\delta]$ is weakly left duo. Then we have the following results*:

(1) *There exists $k \geq 1$ such that*

$$\delta^k(a) = 0,$$
$$[(\delta^{k-1}\sigma) + (\delta^{k-2}\sigma\delta) + \cdots + (\delta\sigma\delta^{k-2}) + (\sigma\delta^{k-1})](a) = 0,$$
$$\cdots$$
$$[(\sigma^{k-1}\delta) + (\sigma^{k-2}\delta\sigma) + \cdots + (\sigma\delta\sigma^{k-2}) + (\delta\sigma^{k-1})](a) = 0$$

for all $a \in R$.

(2) *There exists* $l \geq 1$ *such that* $\sigma^l(a) = a$ *for all* $a \in R$.

(3) *For any* $a \in R$, *there exists* $t \geq 1$ *such that* a^t *is central. Hence* R *is weakly duo.*

(4) *If* R *is reduced, then* σ *is rigid* (*i.e.*, $R[x; \sigma, \delta]$ *is reduced*).

Proof. Let $E = R[x; \sigma, \delta]$.

(1) Note $\sigma(0) = 0 = \delta(0)$, so let $0 \neq a \in R$. Since E is weakly left duo, there exists $k \geq 1$ such that $x^k E \subseteq E x^k$. So, for every $a \in R$, there exists $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in E$ such that $x^k a = f(x) x^k$.

If $\sigma^k(a) = 0$, then the degree of $x^k a$ is less than k, and hence $f(x) = 0$ because the degree of $f(x)x^k$ is equal to or larger than k when it is nonzero. This yields $x^k a = 0$.

Next, suppose $\sigma^k(a) \neq 0$. Then the degree of $x^k a$ is k, and so we have $n = 0$ from the equality

$$\begin{aligned}\delta^k(a) + \cdots + \sigma^k(a)x^k = x^k a &= (a_0 + a_1 x + \cdots + a_n x^n)x^k \\ &= a_0 x^k + a_1 x^{k+1} + \cdots + a_n x^{n+k},\end{aligned}$$

entailing $\delta^k(a) + \cdots + \sigma^k(a)x^k = a_0 x^k$.

But

$$\begin{aligned}x^k a = \delta^k(a) &+ [(\delta^{k-1}\sigma) + (\delta^{k-2}\sigma\delta) + \cdots + (\delta\sigma\delta^{k-2}) + (\sigma\delta^{k-1})](a)x \\ &+ \cdots + [(\sigma^{k-1}\delta) + (\sigma^{k-2}\delta\sigma) + \cdots + (\sigma\delta\sigma^{k-2}) \\ &+ (\delta\sigma^{k-1})](a)x^{k-1} + \sigma^k(a)x^k.\end{aligned}$$

Therefore we obtain the result.

(2) First note that there exists $k \geq 1$ such that $\delta^k(a) = 0$ for all $a \in R$ by (1). Since E is weakly left duo, there exists $h \geq 1$ such that $(1 + x^k)^h a = (a_0 + a_1 x + \cdots + a_n x^n)(1 + x^k)^h$ for some $a_0 + a_1 x + \cdots + a_n x^n \in E$. By applying the proof of Lemma 2.1(1), we can conclude that $\sigma^{kh}(a) = a$ for all $a \in R$ (hence σ is bijective), noting $\delta^{kt}(a) = 0$ for any $t \geq 1$.

(3) Let $0 \neq a \in R$. Since E is weakly left duo, σ is bijective and there exist positive integers h and k such that σ^h is the identity map on R and $\delta^k = 0$ by (1) and (2) above. Set $l = hk$. Since E is weakly left duo, there exists $t \geq 1$ such that $(a + x^l)^t E \subseteq E(a + x^l)^t$. For any $r \in R$, we have $(a + x^l)^t r = f(x)(a + x^l)^t$ for some $f(x) = b_0 + b_1 x + \cdots + b_n x^n \in E$. Then $(a + x^l)^t r = a^t r + \cdots + r x^{lt}$, since $\delta^{lt}(r) = 0$ and $\sigma^{lt}(r) = r$. Thus the degree of $f(x)$ is equal to 0 and so $f(x) = b_0$. This implies that $a^t r = b_0 a^t$ and $r = b_0$, and therefore $a^t r = r a^t$.

