



Ovoidal fibrations in $PG(3, q)$, q even

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MS received 28 June 2019; revised 16 May 2020; accepted 18 May 2020

Abstract. Given a partition of a projective 3-space of odd cardinality by a set of ovoids, a line secant to one of the ovoids of the partition, and its polar relative to the symplectic polarity on the projective 3-space defined by this ovoid, are tangent to distinct ovoids of the partition (Theorem 2). The proof uses the fact that the radical of the linear code generated by the duals of the hyperbolic quadrics in a symplectic generalized quadrangle is of codimension one (Theorem 4).

Keywords. Dual grid; elliptic quadric; Frobenius reciprocity; Tits ovoid.

Mathematics Subject Classification. 05B25, 51E22, 94B05.

1. Introduction and statement of the result

An *ovoid* in $PG(3, q)$, $q > 2$, is a set of $q^2 + 1$ points, no three collinear. Each plane of $P(3, q)$ meets an ovoid in 1 or $q + 1$ points. All ovoids in $PG(3, q)$, q odd, are elliptic quadrics, as shown by Barlotti [4] and Panella [19] (see [18], Theorem 2.1, p. 178). From now on, let $q = 2^n$, $n > 1$. Elliptic quadrics which exist for all q and the Suzuki–Tits ovoids ([25], see [15], 16.4) which exist, if and only if, n is odd, are the only known ovoids in $PG(3, q)$; and these are the only ovoids if $n \leq 6$ [18]. Classification of ovoids in $PG(3, 2^n)$ and their ‘distribution’ in $PG(3, 2^n)$ are fundamental problems in Incidence Geometry. Our object in this note is to make a small contribution (Theorem 2) to the later problem.

Let \hbar be a non-degenerate symplectic bilinear form on $V(q) = \mathbb{F}_q^4$, the vector space of dimension four over \mathbb{F}_q , τ denotes the symplectic polarity on $PG(3, q)$ defined by \hbar and $G \simeq PSp(4, q)$ be the symplectic group defined by \hbar . Then the incidence system $W(q) = W_\tau$ whose point-set is the set P of points of $PG(3, q)$; the line-set $L = L_\tau$ is the set of all absolute lines of $PG(3, q)$ with respect to the symplectic polarity τ (equivalently, the lines on which \hbar restricts to the zero map); and symmetrized inclusion as the *incidence* is a regular generalized quadrangle of order q ([20], p. 37). The group G acts transitively on the sets P and L , preserving the incidence relation.

In view of the uniqueness of a non-degenerate symplectic bilinear form on $V(q)$ up to $GL(V(q))$ -equivalence (see [17], Corollary 9.2, p. 516), $W(q)$ is unique up to $PGL(4, q)$ -

equivalence. A *general linear complex* in $PG(3, q)$ is the set of all absolute lines with respect to a symplectic polarity of $PG(3, q)$.

An *ovoid* of $W(q)$, $q > 2$ and even, is a set of $q^2 + 1$ points, no two on a line of $W(q)$. We note that each ovoid of $PG(3, q)$ is an ovoid of some ‘copy’ of $W(q)$ in $PG(3, q)$ because, as Segre [22] observed, the set of all lines of $PG(3, q)$ tangent to an ovoid \mathcal{O} of $PG(3, q)$ is the complete set of absolute lines of a symplectic polarity of $PG(3, q)$; namely, the involutory bijection between points and planes of $PG(3, q)$ interchanging a point x of $PG(3, q)$ and the plane π_x which is the union of all tangent lines to \mathcal{O} through x . On the other hand, if \mathcal{O} is an ovoid of $W(q)$ defined by a symplectic polarity τ of $PG(3, q)$, then, for each line m of $PG(3, q)$ which is not tangent to \mathcal{O} , either m or its polar m^τ relative to τ shares two points with \mathcal{O} and the other is disjoint from it ([15], Corollary 1 of Theorem 16.1.8). So, \mathcal{O} is also an ovoid of $PG(3, q)$. Thus, the classification of ovoids in $PG(3, q)$ and in $W(q)$ are equivalent problems.

