Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:66
https://doi.org/10.1007/s12044-020-00592-6

Connecting specific maps having two equal-sized faces and its genus

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MS received 14 September 2018; revised 27 November 2019; accepted 13 July 2020

Abstract. Graham Higman introduced the concept of januarial as a specific map having two equal sized faces under the action of $\langle x, y : x^2 = y^k = (xy)^\ell = 1\rangle$ on a finite set. In this paper we take up the question posed by Graham Higman that what is the maximum number of circuits in the subgraph of a simple januarial for any value of k? We describe conditions under which januarials are connected and larger januarials are obtained. In an effort to look at topological features of the connected januarial, we find out genus of the januarial, genera of the two faces and number of circuits.

Keywords. Coset diagrams; januarials; triangle groups; hecke groups and genus.

Mathematics Subject Classification. Primary: 05C25, Secondary: 20G40.

1. Introduction

In 1978, coset diagram was introduced by Graham Higman as a representation of action of the modular group on a set or a space. Several papers, including [2,3,5,9,10] appeared in which these coset diagrams were used to prove interesting topological and group theoretical results. Mushtaq [8] laid the foundations of coset diagrams for the action of the modular group on the Galois fields and real quadratic irrational number fields.

Let $\Delta(2, k, \ell)$ be the abstract group with presentation

$$\Delta(2, k, \ell) = \langle x, y : x^2 = y^k = (xy)^\ell = 1\rangle,$$

where $k, \ell \geq 2 \in \mathbb{Z} \cup \{\infty\}$. Such groups are called triangle groups as they act discretely on the 2-sphere, Euclidean plane or hyperbolic plane with fundamental region a triangle. The $\Delta(2, k, \infty)$ are called *Hecke groups* Δ_k; in particular, for $k = 3$ we obtain the classical modular group.

A *cosetdiagram* Γ is a directed graph in the plane depicting the action of $\Delta(2, k, \ell)$ on some set S as follows: the vertices of the graph are the elements of S; if two vertices are interchanged by the action of x, then there is an undirected x -edge connecting them. There is a directed y-edge from u to v, when u is sent to v under the action of y. We omit the

Published online: 18 November 2020

orientations on the y-edges of a coset diagram with the understanding that the boundary edges of each y-face are oriented anticlockwise around the face.

The genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. The genus of a graph is the minimal integer m such that the graph can be drawn on the surface of genus m without crossings [11].

Suppose $\Delta(2, k, \ell)$ acts on a set S such that there are two orbits of the subgroup $\langle xy\rangle$, each consisting of $|S|/2$ elements. A *janualial* J is the embedded coset diagram, denoted by J_k. In particular, a janualial is a diagram with two xy-faces. For a nice introduction to janualials, see [1], from which the following concepts are taken.

Let S_1 and S_2 be the discs obtained by taking the closures of the two xy-faces. Let J be a janualial arising from the coset diagram Γ.

It is convenient to collapse superfluous structure in a diagram. Collapsing each y-face to a point gives the *companion diagram* J'. Let S_1', S_2' be the result of applying this to the discs S_1, S_2.

Let g be the genus of the surface in which the janualial J is embedded. Let R_i be a small closed neighbourhood of S_i' in J'. Then R_i is a closed surface with boundary. Let g_i be its genus and h_i the number of connected components in the boundary.

The *common graph* Υ is the intersection $S_1' \cap S_2'$. The following is in [1, Lemma 3.5]:

Lemma 1. *Let P_1 (resp. P_2) be the set of all paths that traverse successive edges in Υ in the directions they are traversed by S_1' (resp. S_2') in such a way that whenever such a path reaches a vertex, it continues along the right-most of the other edges in Υ incident with that vertex. (The next edge is necessarily traversed by S_1 (resp. S_2) in that direction.) All such paths close up into circuits, and P_1 (resp. P_2) partitions Υ, in the sense that the union of the circuits is Υ and no two share an edge. The cardinality of P_1 (resp. P_2) is h_2 (resp. h_1).*

A janualial is of *simple type* if Υ is composed of h disjoint simple circuits. In this case, $h = h_1 = h_2$ and we denote it by (h, g_1, g_2). Otherwise, the janualial is of *general type* $((h_1, g_1), (h_2, g_2))$.

