Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:62
https://doi.org/10.1007/s12044-020-0576-7

Check for updates

# Approximate controllability for finite delay nonlocal neutral integro-differential equations using resolvent operator theory

KAMAL JEET

Indian Institute of Technology Roorkee, Roorkee 247 667, India
E-mail: kamaljeetp2@gmail.com

MS received 29 November 2018; revised 9 February 2020; accepted 11 March 2020

**Abstract.** In this paper, our purpose is to study the approximate controllability of abstract nonlocal neutral integro-differential equations with finite delay in a Hilbert space using the resolvent operator theory. We derive a variation of parameters formula for representing a solution of the given neutral integro-differential system in the form of resolvent operators and then define a mild solution of the system. We also study the existence of a mild solution of the system with the help of resolvent operator theory. The fractional power theory, $\alpha$-norm, resolvent operator theory, semigroup theory and Krasnoselskii's fixed point theorem are used to prove the approximate controllability of the system. Finally, we illustrate our results with the help of an example.

**Keywords.** Approximate controllability; semigroup; resolvent operator; $\alpha$-norm; finite delay; Krasnoselskii's fixed point theorem.

**Mathematics Subject Classification.** 93B05, 34K30, 34G20.

## 1. Introduction

In the few past years, the neutral integro-differential equations have been extensively studied by various authors (see [1,5,13,15,16,25] and references therein). The integro-differential equations of neutral type are used to describe many physical phenomena arising in fluid dynamics, electronics, chemical kinetics and so on. Grimmer *et al.* [9] used the resolvent operator theory to establish the existence results for the integro-differential equations

$$\begin{cases} y'(t) = -Ay(t) + \int_0^t \gamma(t-s)y(s)\mathrm{d}s + f(t), \quad t \in [0, \infty), \\ y(0) = y_0 \in Y, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^+ \to Y$, $Y$ a Banach space. The resolvent operator solves the differential equations (1.1) in weak as well as strict sense. In [10,11], Grimmer represented the solutions of integro-differential equations in the form of resolvent operators. Santos *et al.* [13,25] investigated the existence and regularity of a mild solution for an abstract integro-differential equation of neutral type using resolvent operators. We refer the readers to the papers



Published online: 07 October 2020

[1,5,14,16] and the books [12,23] for the study of abstract integro-differential equations via analytic resolvent operators.

In a mathematical control problem, we obtain an appropriate control function such that we can drive the state of a dynamical system to the desired final state. The controllability problem of various nonlinear systems is an important and interesting subject for many researchers. That is why many authors have analyzed the controllability of various nonlinear systems during the last decade, see for instance [8,17–21,24,26–29,31] and references therein. The approximate controllability enables us to steer the system to an arbitrarily small neighborhood of a final state, whereas the exact controllability steers to an exact final state. However, if the semigroup related to the system is compact then we can not establish the exact controllability in an infinite-dimensional space. In this case, the approximate controllability is more appropriate to the dynamical systems. In [2,6,7,19,20,24,29], the authors discussed the approximate controllability of nonlinear systems with the fact that associated linear system is approximately controllable. Kamaljeet and Bahuguna [15] discussed the approximate controllability for the neutral fractional integro-differential system with time-varying delays using $\alpha$-norm and Krasnoselskii's fixed point theorem. Mokkedem and Fu [21] investigated the approximate controllability of the neutral integro-differential system of finite delay using resolvent operator theory. Yan and Lu [30] applied the properties of analytic $\alpha$-resolvent operators to establish the approximate controllability of stochastic fractional integro-differential inclusions of neutral delay.

In this paper, we consider a nonlocal abstract neutral integro-differential equation of the following type:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)+\int_0^t K(t-s)y(s)\mathrm{d}s\right) = -Ay(t)+\int_0^t \gamma(t-s)y(s)\mathrm{d}s \\ \qquad\qquad +Wu(t)+f(t,y_t), \quad t\in J=[0,b], \\ \qquad\qquad y(r)=\zeta(r)+\eta(y)(r), \quad r\in[-\tau,0], \end{cases} \tag{1.2}$$

where $-A : D(A) \subseteq Y \to Y$ generates an analytic and compact semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators in a Hilbert space $Y$; $\{\gamma(t)\}_{t\geq 0}$ is a family of closed linear operators on $D(A)$; $\{K(t)\}_{t\geq 0}$ is a family of bounded linear operators on $Y$; $u(\cdot)$ is the control function in a Hilbert space $L^2(J,U)$, $U$ is another Hilbert space; $W\colon U \to Y$ is a bounded linear operator; functions $f : J\times\mathcal{D}\to Y_\alpha$, $\eta : \mathcal{C}\to\mathcal{D}$ are continuous and nonlinear, here $0<\alpha\leq 1$, $\mathcal{D}=C([-\tau,0],Y_\alpha)$, $\mathcal{C}=C([-\tau,b],Y_\alpha)$; and $\zeta\in\mathcal{D}$. We shall specify the term $Y_\alpha$ in the next section. The time history function $y_t\in\mathcal{D}$ for $y\in\mathcal{C}$ and $t\in J$ is defined as

$$y_t(\sigma)=y(t+\sigma), \quad \sigma\in[-\tau,0].$$

The work of this paper is motivated by [20,21,25]. In the present paper, we apply the resolvent operator theory to study the approximate controllability of the abstract neutral integro-differential system (1.2) with a nonlocal initial condition in a Hilbert space. We first represent the solution of the system (1.2) in the form of resolvent operators and then define a mild solution of the system. We also study the existence of a mild solution of the system (1.2) through the resolvent operator theory. The fractional power theory, $\alpha$-norm, resolvent operator theory, semigroup theory and Krasnoselskii's fixed point theorem are also used to prove our main results. The approximate controllability of a nonlocal neutral integro-differential system (1.2) has not yet been investigated using resolvent operator theory to the best of our knowledge.