(4) Suppose that $a\sigma(a) = 0$ for $a \in R$. Note that σ is an automorphism, by (2). Since E is weakly left duo, there exists $k \geq 1$ such that $a^k x = p(x) a^k$ for some $p(x) \in E$. So, by the method in the proof of [10, Theorem 1], we can write $p(x) = b_0 + x b_1 + \cdots + x^n b_n$, where $b_j \in R$. Then $a^k x = (b_0 + x b_1 + \cdots + x^n b_n) a^k$, and so we may assume that $n = 1$ since σ is an automorphism. Thus $b_j a^k = 0$ for $j \geq 2$ and $b_1 a^k \neq 0$. Then $a^k x = (b_0 + x b_1) a^k = b_0 a^k + \delta(b_1 a^k) + \sigma(b_1 a^k) x$, entailing that $a^k = \sigma(b_1)\sigma(a^k)$. Since R is reduced and $a\sigma(a) = 0$, $a\sigma(b_1)\sigma(a)^k = a(\sigma(b_1)\sigma(a^k)) = a^{k+1} = 0$. Thus $a = 0$, showing that σ is rigid. □

In [21, Theorem 2], Marks proved that if $R[x;\sigma,\delta]$ is left duo, then $R[x;\sigma,\delta]$ is commutative (i.e., R is commutative, σ is the identity map on R, and δ is the zero map). In the following, we obtain that σ is the identity map on R, also by applying the method in the proof of Theorem 3.1(1).

Remark 3.2. Suppose that $E = R[x;\sigma,\delta]$ is left duo, where R is a ring, σ is an endomorphism of R, and δ is a σ-derivation of R. Let $0 \neq a \in R$. Since E is left duo, we have $k = 1$ in the proof of Theorem 3.1(1), and hence $\delta(a) = 0$. Also since E is left duo, there exists $f(x) = a_0 + \cdots + a_n x^n \in E$ such that $(1+x)a = f(x)(1+x)$. Here $(1+x)a = a+\delta(a)+\sigma(a)x = a+\sigma(a)x$ because $\delta(a) = 0$. But $(a_0+\cdots+a_n x^n)(1+x) = a_0 + \cdots + a_n x^{n+1}$, and $n = 0$ follows. This yields $a + \sigma(a)x = a_0 + a_0 x$ and so we have $a = a_0 = \sigma(a)$. Thus σ is the identity.

For the reducedness of $R[x;\sigma,\delta]$, we need a restriction on the endomorphism σ when $R[x;\sigma,\delta]$ is weakly right duo.

Theorem 3.3. *Let R be a ring, σ be an endomorphism of R, and δ be a σ-derivation of R. Suppose that $R[x;\sigma,\delta]$ is weakly right duo.*

(1) *If σ is injective, then σ is surjective and there exists $h \geq 1$ such that $\sigma^h(a) = a$ for all $a \in R$.*
(2) *If σ is injective, then for any $a \in R$, there exists $t \geq 1$ such that a^t is central. Hence R is weakly duo.*
(3) *If R is reduced and σ is injective, then σ is rigid (i.e., $R[x;\sigma,\delta]$ is reduced).*

Proof. Let $E = R[x;\sigma,\delta]$.