The following are Lemmas 3.1, 3.4 and 3.6 of [1]:

Lemma 2. *For a janualial the genus $g = \frac{1}{2}(E - V)$, where E is the number of x-edges which are not loops and V the number of y-faces.*

Lemma 3. *The genus of a janualial J of simple type is $g = g_1 + g_2 + h - 1$.*

Lemma 4. *The genus g_i satisfies $2 - 2g_i = V_i - E_i + h_i + 1$, where V_i and E_i denote the number of vertices and edges in the subgraph of J' visited by S_i'.*

The topics like decomposition of any compound into its basic components and unifying two or more structures to form a new structure have always been of great interest for researchers. Groups are chained through direct and free products, whereas graphs are tied through different techniques, such as connecting, joining and stitching. Since the modular group $PSL(2, \mathbb{Z}) = \langle x, y : x^2 = y^3 = 1\rangle$ is a free product of two cyclic groups C_2 and C_3, the Cayley graph of C_2 is an edge and that of C_3 is a triangle and the Cayley graph of the modular group is formed by connecting infinite number of triangles through the edges.

Different methods of connecting/ joining the coset graphs have been introduced in [3,4,6] and [7].

In this paper, we introduce a novel technique of connecting two or more janualials to form janualials for $\Delta(2, k, \ell)$, where ℓ is sufficiently large. We also study the effect of connecting janualials on their topological aspects. In Sect. 3, we answer the following question raised by Graham Higman in [1]:

Problem 5. *How large can h be, for a given k for janualials of simple type*?

2. Ways of connecting janualials

Our aim in this section is to combine two maps both having two equal-sized faces such that we obtain a map with two same-sized faces. For simplicity, we have considered orbits or cycles of $\langle xy \rangle$ instead of faces.

Two janualials can either be connected through fixed points or by removal and addition of edges or using both end points and edges. In the following subsections, first we study the changes that occur in the orbits of $\langle xy \rangle$ after the connections, then we will sort out the ways where we can get janualials.

2.1 *Connection through fixed points*

Following are the types of connections for cosets graphs using the fixed points.

(1) Use the fixed point c of x of one graph and the fixed point c' of x in the other graph. Insert an x-edge cc' joining the two fixed points and hence connecting the diagrams, as depicted in Figure. 1.
Here the cycles of xy $(a \cdots c)$ and $(a' \cdots c')$ containing both the fixed points are fused together as a cycle $(a \cdots ca' \cdots c')$ of xy in the resulting graph.
(2) Use the fixed point d of y of one graph and the fixed point d' of y in the other graph. Remove the fixed points and the edges dc and $d'c'$ respectively joining them to the graphs and connect the graphs by inserting an x-edge cc', as depicted in Figure 2.
Now the cycles of xy $(da \cdots c)$ and $(d'a' \cdots c')$ containing both the fixed points are joined as a cycle $(a \cdots ca' \cdots c')$ of xy with the omission of fixed points d and d' of y in the resulting graph.
(3) Or use the fixed point $c'$of x of one graph and the fixed point d of y in the other graph. Remove the fixed point d of y and the x-edge cd, then insert an edge cc' connecting the graphs, as shown in Figure 3.
Moreover, the cycles of xy $(da \cdots c)$ and $(a' \cdots c')$ containing both the fixed points add up as a cycle $(a \cdots ca' \cdots c')$ of xy with the omission of fixed point d of y in the resulting graph.

The following theorem gives the condition for connecting janualials using the fixed points.

Theorem 6. *When two janualials are connected using fixed points, then the resulting diagram is a janualial if both of them have at least two fixed points such that one lie in each orbit of* $\langle xy \rangle$ *and both the connections are of same type.*

Proof. Let the two janualials be J and J'. Let the cardinality of two orbits of $\langle xy \rangle$ in J be m and that of J' be n. There are four possibilities.

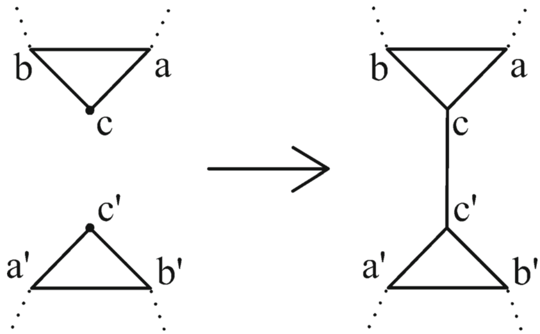


Figure 1. Connecting through fixed points of x.

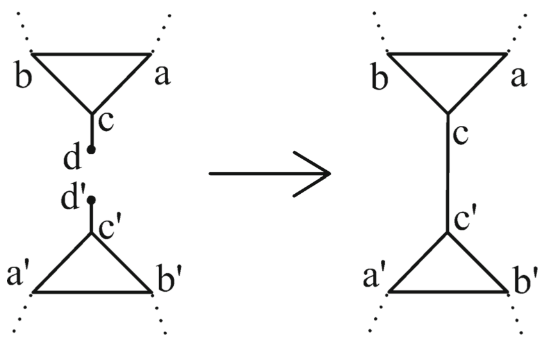


Figure 2. Connecting through fixed points of y.

