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Proof of a supercongruence conjectured by Sun through a q-microscope

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Abstract. In 2011, Sun (*Sci. China Math.* **54** (2011) 2509–2535) made the following conjecture: for any odd prime p and odd integer m,

$$\frac{1}{m^2\binom{m-1}{(m-1)/2}}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}}{8^k}-\left(\frac{2}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}}{8^k}\right)\equiv 0 \pmod{p^2}.$$

By applying the creative microscoping method introduced by Guo and Zudilin (*Adv. Math.* **346** (2019) 329–35), we confirm the above conjecture of Sun.

Keywords. Cyclotomic polynomial; q-binomial coefficient; q-congruence; super congruence; creative microscoping.

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1. Introduction

During the past decade, congruences and supercongruences have been studied by quite a few authors. In 2011, Sun [14, Equation (1.6)] proved that, for any odd prime p,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p}\right)+\left(\frac{-2}{p}\right)\frac{p^2}{4}E_{p-3} \pmod{p^3},$$

where $\left(\frac{m}{n}\right)$ denotes the Jacobi symbol and E_n is the nth Euler number. Later, Sun [15, Equation (1.7)] further proved that

$$\sum_{k=0}^{(p^r-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p^r}\right) \pmod{p^2}. \tag{1.1}$$

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Recently, Sun [16, Conjecture 4(ii)] proposed the following conjecture: for any odd prime p and odd integer m,

$$\frac{1}{m^2\binom{m-1}{(m-1)/2}}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}}{8^k}-\left(\frac{2}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}}{8^k}\right)\equiv 0 \pmod{p^2}, \tag{1.2}$$

which is clearly a generalization of (1.1).

In the past few years, q-analogues of congruences and supercongruences have caught the interest of a lot of people (see [3–13,17]). In particular, Guo and Liu [8] gave the following q-analogue of (1.1): for odd $n>1$,

$$\sum_{k=0}^{(n-1)/2} q^{k^2}\frac{(q;q^2)_k}{(q^4;q^4)_k}\equiv(-q)^{(1-n^2)/8} \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here and in what follows, $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$, $n=0,1,\ldots,$ or $n=\infty$, is the *q-shifted factorial*, and $\Phi_n(q)$ is the *n-th cyclotomic polynomial* in q given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an nth primitive root of unity. Moreover, Gu and Guo [4] gave some different q-analogues of (1.1), such as

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k q^{2k}}{(q^2;q^2)_k(-q;q^2)_k}\equiv\left(\frac{2}{n}\right)q^{2\lfloor (n+1)/4\rfloor^2} \pmod{\Phi_n(q)^2}, \tag{1.4}$$

where $\lfloor x\rfloor$ denotes the largest integer not exceeding x. In order to prove Sun's conjecture (1.2), we need the following new q-analogue of (1.1).

Theorem 1.1. *Let $n>1$ be an odd integer. Then*

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k(-1;q^4)_k}{(-q;q^2)_k(q^4;q^4)_k}q^{2k}\equiv\left(\frac{2}{n}\right) \pmod{\Phi_n(q)^2}. \tag{1.5}$$

Recall that the *q-integer* is defined by $[n]_q=1+q+\cdots+q^{n-1}$ and the *q-binomial coefficient* ${m \brack n}_q$ is defined as

$${m \brack n}_q=\begin{cases}\dfrac{(q;q)_m}{(q;q)_n(q;q)_{m-n}}, & \text{if } 0\leqslant n\leqslant m,\\ 0, & \text{otherwise}.\end{cases}$$

On the basis of (1.5), we are able to give the following q-analogue of (1.2).

Theorem 1.2. *Let m and n be positive odd integers with $n > 1$. Then*

$$\frac{1}{[m]_{q^n}^2 \left[\begin{smallmatrix} m-1 \\ (m-1)/2 \end{smallmatrix}\right]_{q^n}} \Bigg(\sum_{k=0}^{(mn-1)/2} \frac{(q;q^2)_k(-1;q^4)_k}{(-q;q^2)_k(q^4;q^4)_k} q^{2k} - \left(\frac{2}{n}\right) \sum_{k=0}^{(m-1)/2} \frac{(q^n;q^{2n})_k(-1;q^{4n})_k}{(-q^n;q^{2n})_k(q^{4n};q^{4n})_k} q^{2nk} \Bigg) \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.6}$$

Moreover, the denominator of (the reduced form of) the left-hand side of (1.6) *is relatively prime to $\Phi_{n^j}(q)$ for any index $j \geqslant 2$.*

It is well known that $\Phi_n(1) = p$ if n is a prime power p^r ($r \geqslant 1$) and $\Phi_n(1) = 1$ otherwise. Moreover, the denominator of (1.6) is no doubt a product of cyclotomic polynomials. This means that (1.2) immediately follows from (1.6) by letting $n = p$ and taking the limits as $q \to 1$.

