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Twistor space of a generalized quaternionic manifold

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Abstract. We first make a little survey of the twistor theory for hypercomplex, generalized hypercomplex, quaternionic or generalized quaternionic manifolds. This last theory was initiated by Pantilie (*Ann. Mat. Pura. Appl.* **193** (2014) 633–641), and allows one to extend the Penrose correspondence from the quaternion to the generalized quaternion case. He showed that any generalized almost quaternionic manifold equipped with an appropriate connection admit a twistor space which comes naturally equipped with a tautological almost generalized complex structure. But he has left open the problem of the integrability. The aim of this article is to give an integrability criterion for this generalized almost complex structure and to give some examples especially in the case of generalized hyperkähler manifolds using the generalized Bismut connection, introduced by Gualtieri (Branes on Poisson varieties, The many facets of geometry: a tribute to Nigel Hitchin (2010) (Oxford: Oxford University Press) pp. 368–395).

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1. Introduction

In this article, we review some properties of KT and HKT-manifolds, which closely resemble those of Kähler and hyperkähler ones respectively. In physics, KT and HKT-manifolds arise as target spaces of two-dimensional supersymmetric sigma models with Wess–Zumino term [8]. Another application of these geometries is in the context of black-holes, where the moduli spaces of a class of black-hole supergravity solutions are HKT-manifolds [11]. Homogeneous manifolds have been investigated, and they have found many applications in physics in the context of sigma models and supergravity theory [17]. In mathematics, these notions are closely related with generalized geometry introduced by Hitchin [14] and clarified by Gualtieri [12]. In sections 3 and 5, we will recall these relations.

In section 6, we will make a little survey of twistor theory for hypercomplex and almost quaternionic manifolds. The idea of a twistor space is to encode the geometric properties of the target manifold M in term of holomorphic structure of Z.

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Theorem 1 **[1,2,25,26]**. *Let $n \geq 1$ and let (M, Q) be an almost quaternionic $4n$-manifold. If ∇ is a connection on TM compatible with Q then its twistor space admit a natural almost complex structure $\mathbb{J}_\nabla$ which is integrable if and only if, with respect to all local almost complex structures J living in Q and all sections X, Y of TM:*

(1) *The torsion T of ∇ satisfies*

$$T(JX, JY) - JT(JX, Y) - JT(X, JY) - T(X, Y) = 0.$$

(2) *The curvature R of ∇ satisfies*

$$(R(X \wedge Y - JX \wedge JY) + JR(JX \wedge Y + X \wedge JY)) \cdot J = 0.$$

This theorem allows to give many examples of complex twistor space, in particular, we have as follows.

Theorem 2 **[2,25,26]**. *Let $n > 1$ and let (M, Q) be an almost quaternionic $4n$-manifold. If ∇ is a torsion free connection on TM compatible with Q then $\mathbb{J}_\nabla$ is integrable.*

The purpose of this article is to extend these theorems in the context of generalized geometry. Indeed, Pantilie [22] noticed that we can still define a twistor space $\mathcal{Z}(\mathcal{Q})$ for any generalized almost quaternionic manifold $(M, \mathcal{Q})$, and when M admit a (usual) connection ∇ on $\mathbb{T}M := TM \oplus T^\star M$ compatible with $\mathcal{Q}$ and the inner product, then its twistor space admits a natural generalized almost complex structure $\mathbb{J}_\nabla$. In his paper, Pantilie has left open the question of the integrability. Theorem A fills the gap by showing that the integrability of $\mathbb{J}_\nabla$ depends on the generalized torsion and the generalized curvature of ∇ seen as a generalized connection on $\mathbb{T}M$ (see section 6 for precise definitions).

Theorem A. *Let $n \geq 1$ and let $(M, \mathcal{Q})$ be a generalized almost quaternionic $4n$-manifold. If ∇ is a (usual) connection ∇ on $\mathbb{T}M$ compatible with $\mathcal{Q}$ and the inner product, then its twistor space admit a natural generalized almost complex structure $\mathbb{J}_\nabla$ which is integrable if and only if with respect to all local generalized almost complex structures u living in $\mathcal{Q}$ and all sections $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of $\mathbb{T}M$:*

$(C1)$ *The generalized torsion $\mathcal{T}$ of ∇ satisfies*

$$\mathcal{T}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) - \mathcal{T}(\mathcal{X}, u\mathcal{Y}, u\mathcal{Z}) - \mathcal{T}(u\mathcal{X}, \mathcal{Y}, u\mathcal{Z}) - \mathcal{T}(u\mathcal{X}, u\mathcal{Y}, \mathcal{Z}) = 0.$$

$(C2)$ *The generalized curvature $\mathcal{R}$ of ∇ satisfies*

$$(\mathcal{R}(\mathcal{X} \wedge \mathcal{Y} - u\mathcal{X} \wedge u\mathcal{Y}) + u\mathcal{R}(u\mathcal{X} \wedge \mathcal{Y} + \mathcal{X} \wedge u\mathcal{Y})) \cdot u = 0.$$

This theorem enables us to give some new examples of generalized complex twistor space. In particular, as follows.

Theorem B. *Let $n \geq 1$. If $(M, G, \mathcal{I}, \mathcal{J}, \mathcal{K})$ is a twisted generalized hyperkähler $4n$-manifold and if D is the generalized Bismut connection introduced by Gualtieri [13], then the generalized almost complex structure $\mathbb{J}_D$ on its twistor space is integrable.*

This result is motivated by the fact that generalized hyperkähler structures appear in some branches of theoretical physics, such as string theory or in the context of certain supersymmetric sigma models [8,15,16].

Another example, is given in the case of the Levi–Civita connection (Theorem C).

Finaly we give a more specific necessary and sufficient condition for integrability in the special case of generalised torsion-free connection under some fairly general additional assumptions (Theorem D).

2. KT-manifold

Let (M, I, g) be a complex hermitian manifold and let $E \longrightarrow M$ be a fiber bundle. We denote by $\Gamma(E)$ the set of all smooth sections. A connection $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^\star M \otimes TM)$ is called *Hermitian* if $\nabla I = \nabla g = 0$. Let $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ be the torsion tensor of type $(1, 2)$. We denote by the same letter the torsion tensor of type $(0, 3)$ given by $T(X, Y, Z) = g(X, T(Y, Z))$.

DEFINITION

A hermitian connection is called *a Bismut connection* if T is skew-symmetric. The 3-form T is then called the torsion form of the Bismut connection.

PROPOSITION 1 [7]

Let (M, g, I) be a complex hermitian manifold and w the associated hermitian form. There exist a unique Bismut connection ∇^B and the torsion form is equal to Idw, that is,

$$T(X, Y, Z) = dw(IX, IY, IZ).$$

If we denote by ∇^g the Levi–Civita connection of g, we have $\nabla^B = \nabla^g + \frac{1}{2}g^{-1}T$, that is,

$$g(\nabla^B_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}T(X, Y, Z),$$

for all vector fields X, Y, Z.

Clearly if $dw = 0$, then the Bismut connection is torsion-free and thus coincides with the Levi–Civita connection: the manifold (M, g, I) is therefore Kähler.

Connection with skew-symmetric torsion play an important role in string physics. In the physics literature, a complex hermitian manifold (M, g, I) with a Bismut connection is called a *KT-manifold* (Kähler with torsion manifold). If, in addition, the torsion 3-form is closed then (M, g, I) is said to be a *strong KT-manifold*. By Proposition 1, a manifold is therefore strong KT if and only if $\partial\bar{\partial} w = 0$. For complex surfaces, this is equivalent to Gauduchon metric. The strong KT-manifolds have been recently studied by many authors and they have also applications in type II string theory and in 2-dimensional supersymmetric σ-models [8,18,24]. They also have relations with twisted generalized Kähler geometry as we are now going to see.

3. Generalized complex structure

3.1 *Courant bracket*

Let $X, Y \in \Gamma(TM)$ be two vector fields and $\xi, \eta \in \Gamma(T^\star M)$ be two 1-form. On $\mathbb{T}M := TM \oplus T^\star M$, there is an inner product

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X)),$$

and a Courant bracket, which is a skew-symmetric bracket defined by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi),$$

where $[X, Y]$ is the Lie bracket. The Courant bracket on $\mathbb{T}M$ can be twisted by a real closed 3-form h defining another bracket [12,27]

$$[X+\xi, Y+\eta]_h = [X+\xi, Y+\eta] + i_Y i_X h.$$

In fact, this bracket defined a Courant algebroïd structure on $\mathbb{T}M$.

When b is a 2-form on M, we will denote by $e^b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ the transformation sending $X+\xi$ on $X+\xi+i_X b$. This transformation is orthogonal for the inner product and is an automorphism for the Courant bracket if and only if b is closed.