A *transversal* of a set of lines in $PG(3, q)$ is a line of $PG(3, q)$ meeting each member of the set; and a *regulus* in $PG(3, q)$ is the $(q + 1)$ -set consisting of all transversals of a given set of three mutually skew lines in $PG(3, q)$. Any two triples of lines of a given regulus define the same regulus, called the *opposite regulus* of the given regulus. Thus, any three mutually skew lines of $PG(3, q)$ are in a unique regulus. A *spread* of $PG(3, q)$ is a set of $q^2 + 1$ mutually skew lines. A spread in $PG(3, q)$ is said to be *regular* (see [15], p. 53) if it contains the regulus containing any three of its members. We note that, via Klein correspondence, an elliptic quadric of $PG(3, q)$ is dual to a regular spread in $PG(3, q)$, $PGL(4, q)$ acts transitively on the set of all regular spreads of $PG(3, q)$ and the isotropy group is isomorphic to $PSL(2, q^2)$. For all this, see [15], §15.4 and Theorem 15.3.18.

An *ovoidal fibration* $\mathcal{F} = \{\mathcal{O}_i\}_{i=0}^q$ of $PG(3, q)$ is a set of $q + 1$ ovoids of $PG(3, q)$ partitioning the point set of $PG(3, q)$. We recall that a Singer group in $PGL(4, q)$ is a cyclic group of order $(q^2 + 1)(q + 1)$ acting transitively on the point-set of $PG(3, q)$. Let A be a Singer group in $PGL(4, q)$, and let T, K be its subgroups of orders $q^2 + 1$ and $q + 1$, respectively. Then, the point T -orbits $\{E_i\}_{i=0}^q$ in $PG(3, q)$ is an ovoidal fibration by elliptic quadrics (see [9], Theorem 3, p. 1167; and [6], Theorem 3.7, p. 419; also see [1], Lemma 2, p. 141); the set \mathcal{S} of common tangent lines to the E_i 's is a regular spread in $PG(3, q)$; T acts regularly on the set \mathcal{S} and K acts regularly on each member of \mathcal{S} (see [13], p. 274). More generally, if \mathcal{O} is an ovoid of $PG(3, q)$ and \mathcal{S} is a regular spread of $PG(3, q)$ contained in the general linear complex $L = L(\mathcal{O})$ consisting of all tangent lines to \mathcal{O} (see [22]) and if K is the group of all collineations of $PG(3, q)$ fixing each member of \mathcal{S} , then the order of K is $q + 1$ and the set $\{g(\mathcal{O}) : g \in K\}$ of ovoids in $PG(3, q)$ is an ovoidal fibration of $PG(3, q)$ (see [5], Lemma 2.5, p. 160 and Theorem 3.1, p. 161). Further, \mathcal{O} is the only ovoid of the generalized quadrangle $(P, L) \simeq W(q)$, where P is the point-set of $PG(3, q)$ (see [5], Theorem 1.2, p. 158).

We do not know of any examples of ovoidal fibrations with projectively nonequivalent constituent ovoids (see [7] for a discussion of the case $q = 8$); and of any examples of ovoidal fibrations consisting of projectively equivalent ovoids, but with no transitive action of a subgroup of $PGL(4, q)$ on its constituent ovoids.

PROPOSITION 1

Let $\{\mathcal{O}_i\}_{i=0}^q$ be an ovoidal fibration of $PG(3, q)$ and L_i denote the general linear complex consisting of all tangent lines to \mathcal{O}_i , $0 \leq i \leq q$. Then, the following hold:

- (i) The set \mathcal{S} of common tangent lines to the ovoids \mathcal{O}_i is a regular spread in $PG(3, q)$. Further, $\{L_i\}_{i=0}^q$ are the only general linear complexes in $PG(3, q)$ containing \mathcal{S} ; and $L_i \cap L_j = \mathcal{S}$ for all $0 \leq i \neq j \leq q$.
- (ii) Each line ℓ of $PG(3, q)$ not in \mathcal{S} is tangent to a unique ovoid \mathcal{O}_i , secant to $q/2$ ovoids \mathcal{O}_j , $j \neq i$, and is disjoint from each of the remaining $q/2$ ovoids \mathcal{O}_k of the fibration.