We shall prove Theorem 1.1 in the next section. The proof of Theorem 1.2 will be given in Section 3 by using the method of "creative microscoping" recently introduced by Guo and Zudilin [10]. More precisely, we shall first give a generalization of Theorem 1.2 with an extra parameter a, and Theorem 1.2 then follows from this generalization by taking $a \to 1$. Finally, in Section 4, we give some hints on solving another similar conjecture of Sun.

2. Proof of Theorem 1.1

It is easy to check that

$$(1-q^{n-2j+1})(1-q^{n+2j-1}) + (1-q^{2j-1})^2 q^{n-2j+1} = (1-q^n)^2,$$

and so

$$(1-q^{n-2j+1})(1-q^{n+2j-1}) \equiv -(1-q^{2j-1})^2 q^{n-2j+1} \pmod{\Phi_n(q)^2}$$

since $1-q^n \equiv 0 \pmod{\Phi_n(q)}$. Thus, we have

$$\begin{aligned}(q^{1-n};q^2)_k(q^{1+n};q^2)_k &= (-1)^k q^{k^2-nk} \prod_{j=1}^{k} (1-q^{n-2j+1})(1-q^{n+2j-1}) \\ &\equiv q^{k^2-nk} \prod_{j=1}^{k} (1-q^{2j-1})^2 q^{n-2j+1} \\ &= (q;q^2)_k^2 \pmod{\Phi_n(q)^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k(-1;q^4)_k}{(-q;q^2)_k(q^4;q^4)_k} q^{2k} &\equiv \sum_{k=0}^{(n-1)/2} \frac{(q^{1-n};q^2)_k(q^{1+n};q^2)_k(-1;q^4)_k}{(q;q^2)_k(-q;q^2)_k(q^4;q^4)_k} q^{2k} \\ &= \left(\frac{2}{n}\right) \pmod{\Phi_n(q)^2}.\end{aligned} \tag{2.1}$$

Here the last step in (2.1) follows from a terminating q-analogue of Whipple's ${}_3F_2$ sum [2, Appendix (II.19)]:

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{n+1};q)_k(c;q)_k(-c;q)_k}{(e;q)_k(c^2q/e;q)_k(q;q)_k(-q;q)_k}q^k$$
$$=\frac{(eq^{-n};q^2)_\infty(eq^{n+1};q^2)_\infty(c^2q^{1-n}/e;q^2)_\infty(c^2q^{n+2}/e;q^2)_\infty}{(e;q)_\infty(c^2q/e;q)_\infty}q^{n(n+1)/2}$$

with $n\mapsto(n-1)/2$, $q\mapsto q^2$, $c^2=-1$ and $e=q$.

3. Proof of Theorem 1.2

We first establish the following parametric generalization of Theorem 1.2.

Theorem 3.1. *Let m and n be positive odd integers with $n>1$. Then, modulo*

$$\prod_{j=0}^{(m-1)/2}(1-aq^{(2j+1)n})(a-q^{(2j+1)n}), \tag{3.1}$$

we have

$$\sum_{k=0}^{(mn-1)/2}\frac{(aq;q^2)_k(q/a;q^2)_k(-1;q^4)_k}{(q;q^2)_k(-q;q^2)_k(q^4;q^4)_k}q^{2k}$$
$$\equiv\left(\frac{2}{n}\right)\sum_{k=0}^{(m-1)/2}\frac{(aq^n;q^{2n})_k(q^n/a;q^{2n})_k(-1;q^{4n})_k}{(q^n;q^{2n})_k(-q^n;q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk}. \tag{3.2}$$

Proof. It suffices to prove that both sides of (3.2) are identical for $a=q^{-(2j+1)n}$ and $a=q^{(2j+1)n}$ with $j=0,1,\ldots,(m-1)/2$, i.e.,

$$\sum_{k=0}^{(mn-1)/2}\frac{(q^{1-(2j+1)n};q^2)_k(q^{1+(2j+1)n};q^2)_k(-1;q^4)_k}{(q;q^2)_k(-q;q^2)_k(q^4;q^4)_k}q^{2k}$$
$$=\left(\frac{2}{n}\right)\sum_{k=0}^{(m-1)/2}\frac{(q^{-2jn};q^{2n})_k(q^{(2j+2)n};q^{2n})_k(-1;q^{4n})_k}{(q^n;q^{2n})_k(-q^n;q^{2n})_k(q^{4n};q^{4n})_k}q^{2nk}. \tag{3.3}$$