3.2 *Generalized metric*

Let M be a 2n-manifold, since the bundle $\mathbb{T}M \longrightarrow M$ has a natural inner product, it has structure group $O(2n, 2n)$.

DEFINITION

A generalized metric is a reduction of the structure group from $O(2n, 2n)$ to its maximal compact subgroup $O(2n) \times O(2n)$.

A generalized metric is equivalent to the choice of a $2n$-dimensional subbundle C^+ which is positive definite with respect to the inner product. Let C^- be the (negative definite) orthogonal complement to C^+. Note that the splitting

$$\mathbb{T}M = C^+ \oplus C^-$$

defines a positive definite metric on $\mathbb{T}M$ via

$$G = \langle .,.\rangle|_{C^+} - \langle .,.\rangle|_{C^-}.$$

We denote by the same letter the isomorphism $G : \mathbb{T}M \longrightarrow \mathbb{T}M$ with ± 1 eigenspaces $C^\pm$, which is symmetric $G^\star = G$ and squares to the identity $G^2 = \mathrm{Id}$.

PROPOSITION 2 [12]

A generalized metric is equivalent to specifying a Riemannian metric g and a 2-form b on TM such that

(*i*) $G = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b}$,

(*ii*) $C^\pm = \{X + (b \pm g)X \in \mathbb{T}M | X \in TM\}$.

3.3 *Generalized complex structure*

A generalized almost complex structure on M is an endomorphism $\mathcal{J}$ of $\mathbb{T}M$ which satisfies $\mathcal{J}^2 = -1$ and $\mathcal{J}^\star = -\mathcal{J}$. That is a reduction of the structure group from $O(2n, 2n)$ to $U(n, n)$.

DEFINITION

A generalized almost complex structure $\mathcal{J}$ is said to be a twisted generalized complex structure with respect to a closed 3-form h when its i-eigenbundle $\mathbb{T}^{1,0} \subset \mathbb{T}M \otimes \mathbb{C}$ is involutive with respect to the h-twisted Courant bracket. We also said that $\mathcal{J}$ is twisted integrable or simply integrable when $h = 0$.

Let $\mathcal{N}_h$ be the Nijenhuis tensor of $\mathcal{J}$ defined on sections of $\mathbb{T}M$ by

$$\mathcal{N}_h(\mathcal{X}, \mathcal{Y}) = [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}]_h - \mathcal{J}[\mathcal{J}\mathcal{X}, \mathcal{Y}]_h - \mathcal{J}[\mathcal{X}, \mathcal{J}\mathcal{Y}]_h - [\mathcal{X}, \mathcal{Y}]_h.$$

When $h = 0$, we simply note $\mathcal{N}$.

PROPOSITION 3 [12]

The twisted integrability of $\mathcal{J}$ is equivalent to the vanishing of the Nijenhuis tensor $\mathcal{N}_h$.

3.4 *Generalized Kähler manifold*

Suppose that we have a generalized almost complex structure $\mathcal{J}$. To now reduce the structure group from $U(n, n)$ to $U(n) \times U(n)$, we need to choose a generalized metric G which commutes with $\mathcal{J}$. Note that since $G^2 = \mathrm{Id}$ and $G\mathcal{J} = \mathcal{J}G$, the map $G\mathcal{J}$ squares to $-\mathrm{Id}$ and since G is symmetric and $\mathcal{J}$ is skew, $G\mathcal{J}$ is also skew, and therefore defines another generalized almost complex structure.

DEFINITION [12]

A reduction to $U(n) \times U(n)$ is equivalent to the existence of two commuting generalized almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric. We say that $(G, \mathcal{J}_1)$ is an almost generalized Kähler structure.

Since the bundle C^+ is positive definite while TM is null, the projection $\pi : TM \oplus T^\star M \longrightarrow TM$ induces isomorphisms

$$\pi_\pm : C^\pm \longrightarrow TM.$$

We denote by $P_\pm$ the projection from $\mathbb{T}M$ to $C^\pm$. Since $\mathcal{J}_1$ and G commute, $\mathcal{J}_1$ stabilise $C^\pm$. By projection from $C^\pm$, $\mathcal{J}_1$ induces two almost complex structures on TM, which we denote as $J^\pm$. They are compatible with the induced Riemannian metric g and the associated 2-forms are noted by $w_\pm$. Note that $\mathcal{J}_2 = G\mathcal{J}_1$ implies that $\mathcal{J}_1 = \mathcal{J}_2$ on C^+ and $\mathcal{J}_1 = -\mathcal{J}_2$ on C^-.

PROPOSITION 4 [12]

An almost generalized Kähler structure $(G, \mathcal{J}_1)$ *is equivalent to the specification* (g, b, J^+, J^-) *that is a Riemannian metric g, a 2-form b and two hermitian almost complex structures* $J^{\pm}$ *such that*

(*i*) $G = e^b \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-b}$,
(*ii*) $\mathcal{J}_1 = \pi_+^{-1} J_+ \pi P_+ + \pi_-^{-1} J_- \pi P_-$,
(*iii*) $\mathcal{J}_2 = \pi_+^{-1} J_+ \pi P_+ - \pi_-^{-1} J_- \pi P_-$,
(*iv*) $\mathcal{J}_{\frac{1}{2}} = \frac{1}{2} e^b \begin{pmatrix} J_+ \pm J_- & -(w_+^{-1} \mp w_-^{-1}) \\ w_+ \mp w_- & -(J_+^{\star} \pm J_-^{\star}) \end{pmatrix} e^{-b}$.

DEFINITION [12]

Let $(G, \mathcal{J})$ be an almost generalized Kähler structure on M. When $\mathcal{J}$ and $G\mathcal{J}$ are both (twisted) generalized complex, we say that $(G, \mathcal{J})$ is a (twisted) generalized Kähler structure on M.

3.5 *Relation between KT and generalized Kähler manifold*

Let $(M, G, \mathcal{J})$ be an almost generalized Kähler structure corresponding to the quadruple (g, b, J_+, J_-).

PROPOSITION 5 [12]

$(M, G, \mathcal{J})$ *is a twisted generalized Kähler structure if and only if*

(*i*) $J_{\pm}$ *integrable, and*
(*ii*) $h + db = -J_- dw_- = J_+ dw_+$.

This proposition shows that a twisted generalized Kähler structure on a Riemannian manifold (M, g) is the same as a bihermitian structure (J_+, J_-) such that the corresponding Bismut connections have torsion 3-forms which satisfy $T_+ = -T_-$ and $dT_{\pm} = 0$. In other words, a twisted generalized Kähler structure is a pair of strong KT-structures (J_+, J_-) whose torsion satisfies $T_+ = -T_-$.

PROPOSITION 6 [12]

The torsion $T = -J_- dw_- = J_+ dw_+$ *of a twisted generalized Kähler structure is of type* $(2,1) + (1,2)$ *with respect to both complex structures* $J_{\pm}$. *Equivalently, it satisfies the condition*

$$T(X, Y, Z) - T(X, J_{\pm}Y, J_{\pm}Z) - T(J_{\pm}X, Y, J_{\pm}Z) - T(J_{\pm}X, J_{\pm}Y, Z) = 0$$

for all vector fields X, Y, Z *on* M.

Example. By [28], any real compact Lie group G of even dimension has a natural strong KT-metric and a twisted generalized Kähler structure [12].

Another notion related to physics is the notion of HKT-manifolds, which was initiated by Grantcharov and Poon [9].

4. HKT-manifold

DEFINITION

A Riemannian $4n$-manifold (M, g) admits a *hypercomplex* structure if there exists a triple (I, J, K) of complex structures such that $IJ = -JI = K$. When each complex structure is compatible with the metric, we speak about *hyperhermitian* structures.

Let $H \longrightarrow M$ be the vector bundle defined by $H = \mathrm{Span}(I, J, K)$, the set of all linear combinations of I, J, K. We say that a connection ∇ on TM is compatible with the hypercomplex structure or preserves the hypercomplex structure if $\nabla_X \sigma \in \Gamma(H)$ for all vector field X and all smooth section $\sigma \in \Gamma(H)$. On a hyperhermitian manifold, there are two natural torsion free connections, namely the Levi–Civita and the Obata connection. However, in general, the Levi–Civita connection does not preserve the hypercomplex structure and the Obata connection does not preserve the metric. That is why we are interested in the following type of connections.

DEFINITION

A *HKT-manifold* is a hyperhermitian $4n$-manifold (M, g, I, J, K) with a connection ∇ such that

(i) $\nabla g = 0$,
(ii) $\nabla I = \nabla J = \nabla K = 0$,
(iii) the torsion is totally skew-symmetric.

When the torsion is closed then (M, g, I, J, K) is said to be a *strong HKT-manifold*.