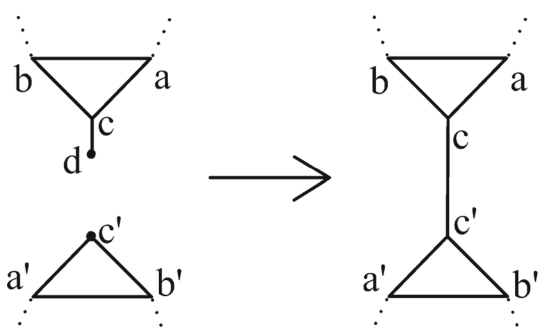


Figure 3. Connecting through fixed points of x and y.

(1) When all the four connecting points are the fixed points of x, then the orbits of $\langle xy\rangle$ in J and J' merge as described above and the resulting diagram will have $m+n$ elements in each orbit of $\langle xy\rangle$. Hence the diagram is janualial in this case.
(2) When all the four connecting points are the fixed points of y, then the orbits of $\langle xy\rangle$ in J and J' merge as described above and the resulting diagram will have $m+n-2$ elements in each orbit of $\langle xy\rangle$. Hence the diagram is januarial in this case.
(3) When the two connecting points of J are fixed points of x and that of the J' are fixed points of y, then the orbits of $\langle xy\rangle$ in J and J' merge as described above and

the resulting diagram will have $m + n - 1$ elements in each orbit of $\langle xy \rangle$. Hence the diagram is januarial in this case.

(4) When one connecting point of both januarials is a fixed point of x and the other is a fixed point of y. If we connect fixed point of x in J to the fixed point of y in J' and fixed point of y in J to the fixed point of x in J', then the orbits of $\langle xy \rangle$ in J and J' merge as described above and the resulting diagram will have $m + n - 1$ elements in each orbit of $\langle xy \rangle$. Hence the diagram is januarial in this case.

□

Note that in the case when one connecting point of both januarials is a fixed point of x and the other is a fixed point of y. If we connect fixed point of x in J to the fixed point of x in J' and fixed point of y in J to the fixed point of y in J' (that is, using different type of connection), then the resulting diagram will have $m + n - 2$ elements in one orbit of $\langle xy \rangle$ and $m + n$ elements in the other. In this case, the diagram is not a januarial.

Next, we discuss the properties of the resulting januarial.

- If we connect two januarials, one of genus g which is the result of the natural action of $\Delta(2, k, \ell)$ on 2ℓ points and the other of genus g' which is the result of the natural action of $\Delta(2, k, \ell')$ on $2\ell'$ points, then depending on the nature of the fixed points, the following cases emerge:
  (1) When all the four connecting points are the fixed points of x, the resulting januarial will be of genus $g+g'+1$ depicting the natural action of $\Delta(2, k, \ell+\ell')$ on $2(\ell+\ell')$ points.
  (2) When all the four connecting points are the fixed points of y, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 2)$ on $2(\ell + \ell' - 2)$ points.
  (3) When the two connecting points of one januarial are fixed points of x and that of the other januarial are fixed points of y, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 1)$ on $2(\ell + \ell' - 1)$ points.
  (4) When one connecting point of both januarials is a fixed point of x and the other is a fixed point of y, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 1)$ on $2(\ell + \ell' - 1)$ points.
- If the two januarials are of general type $((h_1, g_1), (h_2, g_2))$ and $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of type $((h_1 + h'_1, g_1 + g'_1), (h_2 + h'_2, g_2 + g'_2))$.
- If the januarials are of simple type (h, g_1, g_2) and (h', g'_1, g'_2), then the resulting januarial will be of type $(h + h', g_1 + g'_1, g_2 + g'_2)$.
- If one januarial is of simple type (h, g_1, g_2) and the other of general type $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of general type $((h + h'_1, g_1 + g'_1), (h + h'_2, g_2 + g'_2))$.

The above properties follow merely by some calculations using Lemmas 2, 3 and 4 .

2.2 *Connection through edges*

Two coset graphs can be connected by using x-edges of both graphs in two ways.