Clearly, $(mn-1)/2\geqslant((2j+1)n-1)/2$ for $0\leqslant j\leqslant(m-1)/2$, and $(q^{1-(2j+1)n};q^2)_k=0$ for $k>((2j+1)n-1)/2$. By the identity in (2.1), the left-hand side of (3.3) is equal to $\left(\frac{2}{(2j+1)n}\right)$. Likewise, the right-hand side of (3.3) is equal to

$$\left(\frac{2}{n}\right)\left(\frac{2}{2j+1}\right)=\left(\frac{2}{(2j+1)n}\right),$$

where $\left(\frac{2}{1}\right)$ is understood to be 1. This establishes the identity (3.3), and so the q-congruence (3.2) holds. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. It is easy to see that

$$q^N - 1 = \prod_{d|N} \Phi_d(q),$$

and there are $\lfloor m/n^{j-1} \rfloor - \lfloor (m-1)/(2n^{j-1}) \rfloor$ multiples of n^{j-1} in the arithmetic progression $1, 3, \ldots, m$ for any positive integer j. Thus, the limit of (3.1) as $a \to 1$ has the factor

$$\prod_{j=1}^{\infty} \Phi_{n^j}(q)^{2\lfloor m/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor}.$$

On the other hand, the denominator of the left-hand side of (3.2) is divisible by that of the right-hand side of (3.2). The former is equal to $(q^2; q^2)_{mn-1}$ and its factor related to $\Phi_n(q), \Phi_{n^2}(q), \ldots$ is just

$$\prod_{j=1}^{\infty} \Phi_{n^j}(q)^{\lfloor (mn-1)/n^j \rfloor}.$$

Moreover, writing $[m]_q = (q;q)_m/((1-q)(q;q)_{m-1})$, the q-binomial coefficient $\left[{m-1 \atop (m-1)/2}\right]$ as a product of cyclotomic polynomials (see, for example, [1]), and then using the fact $\Phi_{n^j}(q^n) = \Phi_{n^{j+1}}(q)$, we know that the polynomial $[m]_{q^n}^2 \left[{m-1 \atop (m-1)/2}\right]_{q^n}$ only has the following factor:

$$\prod_{j=2}^{\infty} \Phi_{n^j}(q)^{2\lfloor m/n^{j-1} \rfloor - \lfloor (m-1)/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor}$$

related to $\Phi_n(q), \Phi_{n^2}(q), \ldots$.

It is clear that

$$2\lfloor m/n^{j-1} \rfloor - 2\lfloor (m-1)/(2n^{j-1}) \rfloor - \lfloor (mn-1)/n^j \rfloor = 2, \quad \text{for } j = 1$$

and

$$\lfloor (mn-1)/n^j \rfloor = \lfloor (m-1)/n^{j-1} \rfloor, \quad \text{for } j \geqslant 1.$$

Therefore, letting $a \to 1$ in (3.2), we see that the q-congruence (1.6) holds, and the denominator on the left-hand side of (1.6) is relatively prime to $\Phi_{n^j}(q)$ for $j \geqslant 2$, as desired. □

4. Concluding remarks

Sun [16, Conjecture 4(ii)] also made the following conjecture: for any odd prime p and odd integer m,

$$\frac{1}{m^2\binom{m-1}{(m-1)/2}}\left(\sum_{k=0}^{(mp-1)/2}\frac{\binom{2k}{k}}{16^k}-\left(\frac{3}{p}\right)\sum_{k=0}^{(m-1)/2}\frac{\binom{2k}{k}}{16^k}\right)\equiv 0 \pmod{p^2}, \tag{4.1}$$

of which the $m = 1$ case was already proved by Sun [15] himself. Gu and Guo [4] gave the following q-analogue of (4.1) for $m = 1$: for odd $n > 1$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k q^{2k}}{(q^4;q^4)_k(-q;q^2)_k}\equiv\left(\frac{3}{n}\right)q^{(n^2-1)/12} \pmod{\Phi_n(q)^2}. \tag{4.2}$$

However, we cannot utilize (4.2) to give a q-analogue of (4.1), similar to Theorem 1.2, because $(n^2-1)/12$ is not a linear function of n. Anyway, we believe that such a q-analogue of (4.1) should exist, which is left to the interested reader.

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