In contrast to the case of hermitian structure, not every hyperhermitian structure on a manifold admits a compatible HKT-connection but obviously, if such a connection exists, it is unique. Note that a HKT-connection is also the Bismut connection for each complex structure in the given hypercomplex structure. More generally, we have as follows.

PROPOSITION 7 [9]

Let (M, g, I, J, K) be an almost hyperhermitian manifold. It is an HKT-manifold if and only if

$$I dw_I = J dw_J = K dw_K \tag{1}$$

where w_I, w_J, w_K are the associated hermitian forms of I, J and K. A holomorphic characterization has been given in [10] *where the authors proved that* (1) *is equivalent to*

$$\partial_I (w_J + i w_K) = 0.$$

Many examples of HKT-manifolds have been obtained [10,21,29]. *For instance, it has been shown that the geometry of the moduli space of a class of black holes in five dimensions is a HKT-manifold* [11].

We now consider (4,4)-supersymmetry structures on a Riemannian manifold and see the link with HKT-structures. These structures were also introduced by Gates *et al.* [8], and formulated in Hitchin and Gualtieri's language as twisted generalized hyperkähler structures.

5. Generalized hyperkähler manifold

5.1 *Definition*

Let (M, G) be a $4n$-manifold with a generalized metric. A (twisted) *generalized hyperhermitian* structure is a triple $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ of (twisted) generalized complex structures such that

(i) $\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K}$,
(ii) $\mathcal{I}, \mathcal{J}, \mathcal{K}$ commute with G.

DEFINITION

A (twisted) *generalized hyperkähler* structure on M is a triple $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ of (twisted) generalized complex structure each of which forms a generalized Kähler structure with the same generalized metric G and such that

$$\mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} = \mathcal{K}.$$

Example 1. A quaternionic Hopf surface $(\mathbb{H} - \{0\})/\langle q\rangle$ where $q \in \mathbb{R}$, endowed with its two hypercomplex structures (left or right multiplication by i, j or k) is an example of a twisted generalized hyperkähler manifold.

Example 2 [19]. Let $(M, g, I_\pm, J_\pm, K_\pm)$ be a generalized hyperkähler manifold of real dimension 4, and $E \longrightarrow M$ be a smooth complex vector bundle. Denote by $\mathcal{M}$ the moduli space of gauge-equivalence classes of anti-selfdual connections on E. Then $\mathcal{M}$ is equipped with a natural generalized hyperkähler structure.

Example 3 [6]. The Neveu–Schwarz 5-brane solution provides an explicit example of generalized hyperkähler manifold found in string theory.

5.2 *Relation betwenn HKT and generalized hyperkähler structure*

Let $(M, G, \mathcal{I}, \mathcal{J}, \mathcal{K}))$ be an almost generalized hyperkähler structure corresponding to $(g, b, I_+, J_+, K_+, I_-, J_-, K_-)$.

PROPOSITION 8

$(M, G, \mathcal{I}, \mathcal{J}, \mathcal{K})$ is a twisted generalized hyperkähler structure if and only if

(i) $(I_\pm, J_\pm, K_\pm)$ *is a pair of hyperhermitian complex structure on* (M, g), *and*
(ii) $h + db = I_+ dw_{I_+} = J_+ dw_{J_+} = K_+ dw_{K_+} = -I_- dw_{I_-} = -J_- dw_{J_-} = -K_- w_{K_-}$.

In other words, a twisted generalized hyperkähler structure is a pair of strong HKT-structure $(I_\pm, J_\pm, K_\pm)$ *whose torsion satisfies* $T_+ = -T_-$.

COROLLARY

The torsion $T_+ = -T_-$ *of a twisted generalized hyperkähler structure is of type* $(2,1)+(1,2)$ *with respect to each complex structure* $I_\pm$, $J_\pm$ *or* $K_\pm$.

6. Twistor space

In this section, we define the twistor space of a twisted generalized hyperkähler manifold $(M, G, \mathcal{I}, \mathcal{J}, \mathcal{K})$ and more generally to a generalized almost quaternionic manifold. Unlike the approach of Bredthauer [3], still generalized on [4]; in this paper, the twistor space is not an $\mathbb{S}^2 \times \mathbb{S}^2$-fiber bundle but a $\mathbb{S}^2$-bundle exactly as in the original idea of Penrose [23], Atiyah *et al.* [1] and Salamon [25,26]. We first review the results for a quaternionic manifold.

6.1 *Twistor space of a quaternionic manifold*

Let $(M, (I, J, K))$ be an hypercomplex $4n$-manifold. A triple of such complex structures induces a 2-sphere of integrable complex structures

$$\{aI + bJ + cK | a^2 + b^2 + c^2 = 1\}.$$

So it is natural to define the twistor space associated to this hypercomplex structure by

$$Z = M \times \mathbb{S}^2 = \{(m, aI + bJ + cK) | (a, b, c) \in \mathbb{S}^2\}.$$

The idea of a twistor space is to encode the geometric properties of the target manifold M in the holomorphic structure of Z. Indeed, we are now going to define a natural almost complex structure $\mathbb{J}_\nabla$ or simply $\mathbb{J}$ for any connection ∇ on TM that preserves the hypercomplex structure. Such a connection induces a natural connection on $Z \subset \mathrm{End}(TM)$, and a decomposition

$$TZ = H \oplus V$$

of the tangent space of Z into its vertical and horizontal component. Since $\mathbb{S}^2$ has the natural complex structure of $\mathbb{C}P^1$, we take $\mathbb{J}|_V = \mathbb{J}_{\mathbb{C}P^1}$. But H and TM are isomorphic, so we may define $\mathbb{J}|_H$ by letting $\mathbb{J}|_H$ act at $(m, u) \in Z$ like u on $T_m M$.

The construction of a twistor space and its almost complex structure can be easily extended to any *almost quaternionic* $4n$-manifold (M, Q), that is manifold with a rank three subbundle $Q \subset \mathrm{End}(TM) \longrightarrow M$ which is locally spanned by an almost hypercomplex structure (I, J, K). Such a locally defined triple (I, J, K) will be called an admissible basis of Q. A consequence of the definition of an almost quaternionic manifold is that the bundle Q has structure group $SO(3)$. We then have a natural inner product on Q by taking each admissible basis (I, J, K) to be an orthonormal basis. The twistor space $Z(Q)$ of (M, Q) is defined to be the unit sphere bundle of Q. This is a locally trivial bundle over M with fiber $\mathbb{S}^2$. A linear connection ∇ on TM preserves Q means that $\nabla_X \sigma \in \Gamma(Q)$ for all vector field X and smooth section $\sigma \in \Gamma(Q)$. In this case, the same construction as before gives us an almost complex structure $\mathbb{J}_\nabla$ on $Z(Q)$ whose integrability depends on the torsion T and the curvature R of ∇, defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$
$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

By skew symmetry, we write $R(X \wedge Y)$ rather than $R(X, Y)$.

Theorem 3 **[1,2,25,26].** *Let $n \geq 1$ and let (M, Q) be an almost quaternionic $4n$-manifold. If ∇ is a connection on TM compatible with Q then its twistor space admits a natural almost complex structure $\mathbb{J}_\nabla$ which is integrable if and only if, with respect to all local almost complex structures J living in Q and all sections X, Y of TM,*

(1) *The torsion T of ∇ satisfies*

$$T(JX, JY) - JT(JX, Y) - JT(X, JY) - T(X, Y) = 0.$$

(2) *The curvature R of ∇ satisfies*

$$(R(X \wedge Y - JX \wedge JY) + JR(JX \wedge Y + X \wedge JY)) \cdot J = 0.$$

In the particular case of a torsion free connection, we have as follows.

Theorem 4 **[2,25,26].** *Let $n > 1$ and let (M, Q) be an almost quaternionic $4n$-manifold. If ∇ is a torsion free connection on TM compatible with Q, then $\mathbb{J}_\nabla$ is a complex structure on $Z(Q)$.*

Pantilie [22] extended this construction in the context of generalized geometry as we are now going to see.

6.2 *Twistor space of a generalized hypercomplex manifold*

For a generalized hypercomplex manifold $(M, (\mathcal{I}, \mathcal{J}, \mathcal{K}))$, we can still define the associated twistor space by

$$\mathcal{Z} = M \times \mathbb{S}^2 = \{(m, a\mathcal{I} + b\mathcal{J} + c\mathcal{K}) | (a, b, c) \in \mathbb{S}^2\},$$

and we denote by $\pi_{\mathcal{Z}} : \mathcal{Z} \longrightarrow M$, the first projection.