Our object in this note is to prove the following.

Theorem 2. Let $\{\mathcal{O}_i\}_{i=0}^q$ be an ovoidal fibration of $PG(3, q)$, $W(q)$ be the generalized quadrangle with the set $L(\mathcal{O}_0)$ of tangent lines to \mathcal{O}_0 as its set of lines and τ be the symplectic polarity of $PG(3, q)$ defined by \mathcal{O}_0 . Let m be a line of $PG(3, q)$ not in $L(\mathcal{O}_0)$. Then, m and its polar m^τ in $W(q)$ are tangent to distinct ovoids \mathcal{O}_i , each distinct from \mathcal{O}_0 .

An intrinsic description of the ovoids \mathcal{O}_i the lines m and m^τ are tangent to may be interesting.

Our proof of Theorem 2 uses a property of the binary code we now define. Let P, G, τ and L_τ be as in paragraph 2 of Section 1. For any line m of $PG(3, q) \setminus L_\tau$, the union of m and its polar m^τ is called a *dual grid* of $W(q)$. This will be the dual of a hyperbolic quadric in $PG(3, q)$ all whose lines are absolute relative to τ . With the lines of $W(q)$ incident with it as ‘lines’, $m \cup m^\tau$ is a $(1, q)$ -subgeneralized quadrangle of $W(q)$. We denote by \mathbb{D} the set of all dual grids in $W(q)$.

Let \mathbb{F}_2^P and $\mathbb{F}_2^{\mathbb{D}}$ denote the \mathbb{F}_2G -permutation modules on P and \mathbb{D} , respectively, and η denote the \mathbb{F}_2G -module homomorphism taking $m \cup m^\tau \in \mathbb{D}$ to $\sum_{x \in m \cup m^\tau} x \in \mathbb{F}_2^P$. We identify the characteristic function of a subset of P , considered as an element of \mathbb{F}_2^P , with the subset itself. Let \mathcal{C} and \mathcal{D} denote the \mathbb{F}_2G -submodules of \mathbb{F}_2^P whose generators are, respectively, the lines of $W(q)$ and the dual grids of $W(q)$. Then, $\eta(\mathbb{F}_2^{\mathbb{D}}) = \mathcal{D}$. Since any line of $W(q)$ meets a dual grid in zero or two points, \mathcal{D} is contained in the dual code \mathcal{C}^\perp of \mathcal{C} . Further, it is generated by words of \mathcal{C}^\perp of minimum weight ([2], Theorem 1.4).

Lemma 3. The \mathbb{F}_2G -module \mathcal{D} is properly contained in \mathcal{C}^\perp .

We need the following.

Theorem 4. The \mathbb{F}_2G -radical U of \mathcal{D} is of codimension one in \mathcal{D} . Consequently, the sum of two dual grids of $W(q)$ is in U .

Lemma 5.

- (i) The group $PSp(4, q)$, q even, contains a unique conjugacy class of subgroups T of order $q^2 + 1$ and T is cyclic.
- (ii) Let T be a subgroup $PSp(4, q)$ of order $q^2 + 1$, $\{E_i\}_{i=0}^q$ be the point T -orbits in $PG(3, q)$ and \mathcal{S} denote the set of all common tangent lines to E_i 's. Let ℓ be a line of $PG(3, q)$. Then, $\sum_{t \in T} t(\ell) \in \mathbb{F}_2^P$ is P (i.e., the ‘all-one’ vector) or the unique ovoid E_i , the line ℓ is tangent to, according as $\ell \in \mathcal{S}$ or $\ell \notin \mathcal{S}$.

The proofs of all results stated above are given in Section 3.

2. Preliminaries

Let $V(q)$, \hbar , G be as in Section 1 and k be an algebraically closed extension field of \mathbb{F}_q . Let $V = V(q) \otimes_{\mathbb{F}_q} k$. The natural extension of the symplectic form \hbar to V defined above is also denoted by \hbar . Let $Sp(V) \simeq Sp(4, k)$ denote the algebraic group defined by \hbar and σ be the Frobenius map, i.e., algebraic group endomorphism of $GL(4, k)$ which takes each matrix $(a_{i,j})$ to $(a_{i,j}^q)$. Then, G is the subgroup of $Sp(4, k)$ fixed by the n -th power of σ .