(1) Suppose that σ is injective. Assume on the contrary that there exists $p \notin \sigma(R)$. Since E is weakly right duo, there exists $k \geq 1$ such that $px^k = x^k(a_0 + a_1x + \cdots + a_n x^n)$ for some $a_0 + a_1 x + \cdots + a_n x^n \in E$. We can let $a_n \neq 0$ because p is nonzero. Note that

$$x^k(a_0 + a_1x + \cdots + a_n x^n) = \delta^k(a_0) + \cdots + (\delta^k(a_1) + \cdots)x \\ + \cdots + (\sigma^k(a_0) + \cdots)x^k + \cdots + \sigma^k(a_n)x^{k+n}.$$

Since σ is injective, $\sigma^k(a_n) \neq 0$. So we must have $n = 0$ by comparing the degrees of both sides of $px^k = x^k(a_0 + a_1x + \cdots + a_n x^n)$. This yields $px^k = \delta^k(a_0) + \cdots + \sigma^k(a_0)x^k$, and so $p = \sigma^k(a_0)$ and $\delta^k(a_0) = 0$. Here $p = \sigma^k(a_0)$ is contrary to $p \notin \sigma(R)$. Thus σ is surjective.

Now, note that R is a weakly right duo ring if and only if the opposite ring R^{op} of R is a weakly left duo ring, and moreover $E^{\mathrm{op}} \cong R^{\mathrm{op}}[x';\sigma^{-1},-\delta\sigma^{-1}]$ by [6, Lemma 1.5(a)]. Then $R^{\mathrm{op}}[x';\sigma^{-1},-\delta\sigma^{-1}]$ is weakly left duo. By Theorem 3.1(2), there exists $h \geq 1$ such that $(\sigma^{-1})^h(r) = r$ for all $r \in R^{\mathrm{op}}$. Thus $\sigma^h(a) = a$ for all $a \in R$.

(2) Suppose that σ is injective. Then σ is an automorphism, by (1). By the same argument as the proof of (1), we have that $R^{\mathrm{op}}[x';\sigma^{-1},-\delta\sigma^{-1}]$ is weakly left duo, and hence a^t is central in R^{op} for some $t \geq 1$, by Theorem 3.1(3). The proof is completed.

(3) Suppose that R is reduced and σ is injective. Let $a\sigma(a) = 0$ for $a \in R$. Then there exists $k \geq 1$ such that $xa^k = a^k(b_0 + b_1x + \cdots + b_m x^m)$ for some $b_0 + b_1x + \cdots + b_m x^m \in E$. Hence $\sigma(a^k) = a^k b_1$ because $xa^k = \sigma(a^k)x + \delta(a^k)$. By the same computation as in the

proof of Theorem 2.4(1), we have $a = 0$. Thus σ is rigid and so $R[x; \sigma, \delta]$ is reduced by [8, Proposition 5]. □

Remark 3.4. Let R be a ring, σ be an endomorphism of R, and δ be a σ-derivation of R. If $R[x; \sigma, \delta]$ is weakly right duo, then R is a weakly right duo. Suppose that E is weakly right duo and $a \in R$. Then $Ea^k \subseteq a^k E$ for some $k \geq 1$. For any $r \in R$, there exists $b_0 + b_1 x + \cdots + b_m x^m \in E$ such that $ra^k = a^k(b_0 + b_1 x + \cdots + b_m x^m)$. This entails $ra^k = a^k b_0 \in a^k R$, completing the proof.

We use $\ker(\sigma)$ for the kernel of the given endomorphism σ. The following results are motivated by [21, Lemma 3 and Theorem 4].

Lemma 3.5. *Let R be a ring, σ be an endomorphism of R, and δ be a σ-derivation of R. Suppose that $R[x; \sigma, \delta]$ is weakly right duo. Then, for any $i \geq 1$, we have* $\ker(\sigma^i) \subseteq J(R)$.