As in the classical case, any connection on $\mathbb{T}M$ compatible with the generalized hypercomplex structure and the inner product defined a natural generalized almost complex structure $\mathbb{J}_\nabla$ on $\mathcal{Z}$ and the construction is substantially the same. Indeed, since $\mathcal{Z} \subset \mathrm{End}(\mathbb{T}M)$, the connection on $\mathbb{T}M$ can be extended on Z, thus we have an associated decomposition $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ into horizontal and vertical parts. If we denote by $\mathcal{H}^\star$ (resp. $\mathcal{V}^\star$) the element of $T^\star\mathcal{Z}$ null on $\mathcal{V}$ (resp. on $\mathcal{H}$), it gives us a decomposition $\mathbb{T}\mathcal{Z} = (\mathcal{H} \oplus \mathcal{H}^\star) \oplus (\mathcal{V} \oplus \mathcal{V}^\star)$. For all $(m, u) \in \mathcal{Z}$, we have an isomorphism

$$d\pi_{\mathcal{Z}}|_{(m,u)} : \mathcal{H}_{(m,u)} \longrightarrow T_m M$$

which induces an isomorphism

$$d\pi^\star_{\mathcal{Z}}|_{(m,u)} : T^\star_m M \longrightarrow \mathcal{H}^\star_{(m,u)},$$

in such a way that the function $d\pi_{\mathcal{Z}} \oplus d\pi^{\star -1}_{\mathcal{Z}}$ is an isomorphism between $\mathcal{H} \oplus \mathcal{H}^\star$ and $\mathbb{T}M$. So we may define a bundle endomorphism

$$\mathbb{J}|_{\mathcal{H}\oplus\mathcal{H}^\star} : \mathcal{H} \oplus \mathcal{H}^\star \longrightarrow \mathcal{H} \oplus \mathcal{H}^\star, \quad \mathbb{J}|^2_{\mathcal{H}\oplus\mathcal{H}^\star} = -1,$$

where $\mathbb{J}|_{\mathcal{H}\oplus\mathcal{H}^\star}$ acts at $(m, u) \in \mathcal{Z}$ like u on $\mathbb{T}M$. Hence

$$\begin{array}{ccc} \mathcal{H}_{(m,u)} \oplus \mathcal{H}^\star_{(m,u)} & \xrightarrow{\mathbb{J}|_{\mathcal{H}\oplus\mathcal{H}^\star}} & \mathcal{H}_{(m,u)} \oplus \mathcal{H}^\star_{(m,u)} \\ {\scriptstyle d\pi_{\mathcal{Z}}\oplus d\pi^{\star-1}_{\mathcal{Z}}}\downarrow & & \downarrow{\scriptstyle d\pi_{\mathcal{Z}}\oplus d\pi^{\star-1}_{\mathcal{Z}}} \\ \mathbb{T}_m M & \xrightarrow{u} & \mathbb{T}_m M \end{array}.$$

On the other hand, $\mathcal{V}$ is just the tangent space to the fibers and so admits the natural complex structure $\mathbb{J}|_{\mathcal{V}\oplus\mathcal{V}^\star}$ of $\mathbb{C}P^1$. This gives us a natural generalized almost complex structure $\mathbb{J}_\nabla$ on $\mathcal{Z}$, namely $\mathbb{J}|_{\mathcal{H}\oplus\mathcal{H}^\star} \oplus \mathbb{J}|_{\mathcal{V}\oplus\mathcal{V}^\star}$.

6.3 *Twistor space of a generalized quaternionic manifold*

We say that M admit a *generalized almost quaternionic* structure if there exists $\mathcal{Q} \longrightarrow M$ a rank three vector bundle $\mathcal{Q} \subset \mathrm{End}(\mathbb{T}M)$ which is locally spanned by an almost generalized hypercomplex structure. The twistor space $\mathcal{Z}(\mathcal{Q})$ of $(M, \mathcal{Q})$ is still defined to be the unit sphere bundle of $\mathcal{Q}$ for the natural inner product in $\mathcal{Q}$ such that any admissible basis is othonormal. The bundle $\pi_{\mathcal{Z}(\mathcal{Q})} : \mathcal{Z}(\mathcal{Q}) \longrightarrow M$ is a locally trivial bundle with fibre $\mathbb{S}^2$ and structure group $SO(3)$. Moreover, it is not difficult to see that the former construction of the generalized almost complex structure $\mathbb{J}_\nabla$ associated to any connection ∇ on $\mathbb{T}M$ preserving $\mathcal{Q}$ and the inner product, works yet.

Extension. We extend ∇ into a generalized connection on $\mathbb{T}M$ asking $\nabla_{\mathcal{X}} = \nabla_{\pi(\mathcal{X})}$ for all $\mathcal{X} \in \Gamma(\mathbb{T}M)$. Recall that a generalized connection $\mathcal{D}$ on $\mathbb{T}M$ is a map [13]

$$\begin{aligned} \mathcal{D} : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) &\longrightarrow \Gamma(\mathbb{T}M), \\ (\mathcal{X}, \mathcal{Y}) &\longmapsto \mathcal{D}_{\mathcal{X}}\mathcal{Y} \end{aligned}$$

which, for any sections $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of $\mathbb{T}M$ and any functions f and h, satisfies

(1) $\mathcal{D}_{f\mathcal{X}+h\mathcal{Y}}\mathcal{Z} = f\mathcal{D}_{\mathcal{X}}\mathcal{Z} + h\mathcal{D}_Y\mathcal{Z}$,
(2) $\mathcal{D}_{\mathcal{X}}(\mathcal{Y} + \mathcal{Z}) = \mathcal{D}_{\mathcal{X}}\mathcal{Y} + \mathcal{D}_{\mathcal{X}}\mathcal{Z}$,
(3) $\mathcal{D}_{\mathcal{X}}(f\mathcal{Y}) = f\mathcal{D}_{\mathcal{X}}\mathcal{Y} + \pi(\mathcal{X}).f\ \mathcal{Y}$.

For a generalized connection $\mathcal{D}$, the generalized torsion $\mathcal{T}$ is defined, for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathbb{T}M)$, by

$$\mathcal{T}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \langle \mathcal{D}_{\mathcal{X}}\mathcal{Y} - \mathcal{D}_{\mathcal{Y}}\mathcal{X} - [\mathcal{X}, \mathcal{Y}], \mathcal{Z}\rangle + \frac{1}{2}(\langle \mathcal{D}_{\mathcal{Z}}\mathcal{X}, \mathcal{Y}\rangle - \langle \mathcal{D}_{\mathcal{Z}}\mathcal{Y}, \mathcal{X}\rangle).$$

Whereas, the generalized curvature is defined by

$$\mathcal{R}(\mathcal{X}, \mathcal{Y}) = \mathcal{D}_{\mathcal{X}}\mathcal{D}_{\mathcal{Y}} - \mathcal{D}_{\mathcal{Y}}\mathcal{D}_{\mathcal{X}} - \mathcal{D}_{[\mathcal{X},\mathcal{Y}]}.$$

By skew symmetry, sometimes we will use the notation $\mathcal{R}(\mathcal{X} \wedge \mathcal{Y})$ rather than $\mathcal{R}(\mathcal{X}, \mathcal{Y})$.

Remark. Since ∇ is a (usual) connection on $\mathbb{T}M$, its generalized curvature $\mathcal{R}$ is tensorial. Furthermore, ∇ preserve the inner product, so its generalized torsion $\mathcal{T}$ is totally skew.

In his article, Pantilie [22] does not study the integrability of $\mathbb{J}_\nabla$. In the next section, we will give a criterion of integrability for $\mathbb{J}_\nabla$. In particular, we will see that, even if ∇ is a usual connection on the vector bundle $\mathbb{T}M$, the integrability of $\mathbb{J}_\nabla$ depend on its generalized torsion and its generalized curvature.

6.4 *Integrability of the generalized almost complex structure*

Let $(M, \mathcal{Q})$ be a generalized almost quaternionic manifold. The data of a generalized almost hypercomplex structure $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ on an open set $\mathcal{U}$ of M defines a trivialisation $\pi^{-1}_{\mathcal{Z}(\mathcal{Q})}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{S}^2$. The local coordinates of a point in $\mathcal{Z}(\mathcal{Q})$ will be denoted by (m, u). The central theorem of this article is the following.