We now describe an automorphism δ of $Sp(V)$ whose square is σ , following ([12], pp. 58–60). (The argument presented in loc. cit. constructs an outer automorphism for $G = Sp(V(q))$, however the arguments are valid for $Sp(V)$ also.) In view of the uniqueness of a non-degenerate symplectic bilinear form on $V(q)$ up to $GL(V(q))$ -equivalence (see [17], Corollary 9.2, p. 516), we may assume that \hbar is defined by

$$\hbar((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2.$$

Let (e_1, e_2, e_3, e_4) be the standard symplectic ordered basis of V and Q denote the non-degenerate quadratic form on the exterior square Λ^2V of V defined by

$$Q(\sum_{1 \leq i < j \leq 4} \lambda_{ij} e_i \wedge e_j) = \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23}$$

(whose zero set in $P(\Lambda^2V)$ is the well-known *Klein quadric*). Let β denote the polarization of Q and $\gamma = e_1 \wedge e_4 + e_2 \wedge e_3 \in \Lambda^2V$. Then, the restriction of Q to the hyperplane $U = \{x \in \Lambda^2V : \beta(x, \gamma) = 0\}$ of Λ^2V is a non-degenerate quadratic form; and the restriction of β to U is an alternating form with radical $k\gamma$. The alternating form $\bar{\beta}$ induced by β on $\bar{U} = \frac{U}{k\gamma}$ is non-degenerate. Let $w \mapsto \bar{w}$ be the natural homomorphism from U to \bar{U} . Then $(\overline{e_1 \wedge e_2}, \overline{e_1 \wedge e_3}, \overline{e_2 \wedge e_4}, \overline{e_3 \wedge e_4})$ is a symplectic ordered basis of $(\bar{U}, \bar{\beta})$ and the map $\bar{p} : \bar{U} \rightarrow V$ induced by the linear map $p : U \rightarrow V$ defined by

$$\begin{aligned} p(e_1 \wedge e_2) &= e_1, & p(e_1 \wedge e_3) &= e_2, & p(e_2 \wedge e_4) &= e_3, \\ p(e_3 \wedge e_4) &= e_4, & p(\gamma) &= 0 \end{aligned}$$

is an isometric isomorphism.

The map taking $g \in Sp(V)$ to $\bar{p}(\overline{\wedge^2(g)})\bar{p}^{-1} \in Sp(V)$ is an automorphism δ of $Sp(V)$ which, on restriction to G , gives an automorphism of G . Identifying $Sp(V)$ with $Sp(4, k)$ with respect to the ordered basis (e_1, e_2, e_3, e_4) of V , we get that $Sp(4, k)$ is generated by matrices

$$\begin{aligned} x_a(t) &= \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} (t \in k), & x_b(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (t \in k), \\ w_a &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & w_b &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(see [24], Lemma 26, p. 34; also see [10], p. 76). Further, a simple calculation shows that $\delta(x_a(t)) = x_b(t^2)$, $\delta(x_b(t)) = x_a(t)$, $\delta(w_a) = w_b$ and $\delta(w_b) = w_a$. Thus $\delta^2 = \sigma$.

Let $N = \{0, 1, \dots, 2n - 1\}$. Addition in N is always taken modulo $2n$. Let \mathcal{N} denote the set of all subsets I of N containing no consecutive elements (considering zero as the successor of $2n - 1$). For any non-negative integer i and $Sp(V)$ -module M , we denote

by M_i the $Sp(V)$ -module whose k -vector space structure is the same as that of M and an element g of $Sp(V)$ acts on M_i as $\delta^i(g)$ would act on M . For $I \subseteq N$, let V_I denote the kG -module $\otimes_{i \in I} V_i$ (with $V_\emptyset = k$). Then, by Steinberg's tensor product theorem ([23], §11), $\{V_I : I \subseteq N\}$ is a complete set of inequivalent simple kG -modules. For a kG -module M , we denote by $[M : V_I]$ the multiplicity of V_I in a composition series of M . We denote by $\text{rad}(M)$, the *radical* of M (that is, the smallest submodule of M with semisimple quotient). We refer to $M/\text{rad}(M)$ as the *head* of M .