Proof. We apply the proof of [21, Lemma 3]. Suppose that $E = R[x; \sigma, \delta]$ is weakly right duo. Note that R is abelian (hence directly finite). Let $r \in R$, $i \geq 1$, and $a \in \ker(\sigma^i)$. Since E is weakly right duo, there exists $k \geq 1$ such that $x^i(1 - ar)^k \in (1 - ar)^k E$. This yields

$$\sigma^i((1 - ar)^k) = [\sigma^i(1 - ar)]^k = 1^k = 1 \text{ and so}$$
$$\sigma^i((1 - ar)^k) \in (1 - ar)^k R \subseteq (1 - ar)R,$$

because

$$x^i(1 - ar)^k = \sigma^i((1 - ar)^k)x^i + b_{i-1}x^{i-1} + \cdots + b_1 x + \delta^k((1 - ar)^k) \in (1 - ar)^k E,$$

where $b_j \in R$. Thus $1 \in (1 - ar)R$ and $1 - ar \in U(R)$ follows. Therefore $a \in J(R)$ because r is arbitrary. □

In the following, we obtain an information about $R[x; \sigma, \delta]$ when R is reduced.

PROPOSITION 3.6

Let R be a ring, σ be an endomorphism of R, and δ be a σ-derivation of R. Suppose that $R[x; \sigma, \delta]$ is weakly right duo.

(1) *If $J(R) = 0$, then σ is bijective.*
(2) *Let R be reduced. If $a\delta(a) = 0$ for $a \in R$, then $\delta(a) = 0$.*
(3) *Let R be reduced. If $R[x; \sigma, \delta]$ is not reduced, then $0 \neq \ker(\sigma) \subseteq J(R)$.*

Proof. Let $E = R[x; \sigma, \delta]$.

(1) Suppose that $J(R) = 0$. By Lemma 3.5, $\ker(\sigma) = 0$ and so σ is injective. Thus σ is surjective by Theorem 3.3(1), completing the proof.
(2) Let $a\delta(a) = 0$ for $a \in R$. Since E is weakly right duo, there exists $k \geq 1$ such that $xa^k = a^k f(x)$ for some $f(x) = \sum_{i=0}^m b_i x^i \in E$. This yields

$$\delta(a^k) = a^k b_0 \quad \text{and} \quad \sigma(a^k) = a^k b_1.$$

Notice that $\delta(a^n) = \sigma(a^{n-1})\delta(a)$ for any integer $n \geq 1$ because $a\delta(a) = 0$ and R is reduced. Moreover, $0 = ab_0\delta(a) = a^k b_0\delta(a) = \delta(a^k)\delta(a)$. Thus $0 = \delta(a^k\delta(a)) = \sigma(a^k)\delta^2(a) + \delta(a^k)\delta(a) = \sigma(a^k)\delta^2(a)$. From $\sigma(a^k)\delta^2(a) = 0$, we have $\sigma(a)\delta^2(a) = 0$ by reducedness of R. Therefore,

$$0 = \delta(a\delta(a)) = \sigma(a)\delta^2(a) + \delta(a)^2 = \delta(a)^2 \Rightarrow \delta(a) = 0.$$

(3) Suppose that E is not reduced. If σ is injective, then σ is bijective and moreover rigid by Theorem 3.3. This contradicts the assumption. So the proof is done by Lemma 3.5. □

In the following, we see a bundle of useful information about σ and δ when $R[x; \sigma, \delta]$ is weakly right duo.

PROPOSITION 3.7

Let R be a ring, σ be an endomorphism of R, and δ be a σ-derivation of R. If $R[x; \sigma, \delta]$ is weakly right duo, then we have the following results:

(1) *For every $r \in R$, there exists $k \geq 1$ such that $\sigma(r^k)$ and $\delta(r^k)$ are contained in $r^k R$.*
(2) *For every $e^2 = e \in R$, $\delta(e) = 0$ and $\sigma(e) = e$.*
(3) *For every $r \in R$, there exists $k \geq 1$ such that $r^k R$ is a (σ, δ)-ideal of R.*

Proof. Let $E = R[x; \sigma, \delta]$.