Theorem A. *Let $n \geq 1$ and $(M, \mathcal{Q}, \nabla)$ be a generalized almost quaternionic $4n$-manifold with a (usual) connection ∇ on $\mathbb{T}M$ compatible with $\mathcal{Q}$ and the inner product. The generalized almost complex structure $\mathbb{J}_\nabla$ on the twistor space $\mathcal{Z}(\mathcal{Q})$ is integrable if and only if with respect to all local generalized almost complex structures u living in $\mathcal{Q}$ and all sections $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of $\mathbb{T}M$,*

(*C*1) *The torsion $\mathcal{T}$ is of type $(2, 1) + (1, 2)$. Equivalently it satisfies the local condition*:

$$\mathcal{T}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) - \mathcal{T}(\mathcal{X}, u\mathcal{Y}, u\mathcal{Z}) - \mathcal{T}(u\mathcal{X}, \mathcal{Y}, u\mathcal{Z}) - \mathcal{T}(u\mathcal{X}, u\mathcal{Y}, \mathcal{Z}) = 0.$$

(*C*2) *The curvature form $\mathcal{R}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}) := \langle \mathcal{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{U}\rangle$ has no component of type $(4, 0) + (0, 4)$. Equivalently it satisfies the local condition*:

$$(\mathcal{R}(\mathcal{X} \wedge \mathcal{Y} - u\mathcal{X} \wedge u\mathcal{Y}) + u\mathcal{R}(u\mathcal{X} \wedge \mathcal{Y} + \mathcal{X} \wedge u\mathcal{Y})) \cdot u = 0.$$

6.5 *Proof*

We will use the notation $\widehat{\mathcal{X}} \in \Gamma(\mathcal{H} \oplus \mathcal{H}^\star)$ to denote the horizontal lift of a local smooth section of $\mathbb{T}M$, and we will refer to it as a basic section of $\mathcal{H} \oplus \mathcal{H}^\star$.

Because the connection ∇ preserve $\mathcal{Q}$, for all smooth sections $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathbb{T}M)$, and for all $u \in \Gamma(\mathcal{Q})$, we have that $\mathcal{R}(\mathcal{X}, \mathcal{Y}).u \in \Gamma(\mathcal{Q})$ and more precisely it is a vertical element. In order to prove the integrability of $\mathbb{J}$, we consider its Nijenhuis tensor $\mathcal{N}$, and the various components of it in the splitting $\mathbb{T}\mathcal{Z} = (\mathcal{H} \oplus \mathcal{H}^\star) \oplus (\mathcal{V} \oplus \mathcal{V}^\star)$. The computation of these components require the following proposition.

PROPOSITION 9

For all vertical vector fields $A, B \in \Gamma(\mathcal{V})$ and all horizontal lift $\widehat{X + \xi} \in \Gamma(\mathcal{H} \oplus \mathcal{H}^\star)$ one has

(*i*) $[A, B] \in \Gamma(\mathcal{V})$,
(*ii*) $[\widehat{X}, A] \in \Gamma(\mathcal{V})$,
(*iii*) $[\widehat{X} + \widehat{\xi}, \mathbb{J}A] = \mathbb{J}[\widehat{X} + \widehat{\xi}, A]$,
(*iv*) $[\mathbb{J}(\widehat{X} + \widehat{\xi}), \mathbb{J}A] = \mathbb{J}[\mathbb{J}(\widehat{X} + \widehat{\xi}), A]$.

Proof. The first point is a general fact for any vertical distribution, similarly the second point is always true for any basic vector field $\widehat{X}$ and any vertical vector field A [2].

The third formula follows from the parallel transport along horizontal directions with respect to the canonical metric and the orientation of the fibres, hence the vertical complex structure, so $[\widehat{X}, \mathbb{J}A] = \mathbb{J}[\widehat{X}, A]$. It remains to check that $[\widehat{\xi}, A] = 0 = [\widehat{\xi}, \mathbb{J}A]$ which is an immediate consequence of the definition of the Courant; indeed using (i), (ii) and $\widehat{\xi}(A) = 0$, we have

$$\begin{aligned}[\widehat{\xi}, A](\widehat{X} + B) &= -\mathcal{L}_A\widehat{\xi}\ (\widehat{X} + B)\\ &= -d\widehat{\xi}(A, \widehat{X} + B)\\ &= -A.\widehat{\xi}(\widehat{X}) + \widehat{\xi}([A, \widehat{X} + B])\\ &= 0.\end{aligned}$$

For the last formula, pick a local basis $(\mathcal{X}_1, \ldots, \mathcal{X}_{8n})$ of $\Gamma(\mathbb{T}M)$ and note $[\mathbb{J}_{ij}]$ the matrix of $\mathbb{J}$ in the basis $(\widehat{\mathcal{X}}_1, \ldots, \widehat{\mathcal{X}}_{8n})$ of $\mathcal{H} \oplus \mathcal{H}^\star$. Properties of the bracket and the third point give us

$$\begin{aligned}\mathbb{J}[\mathbb{J}\widehat{\mathcal{X}}_j, A] &= \mathbb{J}[\mathbb{J}_{ij}\widehat{\mathcal{X}}_i, A]\\ &= \mathbb{J}\big(\mathbb{J}_{ij}[\widehat{\mathcal{X}}_i, A] - A.\mathbb{J}_{ij}\widehat{\mathcal{X}}_i\big)\\ &= \mathbb{J}\big(\mathbb{J}_{ij}[\widehat{\mathcal{X}}_i, A] - A_{ij}\widehat{\mathcal{X}}_i\big)\\ &= \mathbb{J}\big(\mathbb{J}_{ij}[\widehat{\mathcal{X}}_i, A] - A\widehat{\mathcal{X}}_j\big)\\ &= \mathbb{J}_{ij}[\widehat{\mathcal{X}}_i, \mathbb{J}A] - \mathbb{J}A\widehat{\mathcal{X}}_j\end{aligned}$$

$$\text{and } [\mathbb{J}\widehat{\mathcal{X}}_j, \mathbb{J}A] = [\mathbb{J}_{ij}\widehat{\mathcal{X}}_i, \mathbb{J}A] = \mathbb{J}_{ij}[\widehat{\mathcal{X}}_i, \mathbb{J}A] - \mathbb{J}A\widehat{\mathcal{X}}_j.$$ □

COROLLARY 1

The component $\mathcal{N}(\mathcal{X}, A)$ of the Nijenhuis tensor of $\mathbb{J}$ is null for all $\mathcal{X} \in \Gamma(\mathcal{H} \oplus \mathcal{H}^\star)$ and all $A \in \Gamma(\mathcal{V})$.

Proof. For any smooth function f, we have $\mathcal{N}(f\mathcal{X}, A) = f\mathcal{N}(\mathcal{X}, A)$. By linearity, one can suppose that $\mathcal{X}$ is basic, the corollary is then an immediate consequence of points (iii) and (iv) of Proposition 9. □

PROPOSITION 10

Let $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathbb{T}M)$ be two local sections. According to horizontal and vertical directions at a point $p = (m, u) \in \mathcal{Z}(\mathcal{Q})$, one has

$$[\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}] = \widehat{[\mathcal{X}, \mathcal{Y}]} + \mathcal{R}(\mathcal{X}, \mathcal{Y}) \cdot u.$$

Proof. Identify $\mathbb{H}^n = \mathbb{R}^{4n}$ and let $\mathbb{R}^{4n\star}$ be the dual of $\mathbb{R}^{4n}$. The group $GL(2n, \mathbb{H})$ (respectively $Sp(1)$) act on the right (respectively on the left) on $\mathbb{R}^{4n} \oplus \mathbb{R}^{4n\star}$. We denote by $\mathcal{S}p(2n)$ the subgroup

$$\mathcal{S}p(2n) = GL(2n, \mathbb{H}) \cap O(2n, 2n).$$

Let G be the product $\mathcal{S}p(2n)\mathcal{S}p(1)$ and $\mathcal{P} \longrightarrow M$ be the G-principal bundle. The twistor space $\mathcal{Z}(\mathcal{Q})$ can be considered as the associated fiber bundle of $\mathcal{P}$ with standard fiber $\mathbb{S}^2$. More precisely, the group G acts on the right on $\mathcal{P} \times \mathbb{S}^2$ by

$$\begin{aligned}\mathcal{P} \times \mathbb{S}^2 \times G &\longrightarrow \mathcal{P} \times \mathbb{S}^2,\\ (q, j, g) &\longmapsto (q.g, g^{-1}.j) = (q.g, gjg^{-1})\end{aligned}$$

and $\mathcal{Z}(\mathcal{Q})$ is the quotient of $\mathcal{P} \times \mathbb{S}^2$ by G. Denote by Π the projection

$$\begin{aligned}\Pi : \mathcal{P} \times \mathbb{S}^2 &\longrightarrow \mathcal{Z}(\mathcal{Q}),\\ (q, j) &\longmapsto u = q^{-1}jq.\end{aligned}$$

Start by looking at the case where X, Y are two vector fields on M. As $\widehat{X}, \widehat{Y}$ are basic, then by [2], the horizontal part of $[\widehat{X}, \widehat{Y}]$ is precisely $\widehat{[X, Y]}$. We denote by θ the G-connection on $\mathcal{P}$, $\ker\theta$ the associated horizontal distribution and $\widetilde{X}, \widetilde{Y}$ the horizontal lift of X, Y in $\mathcal{P}$. The vertical part of $[\widetilde{X}, \widetilde{Y}]$ is given by [2,20]

$$(\theta|_{\mathcal{V}})^{-1}(\mathcal{R}(X, Y)),$$

where by definition, $(\theta|_{\mathcal{V}})^{-1}(\mathcal{R}(X, Y))$ is the vertical field on $\mathcal{P}$ defined at the point $q \in \mathcal{P}$ by

$$\left.\frac{d}{dt}\right|_{t=0}(q \cdot \exp(t\mathcal{R}(X, Y))) = q \cdot \mathcal{R}(X, Y).$$

