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On the automorphism group of doubled Grassmann graphs

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Abstract. In this paper, we study a class of regular graphs, which is related to the Grassmann graph. This class of graphs is called the doubled Grassman graph. The Grassmann graph is the class of graphs, which is defined similar to the Johnson graph. This time, however, we are concerned with the subspaces of a vector space, rather than with the subsets of a set. The doubled Grassmann graph is constructed from the Grassmann graph. This graph, was discovered by Biggs and Gardiner. In this paper, we determine the full automorphism group of the doubled Grassmann graph. For determining the automorphism group of the doubled Grassmann graph, we will use the methods used by Mirafzal (*Proc. Indian Acad. Sci. (Math Sci.)* **129**(**3**) (2019), Art. 34, 8).

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1. Introduction and preliminaries

In this paper, a graph $\Gamma = (V, E)$ is considered as a finite undirected simple graph, where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set of Γ. The degree of each vertex $v \in V(\Gamma)$, is the number of neighbours of v in Γ and denoted by $\deg(v)$. A graph $\Gamma = (V, E)$ is called k-regular (or regular graph with degree k), if $\deg(v) = k$ for every $v \in V(\Gamma)$. Let u, v be two vertices in (connected) graph Γ. Then the length of the shortest path from u to v is called the distance between u, v and denoted by $d_\Gamma(u, v)$ (for clarity, we use $d(u, v)$ instead of $d_\Gamma(u, v)$). For all the terminologies and notations not defined here, we follow [2–4,8,10].

Let $q = p^n$, where p is a prime. Then we denote the finite field with q elements, by $\mathbb{F}_q$. Also in this paper, the n-dimensional vector space over $\mathbb{F}_q$ is denoted by $V_n(q)$. Let $k \leq n$, and let V_k be the set of all k-dimensional subspaces (also called k-subspace) of $V_n(q)$. Then the Grassmann graph $G(q, n, k)$ is the graph with vertex-set V_k and two vertices u, w are adjacent if and only if $\dim(u \cap w) = k - 1$. By Brouwer et al. [3], the Grassmann graph $G(q, n, k)$ has $\begin{bmatrix} n \\ k \end{bmatrix}_q$ vertices, where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is q-array Gaussian binomial coefficient, given by

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$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)\cdots(q - 1)}.$$

Also, the Grassmann graph $G(q, n, k)$ is the r-regular graph, where

$$r = q \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q \begin{bmatrix} k \\ k-1 \end{bmatrix}_q.$$

Let $[n] = \{1, \ldots, n\}$ be a set of size n. Let m be an integer such that $2m \leq n$. Then the Johnson graph $J(n, m)$ is the graph with vertex-set consisting of all m-subsets (subsets of size m) of $[n]$, and two vertices u, v are adjacent if and only if $|u \cap v| = m - 1$. The number of vertices of $J(n, m)$ is equal to $\binom{n}{m}$. Furthermore, the Johnson graph $J(n, m)$ is a regular graph with degree $m(n-m)$ (see [3] or [4]). For more information about Johnson graphs, see [1,6].

Let $n = 2k + 1$ be an integer, and $V_n(q)$ be a vector space of dimension n over $\mathbb{F}_q$. Let V_1 and V_2 be the set of all k-dimensional and the set of all $k+1$-dimensional subspaces of $V_n(q)$, respectively. The doubled Grassman graph $G_n(k, k+1)$ is the graph with vertex-set $V = V_1 \cup V_2$, and two vertices u, v are adjacent if and only if $u \leq v$ or $v \leq u$. By definition of the doubled Grassmann graph $G_n(k, k+1)$, it is easy to see that this graph is a regular bipartite graph with vertex-set partition $V = V_1 \cup V_2$ and degree $\left[\begin{smallmatrix} k+1 \\ 1 \end{smallmatrix}\right]_q$. For more information about the doubled Grassmann graphs, see [3,5].

For a positive integer $n > 1$, let $[n] = \{1, 2, \ldots, n\}$ and V be the set of all k-subsets and $(n-k)$-subsets of $[n]$. The bipartite Kneser graph $H(n, k)$ has V as its vertex set, and two vertices A, B are adjacent if and only if $A \subset B$ or $B \subset A$. The bipartite Kneser graph $H(n, k)$ is a regular bipartite graph with the degree $\binom{n-k}{k}$. For more information about bipartite Kneser graphs, see [8,9].