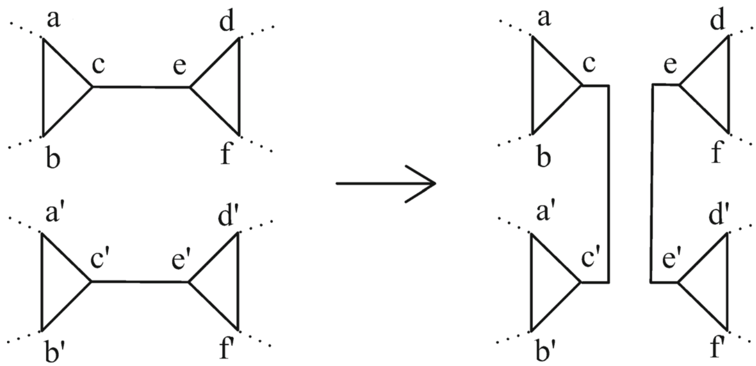


Figure 4. Connecting through moving the edges.

(1) Remove one x-edge from each graph ce and $c'e'$ and connect the end points of these edges across the graphs using two x-edges cc' and ee' as shown in Fig. 4.

  (i) If both the end points of an edge lie in one orbit of $\langle xy\rangle$ in both coset graphs, then the two cycles $(f\cdots ea\cdots c)$ and $(f'\cdots e'a'\cdots c')$ of xy, one in each graph, are reconstructed in two cycles $(a'\cdots c'a\cdots c)$ and $(f\cdots ef'\cdots e')$ of xy in the resulting graph.

  (ii) If both the end points of an edge lie in different orbits of $\langle xy\rangle$ in the two coset graphs, then the four cycles $(f\cdots c)(a\cdots e)$ and $(f'\cdots c')(a'\cdots e')$ of xy, two in each graph, are reconstructed in two cycles $(a'\cdots e'f\cdots c)$ and $(a\cdots ef'\cdots c')$ of xy in the resulting graph, where both the orbits are union of one orbit from each graph.

  (iii) If both the end points of an edge lie in one orbit of $\langle xy\rangle$ in one coset graph and both the end points of an edge lie in the different orbits of $\langle xy\rangle$ in the other coset graph, then the three cycles $(f\cdots ea\cdots c)$ and $(f'\cdots c')(a'\cdots e')$ of xy involved, merge in one cycle $(f\cdots ef'\cdots c'a\cdots ca'\cdots e')$ of xy in the resulting graph.

Therefore Theorem 7 follows.

Theorem 7. *Using edges, janualrial can be connected to the other januarial to give a resultant januarial if both the end points of the edge lie in different orbits of $\langle xy\rangle$ in the two januarials.*

The calculations using Lemmas 2, 3 and 4 give the following topological properties:

- If we connect two januarials, one of genus g which is the result of the natural action of $\Delta(2,k,\ell)$ on 2ℓ points and the other of genus g' which is the result of the natural action of $\Delta(2,k,\ell')$ on $2\ell'$ points, the resulting januarial will be of genus $g+g'$ depicting the natural action of $\Delta(2,k,\ell+\ell')$ on $2(\ell+\ell')$ points.
- If the two januarials are of general type $((h_1,g_1),(h_2,g_2))$ and $((h_1',g_1'),(h_2',g_2'))$, then the resulting januarial will be of type $((h_1+h_1'-1,g_1+g_1'),(h_2+h_2'-1,g_2+g_2'))$.
- If the januarials are of simple type (h,g_1,g_2) and (h',g_1',g_2'), then the resulting januarial will be of type $(h+h'-1,g_1+g_1',g_2+g_2')$.

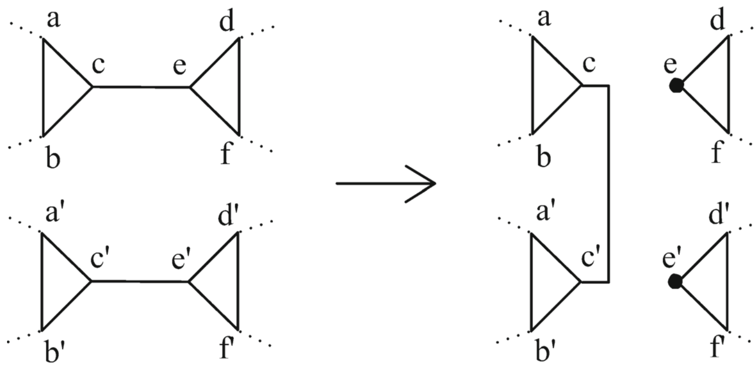


Figure 5. Connecting through moving and omitting the edges.

- If one januarial is of simple type (h, g_1, g_2) and the other of general type $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of general type $((h + h'_1 - 1, g_1 + g'_1), (h + h'_2 - 1, g_2 + g'_2))$.