At $q \in \mathcal{P}$, we have $d\Pi(q \cdot \mathcal{R}(X, Y)) = u\mathcal{R}(X, Y) - \mathcal{R}(X, Y)u = \mathcal{R}(X, Y) \cdot u$. Then at the point $(m, u) \in \mathcal{Z}(\mathcal{Q})$, we have

$$[\widehat{X}, \widehat{Y}] = \widehat{[X, Y]} + \mathcal{R}(X, Y) \cdot u.$$

Now let $X \in TM$ be a vector field and $\xi, \eta \in T^{\star}M$ be two 1-forms on M. Using the definition of the Courant bracket we see that $[\widehat{X}, \widehat{\xi}] = \widehat{[X, \xi]}$ and from $\nabla_{\xi} = 0$, we deduce that $\mathcal{R}(X, \xi) = 0$. Then

$$\begin{aligned}[\widehat{X}, \widehat{\xi}] &= \widehat{[X, \xi]} + \mathcal{R}(X, \xi) \cdot u,\\ [\widehat{\xi}, \widehat{\eta}] &= \widehat{[\xi, \eta]} + \mathcal{R}(\xi, \eta) \cdot u = 0.\end{aligned}$$

□

COROLLARY 2

For all vertical 1*-form* $U^{\sharp}$ *and all* $\mathcal{X} \in \Gamma(\mathcal{H} \oplus \mathcal{H}^{\star})$, *the component* $\mathcal{N}(U^{\sharp}, \mathcal{X})$ *of the Nijenhuis tensor of* $\mathbb{J}$ *is the horizontal form defined for all* $\mathcal{Y} \in \Gamma(\mathcal{H})$ *by*

$$\langle \mathcal{N}(U^{\sharp}, \mathcal{X}), \mathcal{Y}\rangle = U^{\sharp}((\mathcal{R}(\mathcal{X} \wedge \mathcal{Y} - u\mathcal{X} \wedge u\mathcal{Y}) + u\mathcal{R}(u\mathcal{X} \wedge \mathcal{Y} + \mathcal{X} \wedge u\mathcal{Y})).u).$$

Proof. We denote by $\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{Y}}$ the projection of $\mathcal{X}, \mathcal{Y} \in \mathcal{H} \oplus \mathcal{H}^{\star}$ to $\mathcal{H}$. Using the definition of the Courant bracket we know that $[U^{\sharp}, \mathcal{X}] = [U^{\sharp}, \overrightarrow{\mathcal{X}}]$ is a 1-form. More precisely, for two vector fields $A \in \Gamma(\mathcal{V})$ and $\overrightarrow{\mathcal{Y}} \in \Gamma(\mathcal{H})$, at the point $p = (m, u) \in \mathcal{Z}(\mathcal{Q})$ we have

$$\begin{aligned}[U^{\sharp}, \mathcal{X}](A + \overrightarrow{\mathcal{Y}}) &= dU^{\sharp}(\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{Y}} + A)\\ &= \overrightarrow{\mathcal{X}}.U^{\sharp}(A) - U^{\sharp}([\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{Y}} + A])\\ &= \overrightarrow{\mathcal{X}}.U^{\sharp}(A) - U^{\sharp}([\overrightarrow{\mathcal{X}}, A]) - U^{\sharp}(\mathcal{R}(\mathcal{X} \wedge \mathcal{Y}) \cdot u).\end{aligned}$$

The point (iii) of Proposition 9 gives us $[\mathbb{J}U^{\sharp}, \mathcal{X}](A) = \mathbb{J}[U^{\sharp}, \mathcal{X}](A)$, and so $\mathcal{N}(U^{\sharp}, \mathcal{X})$ is the horizontal 1-form defined by

$$\langle \mathcal{N}(U^{\sharp}, \mathcal{X}), \mathcal{Y}\rangle = U^{\sharp}((\mathcal{R}(\mathcal{X} \wedge \mathcal{Y} - u\mathcal{X} \wedge u\mathcal{Y}) + u\mathcal{R}(u\mathcal{X} \wedge \mathcal{Y} + \mathcal{X} \wedge u\mathcal{Y})) \cdot u).$$

□

COROLLARY 3

For all basic sections $\widehat{\mathcal{X}}, \widehat{\mathcal{Y}} \in \Gamma(\mathcal{H} \oplus \mathcal{H}^{\star})$, *at the point* $p = (m, u) \in \mathcal{Z}(\mathcal{Q})$, (*i*) *the vertical part of* $\mathcal{N}(\widehat{\mathcal{X}}, \widehat{\mathcal{Y}})$ *is the following vector field*:

$$-(\mathcal{R}(\mathcal{X} \wedge \mathcal{Y} - u\mathcal{X} \wedge u\mathcal{Y}) + u\mathcal{R}(u\mathcal{X} \wedge \mathcal{Y} + \mathcal{X} \wedge u\mathcal{Y})) \cdot u,$$

(*ii*) *the horizontal part of* $\mathcal{N}(\widehat{\mathcal{X}}, \widehat{\mathcal{Y}})$ *is a* 1*-form defined for any* $\widehat{\mathcal{Z}} \in \Gamma(\mathcal{H})$ *by*

$$\begin{aligned}\langle \mathcal{N}(\widehat{\mathcal{X}}, \widehat{\mathcal{Y}}), \widehat{\mathcal{Z}}\rangle &= \mathcal{T}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) - \mathcal{T}(\mathcal{X}, u\mathcal{Y}, u\mathcal{Z}) \\ &\quad -\mathcal{T}(u\mathcal{X}, \mathcal{Y}, u\mathcal{Z}) - \mathcal{T}(u\mathcal{X}, u\mathcal{Y}, \mathcal{Z}).\end{aligned}$$

Proof. Pick an orthonormal basis $(\mathcal{X}_1, \ldots, \mathcal{X}_{8n})$ of $\mathbb{T}M$ defined over an open set $\mathcal{U}$. The distribution $\mathcal{H} \oplus \mathcal{H}^\star$ is stable for $\mathbb{J}$, we denote by $[\mathbb{J}_{ij}]$ its matrix in the basic $(\widehat{\mathcal{X}_1}, \ldots, \widehat{\mathcal{X}_{8n}})$. By definition,

$$\begin{aligned}[\mathbb{J}\widehat{\mathcal{X}_i}, \mathbb{J}\widehat{\mathcal{X}_j}] &= \overrightarrow{\mathbb{J}\widehat{\mathcal{X}_i}}.(\mathbb{J}_{rj})\ \widehat{\mathcal{X}_r} - \overrightarrow{\mathbb{J}\widehat{\mathcal{X}_j}}.(\mathbb{J}_{li})\ \widehat{\mathcal{X}_l} + \mathbb{J}_{li}\mathbb{J}_{rj}[\widehat{\mathcal{X}_l}, \widehat{\mathcal{X}_r}] \\ &\quad -\mathbb{J}_{ri}d\mathbb{J}_{rj} + \mathbb{J}_{lj}d\mathbb{J}_{li} \\ [\mathbb{J}\widehat{\mathcal{X}_i}, \widehat{\mathcal{X}_j}] + [\widehat{\mathcal{X}_i}, \mathbb{J}\widehat{\mathcal{X}_j}] &= -\overrightarrow{\widehat{\mathcal{X}_j}}.(\mathbb{J}_{li})\ \widehat{\mathcal{X}_l} + \mathbb{J}_{li}[\widehat{\mathcal{X}_l}, \widehat{\mathcal{X}_j}] + \overrightarrow{\widehat{\mathcal{X}_i}}.(\mathbb{J}_{rj})\ (\widehat{\mathcal{X}_r}) \\ &\quad +\mathbb{J}_{rj}[\widehat{\mathcal{X}_i}, \widehat{\mathcal{X}_r}] + d\mathbb{J}_{ji} - d\mathbb{J}_{ij}.\end{aligned}$$

Using Proposition 10, we deduce that at the point $p = (m, u)$, the vertical part of $\mathcal{N}(\widehat{\mathcal{X}_i}, \widehat{\mathcal{X}_j})$ is

$$-(\mathcal{R}(\mathcal{X}_i \wedge \mathcal{X}_j - u\mathcal{X}_i \wedge u\mathcal{X}_j) + u\mathcal{R}(u\mathcal{X}_i \wedge \mathcal{X}_j + \mathcal{X}_i \wedge u\mathcal{X}_j)) \cdot u.$$