By these definitions, we see that the Johnson graphs and Grassmann graphs are defined in a manner similar to each other and so are their related graphs, the bipartite Kneser graphs and the doubled Grassmann graphs. In this paper, we determine the full automorphism group of the doubled Grassmann graphs. The automorphism group of bipartite Kneser graphs are determined in [8]. Furthermore, for automorphism group of the Johnson graph, see [6,7], and for automorphism group of Grassmann graphs, see [3].

Let $\Gamma = (V, E)$ be a graph. Then the mapping $f : V \longrightarrow V$ is called an automorphism of Γ if and only if f is a bijection and preserves the adjacency of vertices in Γ. The set of all automorphisms of Γ with the operation of composition of functions is a group, called the automorphism group of Γ and is denoted by $\mathrm{Aut}(\Gamma)$. Often, determining the automorphism group of graphs is difficult. There are various papers in the literature, and some of the recent works appear in [6–8].

The graph Γ is called vertex-transitive, if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. For $v \in V(\Gamma)$ and $G = \mathrm{Aut}(\Gamma)$, the stabilizer subgroup G_v is the subgroup of G consisting of all automorphisms that fix v. We say that Γ is symmetric (or arc-transitive) if, for all vertices u, v, x, y of Γ such that both $\{u, v\}$ and $\{x, y\}$ are edges in Γ, there is an automorphism α in $\mathrm{Aut}(\Gamma)$ such that $\alpha(u) = x$ and $\alpha(v) = y$. We say that Γ is distance-transitive if, for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, there is an automorphism $\beta \in \mathrm{Aut}(\Gamma)$ satisfying $\beta(u) = x$ and $\beta(v) = y$. In [3], it is proved that the doubled Grassmann graph $G_n(k, k+1)$ is a distance-transitive graph.

2. Main results

In this section, we determine the full automorphism group of the doubled Grassmann graphs. Let $\Gamma = (V, E)$ be a graph and $v \in V(\Gamma) = V$ be a vertex of Γ. If $\operatorname{diam}(\Gamma) = d$, then for each $i = 0, \dots, d$ we denote the set of all vertices at distance i from v, by $\Gamma_i(v)$. In other words,

$$\Gamma_i(v) = \{u \in V \mid d(u, v) = i\}.$$

Let V be a vector space of dimension n with a defined inner product $\langle\ ,\ \rangle$ on V, and W be a subspace of V. Then the orthogonal complement of W with respect to $\langle\ ,\ \rangle$ is a subspace of V which is denoted by $W^{\perp}$ and defined as follows:

$$W^{\perp} = \{v \in V \mid \langle v, w\rangle = 0 \quad \forall\, w \in W\}.$$

Let K be a field. A semilinear transformation of K^n is a set map $f : K^n \longrightarrow K^n$ for which there exists a field automorphism $\tau : K \longrightarrow K$ such that

$$f(c_1v_1 + c_2v_2) = \tau(c_1)f(v_1) + \tau(c_2)f(v_2)(c_1, c_2 \in K, v_1, v_2 \in K^n).$$

A semilinear transformation $f : K^n \longrightarrow K^n$ is a semilinear automorphism if it is bijective. Let $\Gamma\mathrm{L}_n(K)$ be the group of semilinear automorphisms of K^n. This contains a normal subgroup isomorphic to K^*, namely the set of all scalar matrices. The quotient of $\Gamma\mathrm{L}_n(K)$ by this normal subgroup is denoted by $\mathrm{P}\Gamma\mathrm{L}_n(K)$. If $K = \mathbb{F}_q$, then we sometimes write the group $\Gamma\mathrm{L}_n(q)$ and $\mathrm{P}\Gamma\mathrm{L}_n(q)$ instead of $\Gamma\mathrm{L}_n(K)$ and $\mathrm{P}\Gamma\mathrm{L}_n(K)$, respectively.

Let $\Gamma = G(q, n, k)$ be the Grassmann graph. Then by Theorem 9.3.1 in [3], we have

$$\operatorname{Aut}(\Gamma) = \begin{cases} \mathrm{P}\Gamma\mathrm{L}_n(q), & n \neq 2k, \\ \operatorname{Aut}(\mathrm{P}\Gamma\mathrm{L}_n(q)) = \mathrm{P}\Gamma\mathrm{L}_n(q)\cdot 2, & n = 2k. \end{cases}$$

Lemma 1. *Let v be a vertex of $\Gamma = G_n(k, k+1)$, where $n = 2k+1$. Then the mapping $\alpha : v \longmapsto v^{\perp}$ on $V(\Gamma)$ is an automorphism of Γ.*

Proof. By elementary linear algebra theorem,

$$\dim(v + v^{\perp}) = \dim(v) + \dim(v^{\perp}) - \dim(v \cap v^{\perp}).$$

Then $\dim(v^{\perp}) = 2k + 1 - k = k + 1$, and so $v^{\perp} \in V(\Gamma)$ for all $v \in V(\Gamma)$.