We use the following results:

Lemma 6 ([8], Corollary 7.11, p. 148). *Let K be a field of characteristic $p > 0$, let E be a field extension of K and let X be a finite group. Then, for each simple KX -module M , $M^E = M \otimes_K E$ is a direct sum of simple EX -modules, no two of which are isomorphic.*

Lemma 7 ([21], Corollary 6). *Let $K, K' \in \mathcal{N}$ be distinct and f_{\hbar} be a non-degenerate quadratic form on $V(q)$ of index 2 which polarizes to \hbar . Let $O(f_{\hbar}) \subseteq G$ denote the orthogonal group of f_{\hbar} and $\Omega(f_{\hbar})$ its commutator subgroup. Then, V_K and $V_{K'}$ are semisimple $k\Omega(f_{\hbar})$ -modules with no irreducible factors in common.*

3. Proofs

Proof of Theorem 4. Let $g \in \mathbb{D}$ and $L = \text{Stab}_G(g)$. We view $\mathbb{F}_2^{\mathbb{D}}$ as the induced module of the trivial \mathbb{F}_2G -module \mathbb{F}_2 , and write $\text{Ind}_L^G(\mathbb{F}_2) = \mathbb{F}_2^{\mathbb{D}}$. Then

$$[\mathbb{F}_2^{\mathbb{D}}/\text{rad}(\mathbb{F}_2^{\mathbb{D}}) : \mathbb{F}_2] = 1$$

because, by Frobenius reciprocity ([11], p. 99),

$$\dim_{\mathbb{F}_2}(\text{Hom}_{\mathbb{F}_2G}(\text{Ind}_L^G(\mathbb{F}_2), \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(\text{Hom}_{\mathbb{F}_2L}(\mathbb{F}_2, \mathbb{F}_2)) = 1.$$

This shows that the trivial module \mathbb{F}_2 is a summand of the head (that is, the largest semisimple quotient) of the \mathbb{F}_2G -module $\mathbb{F}_2^{\mathbb{D}}$. Let \mathcal{H} denote the set of all hyperbolic quadrics of $W(q)$ and f_{\hbar} be a non-degenerate quadratic form on $V(q)$ of index 2 which polarizes to \hbar . Then, the variety defined by f_{\hbar} is a member of \mathcal{H} . Let I be a nonempty subset of N . Since simple kG -modules are self dual, by Lemma 7,

$$\text{Hom}_{k\Omega(f_{\hbar})}(k, V_I) \cong \text{Hom}_{k\Omega(f_{\hbar})}(V_{I_e}, V_{I_o}) = \{0\},$$

where I_e and I_o denote the set of even and odd elements of I , respectively. This proves that V_I has no nonzero fixed points for the action of $O(f_{\hbar})$.

Since G acts transitively on \mathcal{H} and since the automorphism δ maps the stabilizer of a hyperbolic quadric in $W(q)$ to the stabilizer of a dual grid in $W(q)$ and vice-versa, $\text{Ind}_L^G(k) \cong (\text{Ind}_{O(f_{\hbar})}^G(k))_1$. Thus

$$\dim_k(\text{Hom}_{kG}(\text{Ind}_L^G(k), V_I)) = \dim_k(\text{Hom}_{kG}(\text{Ind}_{O(f_{\hbar})}^G(k), V_{I-1})),$$

where $I - 1 = \{i - 1 : i \in I\}$. Again using Frobenius reciprocity, we have

$$\begin{aligned} \dim_k(\text{Hom}_{kG}(\text{Ind}_L^G(k), V_I)) &= \dim_k(\text{Hom}_{kG}(\text{Ind}_{O(f_h)}^G(k), V_{I-1})) \\ &= \dim_k(\text{Hom}_{kO(f_h)}(k, V_{I-1})) \\ &\leq \dim_k(\text{Hom}_{k\Omega(f_h)}(k, V_{I-1})) = 0. \end{aligned}$$

This proves that the head of the kG -module $k^{\mathbb{D}} \cong \text{Ind}_L^G(k)$ is k . Now Lemma 6 completes the proof of Theorem 4.