For the horizontal part, we consider a local section s of $\mathcal{Z}(\mathcal{Q}) \longrightarrow M$ on $\mathcal{U}$, such that $s(m) = u$ and $(\nabla s)_m = 0$. This gives a local generalized almost complex structure S on $\mathcal{U}$. The horizontal part of $\mathcal{N}(\widehat{\mathcal{X}_i}, \widehat{\mathcal{X}_j})$ restricted to $s(M)$ is equal to the horizontal lift of the Nijenhuis tensor of S. Since the connection ∇ has generalized torsion $\mathcal{T}$ and since $\nabla s = 0$ at m, we have at the point (m, u),

$$\begin{aligned}\langle \mathcal{N}(\widehat{\mathcal{X}_i}, \widehat{\mathcal{X}_j}), \widehat{\mathcal{X}_k}\rangle &= \mathcal{T}(\mathcal{X}_i, \mathcal{X}_j, \mathcal{X}_k) - \mathcal{T}(u\mathcal{X}_i, u\mathcal{X}_j, \mathcal{X}_k) \\ &\quad -\mathcal{T}(u\mathcal{X}_i, \mathcal{X}_j, u\mathcal{X}_k) - \mathcal{T}(\mathcal{X}_i, u\mathcal{X}_j, u\mathcal{X}_k).\end{aligned}$$

Proof of Theorem A. Since fibers of $\mathcal{Z}(\mathcal{Q})$ has the complex structure of $\mathbb{C}P^1$, we get $\mathcal{N}(\mathcal{U}, \mathcal{V}) = 0$ for all $\mathcal{U}, \mathcal{V}$ sections of $\mathcal{V} \oplus \mathcal{V}^\star$. The proof of Theorem A is then an immediate consequence of Corollaries 1, 2 and 3. □

7. Applications

7.1 *Generalized Bismut connection*

Generalized hyperkähler structures are among the simplest examples of generalized quaternionic structure $\mathcal{Q}$. In that case there is a natural connection preserving $\mathcal{Q}$. This connection was introduced by Gualtieri [13] and is called generalized Bismut connection. We start by recalling its construction. Let $G = (g, b)$ be a generalized metric and C^+ the associated maximal-positive-definite subbundle of $\mathbb{T}M$. Let $C : \mathbb{T}M \longrightarrow \mathbb{T}M$ be the automorphism defined by $C(X + \xi) = X - \xi$. Write $\mathcal{X} = \mathcal{X}^+ + \mathcal{X}^-$ for the orthogonal projection of $\mathcal{X} \in \Gamma(\mathbb{T}M)$ to $C^\pm$ and let h be any closed 3-form on M.

PROPOSITION 11 [13]

The operator

$$D_{\mathcal{X}}\mathcal{Y} = [\mathcal{X}^-, \mathcal{Y}^+]_h^+ + [\mathcal{X}^+, \mathcal{Y}^-]_h^- + [C\mathcal{X}^-, \mathcal{Y}^-]_h^- + [C\mathcal{X}^+, \mathcal{Y}^+]_h^+$$

defines a generalized connection on $\mathbb{T}M$, *preserving both the inner product* $\langle ., . \rangle$ *and the positive-definite metric* G. *So* D *preserved* $C^{\pm}$ *and if we denote by* $D^{\pm}$ *the restriction of* D *to* $C^{\pm}$, *then we have*

$$D^{\pm} = \pi_{\pm}^{-1}\nabla^{\pm}\pi_{\pm},$$

where $\nabla^{\pm}$ *are the Bismut connection on* (M, g) *with torsion* $\pm h$. *In particular*, *for any 1-form* ξ *we have* $D_{\xi} = 0$. *We may write* D *explicitly with respect to the splitting* $\mathbb{T}M = TM \oplus T^{\star}M$ *as follows*:

$$D_X = \begin{pmatrix} \nabla_X^g & \frac{1}{2} \wedge^2 g^{-1}(i_X h) \\ \frac{1}{2} i_X h & (\nabla_X^g)^{\star} \end{pmatrix},$$

where ∇^g *is the Levi–Civita connection of* g *and* X *any vector field.*

DEFINITION

This connection D is called by Gualtieri as the generalized Bismut connection associated to G.

This connection enables us to give a new characterisation of twisted generalized Kähler manifold.

PROPOSITION 12 [13]

If $\mathcal{J}$ *is a* G*-orthogonal generalized almost complex structure*, *then* $(\mathcal{J}, G)$ *defines a twisted generalized Kähler structure if and only if*

(1) $D\mathcal{J} = 0$ *and*,
(2) *the generalized torsion* $\mathcal{T}_D$ *is of type* $(2, 1) + (1, 2)$ *with respect to* $\mathcal{J}$.

Theorem 5. *Let* $n \geq 1$. *If* $(M, G, \mathcal{I}, \mathcal{J}, \mathcal{K})$ *is a twisted generalized hyperkähler* $4n$*-manifold and* D *the generalized Bismut connection, then the generalized almost complex structure* $\mathbb{J}_D$ *on* $\mathcal{Z}$ *is integrable.*

Proof. Using Proposition 12, we see that both integrability conditions of Theorem A are trivially true. □

7.2 *Levi–Civita connection*

Theorem C [5]. *Let* $n > 1$ *and let* $(M, g, \mathcal{Q})$ *be a Riemannian* $4n$*-manifold with a generalized almost quaternionic structure such that the Levi–Civita connection* ∇^g *preserve* $\mathcal{Q}$, *then the generalized almost complex structure* $\mathbb{J}_{\nabla^g}$ *on* $\mathcal{Z}(\mathcal{Q})$ *is integrable.*

Remark. The case $n = 1$ is also treated in [5].

7.3 *Generalized torsion free connection*

Let $\mathcal{Q}$ be a generalized almost quaternionic structure on M locally spanned by a generalized almost hypercomplex structure $(\mathcal{I}, \mathcal{J}, \mathcal{K})$. Let G be any generalized metric on M compatible with $\mathcal{Q}$. With respect to the splitting $\mathbb{T}M = C^+ \oplus C^-$, an element $u \in \mathcal{Q}$ takes the form $\begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix}$. By projection from $C^\pm$ to TM, we can consider $u^\pm$ as an almost complex structure on TM. Thus a generalized almost quaternionic structure gives two almost quaternionic structures namely $Q^\pm = \mathrm{Span}(I^\pm, J^\pm, K^\pm)$. We will denote by

$$\begin{aligned} f : Q^- &\longrightarrow \qquad \mathcal{Q} \qquad \longrightarrow Q^+ \\ u^- &\longmapsto u = \begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix} \longmapsto u^+. \end{aligned}$$

This map induces an algebra isomorphism from $\mathrm{Span}(Id) \oplus Q^-$ to $\mathrm{Span}(Id) \oplus Q^+$.

Theorem 6. *Let $n > 1$ and let $(M, \mathcal{Q}, G)$ be a generalized almost quaternionic $4n$-manifold with a generalized metric G compatible with $\mathcal{Q}$ such that $Q^+ = Q^-$. For any (usual) connection ∇ on $\mathbb{T}M$ compatible with $\mathcal{Q}$, and the inner product and which is generalized torsion free: $\mathbb{J}_\nabla$ is integrable if and only if locally*

(1) *there exists a generalized hypercomplex structure such that $\nabla\mathcal{I} = \nabla\mathcal{J} = \nabla\mathcal{K} = 0$,*
(2) *or $f = Id$.*

Remark. From Proposition 4, we see that $f = Id$ correspond to $e^{-b}\mathcal{Q}e^b$ is an almost quaternionic structure; where b is the 2-form associated to G.

Proof of Theorem 6. It is clear that if $\nabla\mathcal{I} = \nabla\mathcal{J} = \nabla\mathcal{K} = 0$ for a generalized torsion free connection, then both integrability conditions of Theorem A are satisfied. As b is not necessarily closed, the integrability of $\mathbb{J}_\nabla$ when $f = Id$ is not so clear and requires the following lemma.