On the other hand if v, w are two vector spaces such that $v \leq w$, then it is easy to see that $\alpha(w) \leq \alpha(v)$. In other words, the mapping α, which is defined in this lemma preserves the adjacency of vertices in Γ and so $\alpha \in \operatorname{Aut}(\Gamma)$. □

Lemma 2. *Let k be an integer and $n = 2k+1$. Let $\Gamma = (V, E) = G_n(k, k+1)$ be the doubled Grassmann graph with vertex-set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$, where $V_1 = \{v \mid v \leq V_n(q), \dim(v) = k\}$ and $V_2 = \{v \mid v \leq V_n(q), \dim(v) = k+1\}$. Let f be an automorphism of Γ such that for a fixed vertex $v \in V_1$, we have $f(v) \in V_1$. Then $f(V_1) = V_1$ and $f(V_2) = V_2$.*

Proof. Let w be an arbitrary vertex of Γ such that $w \in V_1$, We show that $f(w) \in V_1$. It is easy to show that $d_\Gamma(v, w)$ is an even number. Let $0 \leq d_\Gamma(v, w) = 2l \leq D = \text{diam}(\Gamma)$. By induction on distance of two vertices l, we show that $f(w) \in V_1$. If $l = 0$, then the assertion is obvious. Let the assertion be true for every $s \leq l - 1$. Suppose that $d_\Gamma(v, w) = 2l$. Then there is a vertex $u \in V(\Gamma)$ such that $d_\Gamma(u, v) = 2(l - 1)$ and $d_\Gamma(u, w) = 2$. Then by induction, we know that $f(u) \in V_1$ and since $d_\Gamma(u, w) = 2$ therefore, $f(w) \in V_1$. Now it follows that $f(V_1) = V_1$ and consequently, $f(V_2) = V_2$. □

Now from Lemmas 1 and 2, we have the following corollary.

COROLLARY 3

Let $\Gamma = G_n(k, k + 1) = (V, E)$ be the doubled Grassmann graph with the vertex set partition $V = V_1 \cup V_2, V_1 \cap V_2 = \phi$. If f is an automorphism of the graph Γ, then $f(V_1) = V_1$ and $f(V_2) = V_2$, or $f(V_1) = V_2$ and $f(V_2) = V_1$.

Lemma 4. *Let $\Gamma = G_n(k, k+1) = (V, E)$ be the doubled Grassmann graph with partition $V = V_1 \cup V_2, V_1 \cap V_2 = \phi$. If f is an automorphism of the graph Γ such that $f(v) = v$ for every $v \in V_1$, then f is the identity automorphism of Γ.*

Proof. Let $f(v) = v$ for all $v \in V_1$. Then $f(V_1) = V_1$ and by Lemma 2, $f(V_2) = V_2$. Since f is an automorphism of the graph Γ, then for every vertex $w \in V_2$ and neighbour-set $N(w) = \{v |\ v \in V_1,\ v \leftrightarrow w\}$, we have

$$f(N(w)) = \{f(v)|\ v \in V_1,\ v \leftrightarrow w\} = N(f(w)).$$

On the other hand, since $f(v) = v$ for every $v \in V_1$, then $f(N(w)) = N(w)$ and therefore, $N(f(w)) = N(w)$. In other words, w and $f(w)$ are $(k + 1)$-subsets of $V_n(q)$ such that their family of k-subspaces are the same. Now, it is an easy task to show that $f(w) = w$. Therefore, for every vertex $u \in V_\Gamma$, $f(u) = u$. Thus f is the identity automorphism of Γ. □

Theorem 5. *Let k be an integer and $n = 2k + 1$. Let $\Gamma = (V, E) = G_n(k, k + 1)$ be the doubled Grassmann graph with vertex-set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$, where $V_1 = \{v \mid v \leq V_n(q), \dim(v) = k\}$ and $V_2 = \{v \mid v \leq V_n(q), \dim(v) = k + 1\}$. Then* $\text{Aut}(\Gamma) \cong \langle \text{P}\Gamma\text{L}_n(q), \alpha\rangle$, *where α is the automorphism of Γ defined in Lemma 1.*