(1) Let $r \in R$. Since E is weakly right duo, there exists $k \geq 1$ such that $xr^k \in r^k E$. Since $xr^k = \sigma(r^k)x + \delta(r^k) \in r^k E$, we have $\sigma(r^k), \delta(r^k) \in r^k R$.
(2) Let $e^2 = e \in R$. Then e is central. By (1), we first have $\delta(e), \sigma(e) \in eR$ and $\delta(1-e), \sigma(1-e) \in (1-e)R$. Next, we can apply the proof of [21, Proposition 8] to obtain $\delta(e) = 0$ and $\sigma(e) = e$.
(3) Since E is weakly right duo, so is R. Let $r \in R$. Since E is weakly right duo, there exists $k \geq 1$ such that $r^k E$ is an ideal of E and $r^k R$ is an ideal of R. By the proof of (1), we have $\sigma(r^k), \delta(r^k) \in r^k R$. So the result follows because $r^k R$ is an ideal of R. □

We use $\prod$ to denote the direct product of rings. We obtain the following by help of [21, Corollary 9].

COROLLARY 3.8

Suppose that R is a semiperfect ring, σ is an endomorphism of R, and δ is a σ-derivation of R. If $R[x; \sigma, \delta]$ is weakly right duo, then there is a canonical isomorphism

$$R[x; \sigma, \delta] \cong \prod_{i=1}^{n} R_i[x; \sigma_i, \delta_i],$$

where the decomposition of $1 \in R$ into a sum of n local idempotents yields $R = \prod_{i=1}^{n} R_i$ with R_i's local rings.

Proof. Let $R[x; \sigma, \delta]$ be weakly right duo. Then $\sigma(e) = e$ and $\delta(e) = 0$ for all $e^2 = e \in R$, by Proposition 3.7(2). The corollary follows by the proof of [21, Corollary 9]. □

The set of all central idempotents in a ring R is denoted by $B(R)$ and $(B(R))^\sigma = \{e \in B(R) \mid \sigma(e) = e\}$.

COROLLARY 3.9

Let R be a ring, σ be a monomorphism of R, and δ be a σ-derivation of R. If $R[x; \sigma, \delta]$ is a weakly right duo ring, then $B(R[x; \sigma, \delta]) = (B(R))^\sigma$.

Proof. Suppose that $R[x; \sigma, \delta]$ is a weakly right duo ring. Then σ is an automorphism, by Theorem 3.3(1). So this proof comes from Proposition 3.7(2) and [14, Theorem 3.13], noting that $R[x; \sigma, \delta]$ is abelian. □

The following shows that the results (1,2) of Proposition 3.7 and the necessary condition of Lemma 3.5 do not guarantee Ore extensions to be weakly right duo.

Example 3.10. We apply the construction and argument in [21, Example 13]. Let R be the *trivial extension* of any weakly right duo ring S, $T(S, S)$, with addition defined componentwise and multiplication defined by $(t_1, m_1)(t_2, m_2) = (t_1t_2, t_1m_2 + m_1t_2)$ for $t_i, m_i \in S$. Then R is abelian, by [12, Lemma 2] because weakly right duo rings are abelian.

Let σ and δ be the endomorphism and σ-derivation of R in [21, Example 13], respectively; i.e., $\sigma(t, m) = (t, 0)$ and $\delta(t, m) = (0, m)$ for all $(t, m) \in R$. Then (1) and (2) of Proposition 3.7 are held because every idempotent in R is of the form $(e, 0)$ with $e^2 = e \in S$ by [12, Lemma 2]. Moreover $\ker(\sigma) = \cup_{i=1}^{\infty} \ker(\sigma^i) = (0, S) \subseteq J(R)$, and so the necessary condition of Lemma 3.5 is held.