(2) In Figure 5, first remove one x-edge from each graph ce and $c'e'$, then connect the end points of these edges across the graphs using one x-edge cc' and leaving two end points e and e' as either fixed points of x, or attaching x-edges, with a fixed point of y at the other end, to them or leaving one end point as fixed point of x and attaching x-edges, with a fixed point of y at the other end, to the other end point. The effect of connecting the graphs, on the orbits of $\langle xy \rangle$, in the three ways described above remains the same.

(i) If both the end points of an edge lie in one orbit of $\langle xy \rangle$ in both coset graphs, then the two cycles $(f \cdots ea \cdots c)$ and $(f' \cdots e'a' \cdots c')$ of xy, one in each graph, are reconstructed in three cycles $(a' \cdots c'a \cdots c)$ and $(f \cdots e)(f' \cdots e')$ of xy in the resulting graph.

(ii) If both the end points of the edge lie in different orbits of $\langle xy \rangle$ in the two coset graphs, then the four cycles $(f \cdots c)(a \cdots e)$ and $(f' \cdots c')(a' \cdots e')$ of xy involved, merge in one cycle $(a' \cdots e'f' \cdots c'a \cdots ef \cdots c)$ of xy in the resulting graph.

(iii) If both the end points of the edge lie in one orbit of $\langle xy \rangle$ in one coset graph and both the end points of the edge lie in the different orbits of $\langle xy \rangle$ in the other coset graph, then the three $(f \cdots ea \cdots c)$ and $(f' \cdots c')(a' \cdots e')$ of xy involved are deconstructed in two cycles $(a' \cdots e'f' \cdots c'a \cdots c)$ and $(f \cdots e)$ of xy in the resulting graph.

From the arguments in (2), it follows that januarials can not be constructed using this technique.

2.3 *Construction using both fixed point/s and an edge*

If we use an edge of only one graph, we can use either one or two fixed points from the other graph.

(1) In Figure 6, one fixed point c' of x of a coset graph is used. Remove the x-edge ce from the other graph and connect one end c of this x-edge to the fixed point c' by inserting

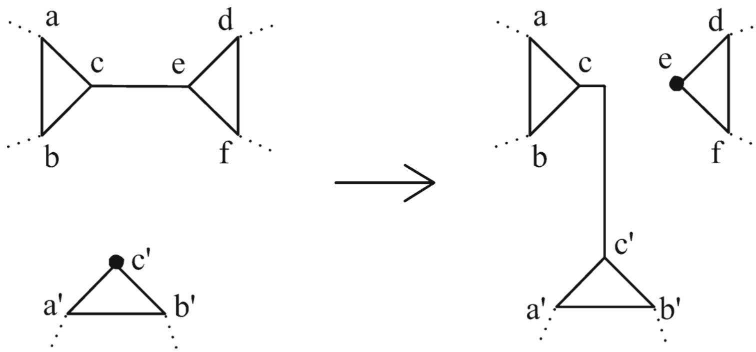


Figure 6. Connecting using one edge and one fixed point.

another x-edge cc' and leaving the other end e as fixed point of x or attaching x-edges, with a fixed point of y at the other end, to this end. If we use one fixed point of y of a coset graph then, first we need to remove the x-edge the other graph and delete the fixed point of y and the x-edge connecting it to the graph. Then we can insert an x-edge between the freed end points of the removed x-edges one of each graph and either leaving the third end point as fixed point of x or attaching x-edges, with a fixed point of y at the other end, to this end.

- (i) If both the end points of an edge lie in the one orbits of $\langle xy\rangle$ in the coset graph, then the two cycles $(f\cdots ea\cdots c)$ and $(a'\cdots c')$ of xy involved are reconstructed in two cycles $(a'\cdots c'a\cdots c)$ and $(f\cdots e)$ of xy in the resulting graph.
- (ii) If both the end points of an edge lie in the different orbits of $\langle xy\rangle$ in the two coset graphs, then the three cycles $(f\cdots c)(a\cdots e)$ and $(a'\cdots c')$ of xy involved, merge in one cycle $(f\cdots ca'\cdots c'a\cdots e)$ of xy in the resulting graph.

From the arguments in (1), it follows that januarials can not be constructed using this technique.

(2) In Figure 7, two fixed points of x, e' and c' of a coset graph are used. Remove the x-edge ce from the other graph and connect the ends c and e of this x-edge to the fixed point c' and e' by inserting the x-edges cc' and ee'. We can also use two fixed points of y or one fixed point of each x and y instead of two fixed point of x.