Proof. Let α be the automorphism of Γ which is defined in Lemma 1. We have seen already that $\text{P}\Gamma\text{L}_n(q)$ and $\langle\alpha\rangle \cong \mathbb{Z}_2$ are subgroups of $G = \text{Aut}(\Gamma)$. It is obvious that $\alpha \notin \text{P}\Gamma\text{L}_n(q)$. Therefore, the group

$$\langle \text{P}\Gamma\text{L}_n(q), \alpha\rangle = S$$

is a subgroup of $G = \text{Aut}(\Gamma)$. We now want to show that $G = S$. Let $f \in G = \text{Aut}(\Gamma)$. We must show that $f \in S$. We have the following two cases:

(1) There is a vertex $v \in V_1$ such that $f(v) \in V_1$. Then by Lemma 2, $f(V_1) = V_1$ and $f(V_2) = V_2$.

(2) There is a vertex $v \in V_1$ such that $f(v) \in V_2$. Then by Lemma 2, $f(V_1) = V_2$ and $f(V_2) = V_1$.

(1) Let $f(V_1) = V_1$. Then, for every vertex $v \in V_1$ we have $f(v) \in V_1$, and therefore, the mapping $g = f|_{V_1} : V_1 \longrightarrow V_1$, is a permutation of V_1, where $f|_{V_1}$ is the restriction of f to V_1. Let $\Gamma_2 = G(q, n, k)$ be the Grassmann graph with the vertex set V_1. Then, the vertices $v, w \in V_1$ are adjacent in Γ_2 if and only if $\dim(v \cap w) = k - 1$. We assert that the permutation $g = f|_{V_1}$ is an automorphism of the graph Γ_2. For proving our assertion, it is sufficient to show that if $v, w \in V_1$ such that $\dim(v \cap w) = k - 1$, then we have $\dim(g(v) \cap g(w)) = k - 1$. As v, w are k-subspaces of $V_n(q)$, then if u is a common neighbour of v, w in the doubled Grassmann graph $\Gamma = G_n(k, k+1)$, then the subspace u contains the subspaces v and w. Now, we can see that the number of vertices u, such that u is adjacent in Γ to both of the vertices v and w is $\frac{q^n - q^{k-1}}{q^k - q^{k-1}} = \frac{q^{n-k+1}-1}{q-1}$. Note that if t is a positive integer such that $k + 1 + t = n - k$, then $t = n - 2k - 1$. Now, if we adjoin to the $(k+1)$-subspace $v + w$ of $V_n(q)$, $q^n - q^{k+1}$ elements of the complement of $v + w$ in $V_n(q)$, then we obtain a subspace u of $V_n(q)$ such that $v + w = u$ and u is a $(k+1)$-subspace of $V_n(q)$. Now, since v and w, have $\frac{q^{n-k+1}-1}{q-1}$ common neighbours in the graph Γ, then the vertices $g(v)$ and $g(w)$ must have $\frac{q^{n-k+1}-1}{q-1}$ neighbours in Γ, and therefore $\dim(g(v) \cap g(w)) = k - 1$. Hence, for constructing a $(k+1)$-subset $u = g(v) + g(w)$, we must adjoin $t = q^n - q^{k+1}$ elements of the complement of $g(v) + g(w)$ in $V_n(q)$, to the basis of $g(v) + g(w)$. Therefore, the number of common neighbours of vertices $g(v)$ and $g(w)$ in the graph Γ is $\frac{q^{n-k+1}-1}{q-1}$. Our argument shows that $g = f|_{V_1}$ is an automorphism of the Grassmann graph $\Gamma_2 = G(q, n, k)$ and therefore, $g \in \mathrm{P\Gamma L}_n(q)$. In other words, we have proven that if f is an automorphism of $\Gamma = G_n(k, k+1)$ such that $f(V_1) = V_1$, then $f \in S$.

(2) We now assume that $f(V_1) \neq V_1$. Then, $f(V_1) = V_2$. Since the mapping α is an automorphism of the graph Γ, then $f\alpha$ is an automorphism of Γ such that $f\alpha(V_1) = f(\alpha(V_1)) = f(V_2) = V_1$. Therefore, by what is proved in (1), we have $f\alpha = \theta$, for some $\theta \in \mathrm{P\Gamma L}_n(q)$. Now since α is of order 2, then $f = \theta\alpha \in S$. □

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