Next we show that $E = R[x; \sigma, \delta]$ is not weakly right duo. Take $f(x) = (1, 1) + (-1, 1)x \in E$. Assume on the contrary that there exists $k \geq 1$ such that $xf(x)^k \in f(x)^k E$. Say, $xf(x)^k = f(x)^k(a_0 + a_1x + \cdots + a_nx^n)$ with $a_0 + a_1x + \cdots + a_nx^n \in E$. Let $a_i = (t_i, m_i)$ for $i = 0, 1, \ldots, n$. But we have

$$f(x)^k = (1, p) + b_1x + \cdots + b_{k-1}x^{k-1} + ((-1)^k, (-1)^{k+1})x^k$$

for some $p \geq 1$ and $b_l \in R$ $(l = 1, \ldots, k-1)$, by using that $\sigma^j(1, 1) = (1, 0)$, $\sigma^j(-1, 1) = (-1, 0)$, and $\delta^j(1, 1) = (0, 1) = \delta^j(-1, 1)$ for all $j \geq 1$. Now we have

$$\begin{aligned}
xf(x)^k &= x[(1, p) + b_1x + \cdots + b_{k-1}x^{k-1} + ((-1)^k, (-1)^{k+1})x^k] \\
&= \delta(1, p) + (\sigma(1, p) + \delta(b_1))x \\
&\quad + \cdots + (\sigma(b_{k-1}) + \delta((-1)^k, (-1)^{k+1}))x^k \\
&\quad + \sigma((-1)^k, (-1)^{k+1})x^{k+1} \\
&= (0, p) + (1, q)x + \cdots + (s, (-1)^{k+1})x^k + ((-1)^k, 0)x^{k+1};
\end{aligned}$$

and

$$\begin{aligned}
&f(x)^k(a_0 + a_1x + \cdots + a_nx^n) \\
&\quad = ((1, p) + b_1x + \cdots + b_{k-1}x^{k-1} \\
&\qquad + ((-1)^k, (-1)^{k+1})x^k)((t_0, m_0) + (t_1, m_1)x + \cdots + (t_n, m_n)x^n)
\end{aligned}$$

$$
\begin{aligned}
&= (t_0, m_0 + pt_0) + \cdots + ((-1)^k, (-1)^{k+1})\delta^k(t_n, m_n)x^n \\
&\quad + \cdots + ((-1)^k, (-1)^{k+1})\sigma^k(t_n, m_n)x^{k+n} \\
&= (t_0, m_0 + pt_0) + \cdots + ((-1)^k, (-1)^{k+1})(0, m_n)x^n \\
&\quad + \cdots + ((-1)^k, (-1)^{k+1})(t_n, 0)x^{k+n} \\
&= (t_0, m_0 + pt_0) + \cdots + (0, (-1)^k m_n)x^n \\
&\quad + \cdots + ((-1)^k t_n, (-1)^{k+1} t_n)x^{k+n},
\end{aligned}
$$

where $\delta(b_1) = (0, q)$ and $\sigma(b_{k-1}) = (s, 0)$. Since $xf(x)^k \neq 0$, we have $a_n = (t_n, m_n) \neq 0$.

Clearly $n \geq 1$. Equating x^{k+1}-coefficients yields $t_n = 0$. Next consider $a_{n-1} = (t_{n-1}, s_{n-1})$, then we also get $t_{n-1} = 0$ because $((-1)^k, 0)$ cannot be equal to

$$((-1)^k, (-1)^{k+1})\sigma^k(t_{n-1}, m_{n-1}) = ((-1)^k t_{n-1}, (-1)^{k+1} t_{n-1}),$$

when t_{n-1} is nonzero and $\delta^u(t_i, m_i) = (0, m_i)$ for all $u \geq 1$. We can proceed in this manner, obtaining $t_s = 0$ for $s \leq n$.