- (i) If both the end points of an edge lie in one orbit of $\langle xy\rangle$ in and both the fixed points of x lie in one orbit of $\langle xy\rangle$ in the coset graphs, then the two cycles $(f\cdots ea\cdots c)$ and $(a'\cdots e'f'\cdot c')$ of xy involved, one in each graph, merge in one cycle $(f\cdots ef'\cdots c'a\cdots ca'\cdot e')$ of xy in the resulting graph.
- (ii) If both the end points of an edge lie in one orbit of $\langle xy\rangle$ and both the fixed points of x lie in different orbits of $\langle xy\rangle$ in the coset graphs, then the three cycles $(f\cdots ea\cdots c)$ and $(a'\cdots c')(f'\cdot e')$ of xy involved are deconstructed in two cycles $(a'\cdots c'a\cdots c)$ and $(f'\cdots e'f\cdot e)$ of xy in the resulting graph.
- (iii) If both the end points of an edge lie in different orbits of $\langle xy\rangle$ and both the fixed points of x lie in one orbit of $\langle xy\rangle$ in the coset graphs, then the three cycles $(f\cdots c)(a\cdots e)$ and $(a'\cdots e'f'\cdot c')$ of xy involved are deconstructed in two cycles $(a'\cdots e'f\cdots c)$ and $(f'\cdots c'a\cdot e)$ of xy in the resulting graph.

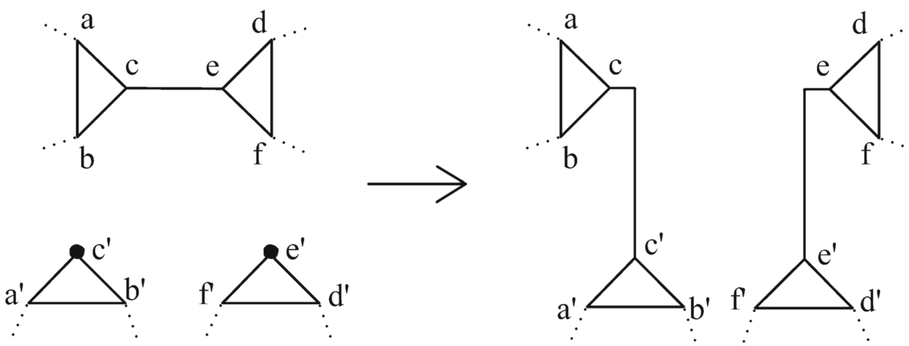


Figure 7. Connecting using one edge and two fixed points.

(iv) If both the end points of an edge lie in different orbits of $\langle xy\rangle$ and both the fixed points of x lie in different orbits of $\langle xy\rangle$ in the coset graphs, then the four cycles $(f\cdots c)(a\cdots e)$ and $(a'\cdots c')(f'\cdot e')$ of xy involved, two in each graph, merge in one cycle $(f\cdots ca'\cdots c'a\cdots ef'\cdot e')$ of xy in the resulting graph.

Theorem 8. *Using edges from one janularial and fixed points from the other, the resulting graph will only be a janularial if*

(1) *both the end points of each of the two edges lie in each of the two orbits of $\langle xy\rangle$ and*
(2) *the other janularial has four fixed points, two in each orbit of $\langle xy\rangle$, with similar nature, that is, either*

- *two fixed points of x in each orbit, or*
- *two fixed points of y in each orbit or*
- *one fixed point of x and one fixed point of y in each orbit.*

Proof. Let the two janularials be J and J'. Let the cardinality of two orbits O_1 and O_2 of $\langle xy\rangle$ in J be m and that of O_1' and O_2' in J' be n. Let both the end points of the edge ab lie in O_1 and that of the edge cd in O_2. Suppose the fixed points of x, a' and b' lie in O_1' and fixed points of x, c' and d' lie in O_2'. If we connect the edge ab to fixed points a', b' as described in 2 of Section 2.3, then by the argument in O_1 and O_1', we will have $m+n$ points in one orbit of the resulting diagram. Similarly connecting the edge cd to the fixed points c' and d' will give $m+n$ points in the other orbit, making the resulting diagram a janularial.

If a', b', c' and d' are fixed points of y, both the orbits of resulting diagram will have $m+n-2$ points and the diagram will be a janularial.

If a' and c' are fixed points of x, b' and d' are fixed points of y, then both the orbits of resulting diagram will have $m+n-1$ points and the diagram will be a janularial. □

Note that if a' and b' are fixed points of x, c' and d' are fixed points of y, then one orbit will have $m+n$ elements and the other will have $m+n-2$ points in the resulting diagram and therefore the diagram will not be a janularial.