Lemma 1. *Let ∇ be a generalized torsion free (usual) connection on $\mathbb{T}M$ compatible with the inner product. On the basis $TM \oplus T^\star M$, we have*

$$\nabla = \begin{bmatrix} \nabla^1 & 0 \\ L & \nabla^2 \end{bmatrix},$$

where

(*i*) *∇^1 is a torsion free connection on TM : $\nabla^1_X Y - \nabla^1_Y X = [X, Y]$.*
(*ii*) *∇^2 is the connection on $T^\star M$ induced by ∇^1, for all $X, Y \in \Gamma(TM)$, for all $\xi \in \Gamma(T^\star M)$, $X \cdot \langle \xi, Y\rangle = \langle \nabla^2_X \xi, Y\rangle + \langle \xi, \nabla^1_X Y\rangle$.*

Proof of the lemma. ∇ is a generalized torsion free connection so $\mathcal{T}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = 0$ $\forall \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathbb{T}M)$. In particular, for $(X, \xi, \eta) \in \Gamma(TM) \times \Gamma(T^\star M) \times \Gamma(T^\star M)$,

$$\begin{aligned} \mathcal{T}(X, \xi, \eta) = 0 &\Longleftrightarrow \langle \nabla_X \xi - [X, \xi], \eta\rangle = 0 \\ &\Longleftrightarrow \nabla_X \xi \text{ is a 1-form.} \end{aligned}$$

On the basis $TM \oplus T^\star M$, a generalized torsion free connection ∇ takes the form

$$\nabla = \begin{bmatrix} \nabla^1 & 0 \\ L & \nabla^2 \end{bmatrix}.$$

On the other hand, for all $X, Y, \xi \in \Gamma(TM) \times \Gamma(TM) \times \Gamma(T^\star M)$, we have

$$\mathcal{T}(X, Y, \xi) = 0 \iff \langle \nabla_X Y - \nabla_Y X - [X, Y], \xi \rangle = 0 \\ \iff \nabla^1 \text{ is torsion free.}$$

But ∇ is compatible with the inner product and so

$$X \cdot \langle \xi, Y \rangle = \langle \nabla^2_X \xi, Y \rangle + \langle \xi, \nabla^1_X Y \rangle. \qquad \square$$

When $f = Id$, any local generalized almost complex structure living in $\mathcal{Q}$ takes form $u = e^b \begin{bmatrix} J & 0 \\ 0 & -J^\star \end{bmatrix} e^{-b}$ for some local almost complex structure J. Using the lemma and the fact that ∇ preserve $\mathcal{Q}$, a little computation shows that

$$\nabla_X u = e^b \begin{bmatrix} \nabla^1_X J & 0 \\ 0 & -(\nabla^1_X J)^\star \end{bmatrix} e^{-b}.$$

In particular, this means that ∇^1 preserve the almost quaternionic structure $Q = e^{-b} \mathcal{Q} e^b$. If we denote by R^1 the curvature of the connection ∇^1 and if we differentiate one more time, we have that

$$\mathcal{R}(X, Y) \cdot u = e^b \begin{bmatrix} R^1(X, Y) \cdot J & 0 \\ 0 & -(R^1(X, Y) \cdot J)^\star \end{bmatrix} e^{-b}.$$

Thus the integrability of $\mathbb{J}_\nabla$ on $\mathcal{Z}(\mathcal{Q})$ is a consequence of the integrability of $\mathbb{J}_{\nabla^1}$ on $Z(Q)$ (cf. Theorems 1 and 2).

It remains to prove the converse. In the basis $C^+ \oplus C^-$ the connection ∇ is partitioned into four blocks:

$$\nabla = \begin{pmatrix} \nabla^+ & \nabla^{+-} \\ \nabla^{-+} & \nabla^- \end{pmatrix}.$$

Since ∇ is compatible with $\mathcal{Q}$, for any vector field X, we have

$$\begin{aligned} \nabla_X \mathcal{I} &= \gamma(X)\mathcal{J} - \beta(X)\mathcal{K}, \\ \nabla_X \mathcal{J} &= -\gamma(X)\mathcal{I} + \alpha(X)\mathcal{K}, \\ \nabla_X \mathcal{K} &= \beta(X)\mathcal{I} - \alpha(X)\mathcal{J}, \end{aligned}$$

where α, β, γ are 1-form. Projecting on $C^\pm$, we get

$$\begin{cases} \nabla^+ I^+ = \gamma J^+ - \beta K^+ \\ \nabla^+ J^+ = -\gamma I^+ + \alpha K^+ \\ \nabla^+ K^+ = \beta I^+ - \alpha J^+ \end{cases} \text{ and } \begin{cases} \nabla^- I^- = \gamma J^- - \beta K^- \\ \nabla^- J^- = -\gamma I^- + \alpha K^- \\ \nabla^- K^- = \beta I^- - \alpha J^- \end{cases}. \tag{2}$$

Let $G = (g, b)$ be the generalized metric compatible with $\mathcal{Q}$. Using the lemma and the fact that $C^\pm = \{X + (b \pm g)X \in \mathbb{T}M / X \in TM\}$, we find

$$\begin{aligned} \nabla^+ &= (\pi_+)^{-1}(\nabla^1 + g^{-1}(\nabla^2(b + g) + L - b(\nabla^1)))\pi, \\ \nabla^- &= (\pi_-)^{-1}(\nabla^1 - g^{-1}(\nabla^2(b - g) + L - b(\nabla^1)))\pi, \\ \nabla^{-+} &= (\pi_+)^{-1}(\nabla^1 - g^{-1}(\nabla^2(b + g) + L - b(\nabla^1)))\pi, \\ \nabla^{+-} &= (\pi_-)^{-1}(\nabla^1 + g^{-1}(\nabla^2(b - g) + L - b(\nabla^1)))\pi. \end{aligned}$$

But f is an automorphism of $\mathbb{S}^2$, so it is a rotation. In a suitable basis we can write

$$\begin{cases} I^+ = f(I^-) = I^- \\ J^+ = f(J^-) = cJ^- + sK^-, \text{ with } \mathrm{c}^2 + s^2 = 1. \\ K^+ = f(K^-) = -sJ^- + cK^- \end{cases} \tag{3}$$

On the other hand, from $\nabla\mathcal{I} \in \mathcal{Q}$ and $I^+ = I^-$, we deduce that $\nabla^{+-}I^+ = 0$ and $\nabla^{-+}I^+ = 0$, and so $(\nabla^2 b + L - b(\nabla^1))I^+ = 0$. In particular, we deduce that $\nabla^+ I^+ = \nabla^- I^-$. Now using (2) and (3), we have

$$\begin{cases} c\gamma + s\beta = \gamma \\ s\gamma - c\beta = -\beta \end{cases} \iff \begin{cases} (c-1)\gamma + s\beta = 0, \\ s\gamma + (1-c)\beta = 0. \end{cases}$$

On each point, either $f = Id$ or $\gamma = \beta = 0$. We suppose that $f \neq Id$ and denote by R^+ the curvature of the connection ∇^+. In this case, we have

$$\begin{cases} \nabla^+_X I^+ = 0 \\ \nabla^+_X J^+ = \alpha(X)\ K^+ \\ \nabla^+_X K^+ = -\alpha(X)\ J^+ \end{cases} \implies \begin{cases} R^+(X,Y)\cdot I^+ = 0, \\ R^+(X,Y)\cdot J^+ = d\alpha(X,Y)\ K^+, \\ R^+(X,Y)\cdot K^+ = -d\alpha(X,Y)\ J^+. \end{cases}$$

But if $\mathbb{J}_\nabla$ is integrable, the condition $(C2)$ of Theorem A must be true. In particular, for $\mathcal{X} = \pi_+^{-1}(X)$ and $\mathcal{Y} = \pi_-^{-1}(Y)$, the projection on C^+ gives

$$\begin{cases} R^+(J^+X\wedge Y + X\wedge J^-Y)\cdot J^+ = 0 \\ R^+(K^+X\wedge Y + X\wedge K^-Y)\cdot K^+ = 0 \end{cases} \iff \begin{cases} d\alpha(J^+X,Y) = -d\alpha(X,J^-Y), \\ d\alpha(K^+X,Y) = -d\alpha(X,K^-Y), \end{cases}$$

so

$$\begin{cases} d\alpha((cJ^- + sK^-)X,Y) = -d\alpha(X,J^-Y) & (L1) \\ d\alpha((-sJ^- + cK^-)X,Y) = -d\alpha(X,K^-Y) & (L2) \end{cases}$$
$$\implies d\alpha(J^-X,Y) = -d\alpha(X,(cJ^- - sK^-)Y) \quad (L3) = (cL1 - sL2)$$

By symmetry, we also have

$$d\alpha(J^-X,Y) = -d\alpha(X,J^+Y)$$
$$\implies d\alpha(J^-X,Y) = -d\alpha(X,(cJ^- + sK^-)Y) \quad (L4)$$

Taking $(L3) - (L4)$, we have

$$0 = -2s\ d\alpha(X,K^-Y) \implies 0 = d\alpha(X,Y) \quad \forall X,Y \in TM.$$

Thus, α is closed so locally exact: $\alpha = d\theta$ for some locally defined function θ on M, and it is easy to check that $(\mathcal{I}, \cos\theta\,\mathcal{J} - \sin\theta\,\mathcal{K}, \sin\theta\,\mathcal{J} + \cos\theta\,\mathcal{K})$ is a generalized hypercomplex structure such that $\nabla\mathcal{I} = \nabla\mathcal{J} = \nabla\mathcal{K} = 0$. □

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