Assume here that there exists some nonzero t_i. Let v be largest with respect to $t_v \neq 0$. Then we get $v = 1$ and

$$((-1)^k, 0) = ((-1)^k, (-1)^{k+1})\sigma^k(t_v, m_v) = ((-1)^k t_v, (-1)^{k+1} t_v),$$

entailing $t_v = 0$. This induces a contradiction. Consequently, $t_i = 0$ for all $i = 0, 1, \ldots, n$. This implies $xf(x)^k \notin f(x)^k E$, contrary to $xf(x)^k \in f(x)^k E$. Therefore E is not weakly right duo.

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References

[1] Amitsur S A, Radicals of polynomial rings, *Can. J. Math.* **8** (1956) 355–361
[2] Armendariz E P, A note on extensions of Baer and P.P.-rings *J. Austral. Math. Soc.* **18** (1974) 470–473
[3] Bedi S S and Ram J, Jacobson radical of skew polynomial rings and skew group rings, *Israel J. Math.* **35** (1980) 327–338
[4] Brungs H H, Three questions on duo rings, *Pac. J. Math.* **58** (1975) 345–349
[5] Feller E H, Properties of primary noncommutative rings, Trans. *Am. Math. Soc.* **89** (1958) 79–91
[6] Goodearl K R, Prime Ideals in Skew Polynomial Rings and Quantized Weyl Algebras, *J. Algebra* **150** (1992) 324–377
[7] Goodearl K R and Warfield R B, An Introduction to Noncommutative Noetherian Rings (2004) (Cambridge: Cambridge University Press)
[8] Hong C Y, Kim N K and Kwak T K, Ore extensions of Baer and p.p.-rings , *J. Pure Appl. Algebra* **151** (2000) 215–226

[9] Hong C Y, Kim N K and Kwak T K, On skew Armendariz rings, *Commun. Algebra* **31** (2003) 103–122
[10] Hong C Y, Kim N K and Lee Y, Ore extensions of quasi-Baer rings, *Commun. Algebra* **37** (2009) 2030–2039
[11] Hong C Y, Kim N K and Lee Y, Radicals of skew polynomial rings and skew Laurent polynomial rings, *J. Algebra* **331** (2011) 428–448
[12] Huh C, Kim H K and Lee Y, p.p. rings and generalized p.p. rings, *J. Pure Appl. Algebra* **167** (2002) 37–52
[13] Jordan D A, Bijective extensions of injective ring endomorphisms, *J. Lond. Math. Soc.* **25** (1982) 435–448
[14] Kamal A A M, Idempotents in polynomial rings, *Acta. Math. Hung.* **59** (1992) 355–363
[15] Kim H K, Kim N K and Lee Y, Weakly duo rings with nil Jacobson radical, *J. Kor. Math. Soc.* **42** (2005) 455–468
[16] Krempa J, Some examples of reduced rings, *Algebra Colloq.* **3** (1996) 289–300
[17] Lam T Y, Leroy A and Matczuk J, Primeness, semiprimeness and prime radical of ore extensions, *Commun. Algebra* **25** (1997) 2459–2506
[18] Leroy A, Matczuk J and Puczylowski E R, Quasi-duo skew polynomial rings, *J. Pure Appl. Algebra* **212** (2008) 1951–1959
[19] Leroy A, Matczuk J and Puczyowski E R, A description of quasi-duo $\mathbb{Z}$-graded rings, *Commun. Algebra* **38** (2010) 1319–1324
[20] Marks G, Reversible and symmetric rings, *J. Pure Appl. Algebra* **174** (2002) 311–318
[21] Marks G, Duo rings and Ore extensions, *J. Algebra* **280** (2004) 463–471
[22] McConnell J C and Robson J C, Noncommutative Noetherian rings (1987) (Chichester: Wiley)
[23] Thierrin G, On duo rings, *Can. Math. Bull.* **3** (1960) 167–172
[24] Yao X, Weakly right duo rings, *Pure Appl. Math. Sci.* **21** (1985) 19–24
[25] Yu H-P, On quasi-duo rings, *Glasgow Math. J.* **37** (1995) 21–31

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