Following are the properties of the resulting janularial:

- If we connect two janularials, one of genus g which is the result of the natural action of $\Delta(2,k,\ell)$ on 2ℓ points and the other of genus g' which is the result of the natural action of $\Delta(2, k, \ell')$ on $2\ell'$ points, then depending on the nature of the fixed points the following cases emerge:

(1) When we have two fixed points of x in each orbit of $\langle xy \rangle$ of the second januarial, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell')$ on $2(\ell + \ell')$ points.
(2) When we have two fixed points of y in each orbit of $\langle xy \rangle$ of the second januarial, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 2)$ on $2(\ell + \ell' - 2)$ points.
(3) When we have one fixed point of x and one fixed point of y in each orbit of $\langle xy \rangle$ of the second januarial, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 1)$ on $2(\ell + \ell' - 1)$ points.

- If the two januarials are of general type $((h_1, g_1), (h_2, g_2))$ and $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of type $((h_1 + h'_1, g_1 + g'_1), (h_2 + h'_2, g_2 + g'_2))$.
- If the januarials are of simple type (h, g_1, g_2) and (h', g'_1, g'_2), then the resulting januarial will be of type $(h + h', g_1 + g'_1, g_2 + g'_2)$.
- If one januarial is of simple type (h, g_1, g_2) and the other of general type $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of general type $((h + h'_1, g_1 + g'_1), (h + h'_2, g_2 + g'_2))$.

Theorem 9. *Using an edge and two fixed points from both the januarials, the resulting graph will be a januarial if*

(1) *both the end points of each of the two edges from the two januarials lie in one of the orbits of $\langle xy \rangle$ and*
(2) *the four fixed points, two in each januarial lie in the other orbit of $\langle xy \rangle$ respectively with similar nature, that is, either*
   - *two fixed points of x in each januarial, or*
   - *two fixed points of y in each januarial or*
   - *one fixed point of x and one fixed point of y in each januarial.*

The proof of this theorem follows the same reasoning as in the proof of Theorem 8. Next we discuss the topological properties of the resulting januarials for different cases:

- If we connect two januarials, one of genus g which is the result of the natural action of $\Delta(2, k, \ell)$ on 2ℓ points and the other of genus g' which is the result of the natural action of $\Delta(2, k, \ell')$ on $2\ell'$ points, then depending on the nature of the fixed points the following cases emerge:

(1) When we have two fixed points of x in each januarial, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell')$ on $2(\ell + \ell')$ points.
(2) When we have two fixed points of y in each januarial, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 1)$ on $2(\ell + \ell' - 1)$ points.
(3) When we have one fixed point of x and one fixed point of y in each januarial, the resulting januarial will be of genus $g + g' + 1$ depicting the natural action of $\Delta(2, k, \ell + \ell' - 1)$ on $2(\ell + \ell' - 1)$ points.

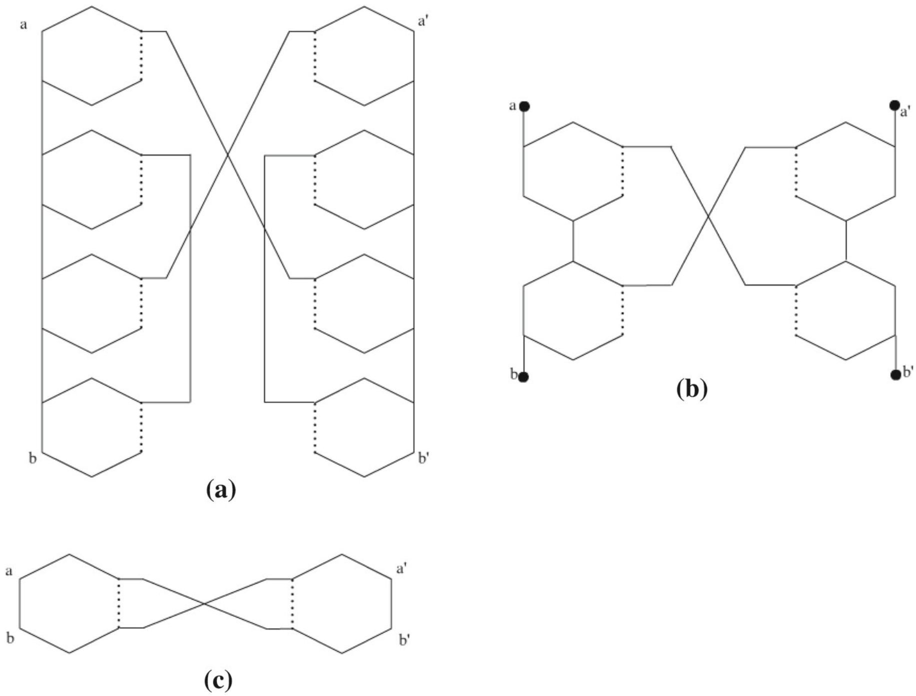


Figure 8. A simple k-janualial on **(a)** $8k$, **(b)** $4k$ and **(c)** $2k$ points.