Proof of Proposition 1. The statement (i) follows from ([3], Theorem 2.2 and Corollary 3.4) and ([5], Lemma 2.4, p. 159). Since $|L_i| = (q^2 + 1)(q + 1)$, $L_i \cap L_j = \mathcal{S}$ for all $i, j, i \neq j$, and $|\mathcal{S}| = q^2 + 1$, each line ℓ of $PG(3, q)$ not in \mathcal{S} is in L_i for a unique i and $|\ell \cap \mathcal{O}_j| \in \{0, 2\}$ for all $j \neq i$. Thus (ii) follows.

Proof of Lemma 5. Since G has a cyclic subgroup of order $q^2 + 1$ (see [14], Theorem 5.6, p. 513) and a subgroup of order $q^2 + 1$ of G is a Hall π -subgroup, (i) follows from Wielandt's theorem (see [16], Satz 5.8, p. 285). For (ii), see ([1], Lemma 2, p. 141).

Proof of Theorem 2. Let \mathcal{S} be the set of common tangent lines to the \mathcal{O}_i 's. Then, \mathcal{S} is a regular spread in $PG(3, q)$, by Proposition 1(i). Let H be the subgroup of $PGL(4, q)$ fixing each member of \mathcal{S} . Since $PGL(4, q)$ acts transitively on the set of regular spreads of $PG(3, q)$, H is contained in a Singer subgroup A of $PGL(4, q)$ (see [13], p. 274). Consider the unique subgroup T of A of order $q^2 + 1$. The point-orbits of T are elliptic ovoids E_i . They form an ovoidal fibration of $PG(3, q)$ and \mathcal{S} is the set of common tangents to E_i 's (see [1], Lemma 2, p. 141). Let L_i be the set of all tangent lines to \mathcal{O}_i and $W(q) = W(L_0)$. Since \mathcal{S} is contained in exactly $q + 1$ general linear complexes (see Proposition 1(i)), we may assume that L_i is also the set of all tangent lines to E_i . Thus \mathcal{O}_0 and E_0 are ovoids of $W(L_0)$ and T is a subgroup of $G (= PSp(4, q)$ defined by $W(q)$). Now, assume that $m \cup m^\tau$ is a dual grid of $W(q)$ such that m and m^τ are tangents to \mathcal{O}_i for some $i \neq 0$. Then, both are tangents to E_i also. Since $m, m^\tau \notin \mathcal{S}$, by Lemma 5,

$$\sum_{t \in T} t(m + m^\tau) = E_i + E_i = 0.$$

On the other hand, since for each $t \in T$, $t(m + m^\tau) \in \mathcal{D}$ and the codimension of U in \mathcal{D} is one (Theorem 4), $\sum_{t \in T} t(m + m^\tau) = m + m^\tau \pmod{U}$. Since G is transitive on \mathbb{D} , U can not contain any dual grid of $W(q)$. This completes the proof of the theorem.

Proof of Lemma 3. Let T be a cyclic subgroup of order $q^2 + 1$ contained in the stabilizer in G of an elliptic ovoid \mathcal{O}_0 of $W(q)$. Let $\{\mathcal{O}_i\}_{i=0}^q$ be the point T -orbits in $PG(3, q)$ and \mathcal{S} denote the set of all common tangent lines to \mathcal{O}_i 's (see [1], Lemma 2, p. 141). Consider the \mathbb{F}_2 -linear map $\sigma : \mathbb{F}_2^P \rightarrow \mathbb{F}_2^P$ defined by $\sigma(w) = \sum_{t \in T} t(w)$. As we noted earlier, $\mathcal{D} \subseteq \mathcal{C}^\perp$. Assume that $\mathcal{C}^\perp = \mathcal{D}$. Then $\mathcal{C} = (\mathcal{C}^\perp)^\perp = \mathcal{D}^\perp \supseteq \mathcal{D}$, as any two dual grids of $W(q)$ intersect in zero or two points. Now, $\sigma(\mathcal{C}) = \{\emptyset, P, \mathcal{O}_0, P \setminus \mathcal{O}_0\}$ (see Lemma 5); however if $m \cup m^\perp$ is a dual grid of $W(q)$, then by Lemma 5 and Theorem 2, $\sigma(m + m^\perp) = \mathcal{O}_i + \mathcal{O}_j$ for distinct i and j ($i > 0, j > 0$), a contradiction.