- If the two januarials are of general type $((h_1, g_1), (h_2, g_2))$ and $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of type $((h_1 + h'_1, g_1 + g'_1), (h_2 + h'_2, g_2 + g'_2))$.
- If the januarials are of simple type (h, g_1, g_2) and (h', g'_1, g'_2), then the resulting januarial will be of type $(h + h', g_1 + g'_1, g_2 + g'_2)$.
- If one januarial is of simple type (h, g_1, g_2) and the other of general type $((h'_1, g'_1), (h'_2, g'_2))$, then the resulting januarial will be of general type $((h + h'_1, g_1 + g'_1), (h + h'_2, g_2 + g'_2))$.

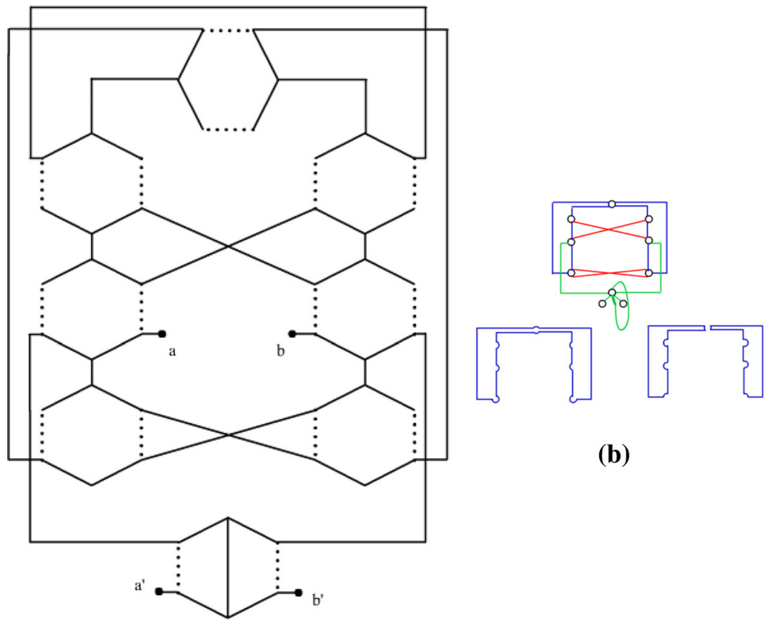
3. Maximum circuits for given *k*

The following result answers Graham Higman's question:

Theorem 10. *For any value of k, there exist simple januarials for all $h \in \mathbb{N}$.*

Proof. Consider the diagrams in Figure 8 for the action of (a) $\Delta(2, k, 4k)$ on $8k$ points, (b) $\Delta(2, k, 2k)$ on $4k$ points and (c) $\Delta(2, k, k)$ on $2k$ points.

They are k-januarials of simple type with $h = 2$, 1 and 1 respectively. The vertices a, b lie in one orbit of $\langle xy \rangle$ and a', b' lie in the other one. By Theorem 6, connecting the januarials using these vertices will result in a k-januarial of simple type and the number of circuits in the resulting k-januarial is the sum of number of circuits in the basic diagrams (by the arguments in Section 2.1).



(a) (b)

Figure 9. **(a)** The k-janualial of general type with $h_1 = 1, h_2 = 2$ and **(b)** the two partitions of the subgraph Υ.

Using the above technique, n-copies of (a), m-copies of (b), r-copies of (c) can be connected, with the resulting janualial to be of simple type having $2n + m + r$ number of circuits. Therefore, we can construct k-janualials of simple type for all $h \in \mathbb{N}$. □

For the janualials of general type, we have as follows.

COROLLARY 11

For any value of k, there exist janualials of general type $((h_1, g_1), (h_2, g_2))$ *for all* $h_{i's} \in \mathbb{N}$.

For instance, consider the example in Figure 9a. It is a k-janualial of general type with $h_1 = 1, h_2 = 2$ and the difference $h_1 - h_2 = 1$, which can be seen in the two partitions of the subgraph Υ in Figure 9b.

By connecting n-copies of the above diagram, we can construct a k-janualial of general type with $h_1 = n, h_2 = 2n$ and $h_1 - h_2 = n$.

4. Conclusion

Section 3 answers Graham Higman's question that for any value of k, we can have janualials with arbitrarily large values of h, h_1, h_2 and the difference $h_1 - h_2$. All the possible ways in which janualials can be connected are discussed in Section 2.

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Communicating Editor: B Sury