References

- [1] Bagchi B and Sastry N S N, Even order inversive planes, generalized quadrangles and codes, *Geom. Dedicata* **22** (1987) 137–147
- [2] Bagchi B and Sastry N S N, One-step completely orthogonalizable codes from generalized quadrangles, *Inform. and Comput.* **77** (1988) 123–130
- [3] Bagchi B and Sastry N S N, Ovoidal packings of $P(3, q)$ for even q , *Discrete Math.* **313** (2013) 2213–2217; Corrigendum, *Discrete Math.* **341** (2018) 1194
- [4] Barlotti A, Un'estensione del teorema di Segre-Kustaanheimo, *Boll. Un. Mat. Ital.* **10** (1955) 498–506
- [5] Butler D K, On the intersection of ovoids sharing a polarity, *Geom. Dedicata* **135** (2008) 157–165
- [6] Cossidente A and Storme L, Caps on elliptic quadrics, *Finite Fields Appl.* **1** (1995) 412–420
- [7] Cossidente A and Vereecke S K J, Some geometry of the isomorphism $Sp(4, q) \cong O(5, q)$, q even, *J. Geom.* **70** (2001) 28–37
- [8] Curtis C W and Reiner I, Methods of representation theory, Vol I (1981) (New York: Wiley-Interscience Pub.)
- [9] Ebert G L, Partitioning projective geometries into caps, *Canad. J. Math.* **XXXVII** (6) (1985) 1163–1175
- [10] Enomoto H, The characters of the finite symplectic group $Sp(4, q)$, $q = 2^f$, *Osaka J. Math.* **9** (1972) 75–94
- [11] Feit W, The representation theory of finite groups (1982) (Amsterdam: North-Holland Publishing Company)
- [12] Flesner D E, The geometry of subgroups of $PSp_4(2^n)$, *Illinois J. Math.* **19** (1975) 48–70
- [13] Glynn D G, On the set of lines of $PG(3, q)$ corresponding to a maximal cap contained in the Klein quadric of $PG(5, q)$, *Geom. Dedicata* **26** (1988) 273–280
- [14] Hestenes M D, Singer Groups, *Can. J. Math.* **XXII**(3) (1970) 492–513
- [15] Hirschfeld J W P, Projective spaces of three dimensions (1985) (Oxford: Clarendon Press)
- [16] Huppert B, Endliche Gruppen I (1967) (Berlin: Springer-Verlag)
- [17] Lang S, Algebra, Second Edition (1984) (New York: Addison-Wessley)
- [18] O'Keefe C M, Ovoids in $PG(3, q)$, a survey, *Discrete Math.* **151** (1996) 175–188
- [19] Panella G, Caratterizzazione delle quadriche di uno spazio (tridimensionale) lineare sopra un corpo finito. *Boll. Un. Mat. Ital.* **10** (1955) 507-513
- [20] Payne S E and Thas J A, Finite generalized quadrangles, Research Notes in Mathematics, vol. 110 (1984) (Boston/London/Melbourne: Pitman Advanced Publishing Program)
- [21] Sastry N S N and Shukla R P, Structure of a code related to $Sp(4, q)$, q even, *Proc. Indian Acad. Sci.(Math. Sci.)* **117**(4) (2007) 457-470
- [22] Segre B, Introduction to Galois Geometries, *Mem. Acad. Naz. Lincei* **8** (1967) 133-236
- [23] Steinberg R, Representations of Algebraic groups, *Nagoya J. Math.* **22** (1963) 33-56
- [24] Steinberg R, Lectures on Chevalley Groups, Mimeographed Notes (1968) (New Haven, Conn.: Yale Univ. Math. Dept.)
- [25] Tits J, Ovoides et groupes de Suzuki, *Arch. Math.* **13** (1962) 187-198

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