This paper is organized as follows: In section 2, we introduce some notations, basic definitions, hypotheses, and preliminary results. We also derive a variation of parameters formula for a solution of the system (1.2). In section 3, we establish our main results on the existence of a mild solution and the approximate controllability of the system (1.2) using the resolvent operator theory. Finally, in section 4, we illustrate our results with the help of an example.

## 2. Preliminaries

Let $(Y, \langle\cdot,\cdot\rangle)$ be a Hilbert space and the norm on $Y$ be defined by $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$, and $\mathcal{L}(Y)$ be a space of bounded linear operator from $Y$ to $Y$ with operator norm. Let us assume that $-A$ generates a compact and analytic semigroup $\{T(t),\ t \geq 0\} \subset \mathcal{L}(Y)$. Thus $\sup_{t\in J}\|T(t)\| \leq M_0$ holds for some $M_0 > 1$.

Let $0 \in \rho(A)$ (resolvent set of $A$), without loss of generality. We can now define $A^\alpha$, $0 < \alpha \leq 1$, that is a closed linear operator with domain $D(A^\alpha) \subseteq Y$. Moreover, $D(A^\alpha)$ is dense in $Y$. Henceforth, we denote by $Y_\alpha$ the Hilbert space $D(A^\alpha)$ with the norm $\|y\|_\alpha = \|A^\alpha y\|,\ y \in Y_\alpha \subset Y$. To explore more information on this topic, one can follow the book [22]. The following properties of $A^\alpha$ are obtained from [22, Lemma 2.6.3 and Theorem 2.6.8].

**Theorem 2.1.** *The operator $A^\alpha$ has the following properties*:

(*a*) *For any* $\beta \geq \alpha > 0$, $D(A^\beta) \subseteq D(A^\alpha)$.
(*b*) *If* $\alpha_1, \alpha_2 \in \mathbb{R}$ *and* $\alpha = \max\{\alpha_1+\alpha_2, \alpha_1, \alpha_2\}$, *then*

$$A^{\alpha_1+\alpha_2}y = A^{\alpha_1}.A^{\alpha_2}y, \quad y \in D(A^\alpha).$$

(*c*) *There is a constant* $C_\alpha > 0$ *such that*

$$\|A^{-\alpha}\| \leq C_\alpha, \quad 0 < \alpha \leq 1.$$

A brief summary on the resolvent operator theory is given as follows (for more details, see [13,25]).

DEFINITION 2.1

A one-parameter family $\{\mathscr{R}(t)\}_{t\geq 0} \subset \mathcal{L}(Y)$ is called a resolvent operator of the abstract integro-differential system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(y(t) + \int_0^t K(t-r)y(r)\mathrm{d}r\right) = -Ay(t) + \int_0^t \gamma(t-r)y(r)\mathrm{d}r, \quad t \in J, \\ y(0) = y_0 \in Y \end{cases} \tag{2.1}$$

if the following statements hold:

(a) The map $\mathscr{R}(\cdot)\colon [0,\infty) \to \mathcal{L}(Y)$ is strongly continuous, exponentially bounded, and satisfies $\mathscr{R}(0)v = v$ for each $v \in Y$, i.e. $\mathscr{R}(0) = I$, the identity operator.

(b) For any $v \in D(A)$, $\mathscr{R}(\cdot)v \in C([0,\infty), D(A)) \cap C^1((0,\infty), Y)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{R}(t)v + \int_0^t K(t-r)\mathscr{R}(r)v\,\mathrm{d}r\right) = -A\mathscr{R}(t)v + \int_0^t \gamma(t-r)\mathscr{R}(r)v\,\mathrm{d}r \tag{2.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{R}(t)v + \int_0^t \mathscr{R}(t-r)K(r)v\,\mathrm{d}r\right) = -\mathscr{R}(t)Av + \int_0^t \mathscr{R}(t-r)\gamma(r)v\,\mathrm{d}r, \tag{2.3}$$

for each $t \geq 0$.

Throughout the paper, we assume that the following hypothesis hold:

(A1) The operator $-A\colon D(A) \subseteq Y \to Y$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $Y$. $-A$ is closed linear densely defined on $Y$ and $\rho(-A) \supset \Lambda_\theta = \{\omega \in \mathbf{C} \setminus \{0\}\colon |\arg(\omega)| < \theta\}$ and $\|R(\omega,-A)\| \leq M_0|\omega|^{-1}$ for each $\omega \in \Lambda_\theta$ for some constants $M_0 > 0$ and $\theta \in (\pi/2, \pi)$, where $R(\omega,-A)$ is the resolvent of $-A$.

(A2) The map $K\colon [0,\infty) \to \mathcal{L}(Y)$ is strongly continuous. $\hat{K}(\omega)v$ is absolutely convergent for any $v \in Y$ if $\mathbf{Re}(\omega) > 0$, here $\hat{K}(\omega)$ denotes the Laplace transform of $K$. There is a constant $\alpha > 0$ and an analytic extension of $\hat{K}(\omega)$ (still denoted by $\hat{K}(\omega)$) to $\Lambda_\theta$ such that $\|\hat{K}(\omega)\| \leq K_0|\omega|^{-\alpha}$ for each $\omega \in \Lambda_\theta$ and $\|\hat{K}(\omega)v\| \leq K_1|\omega|^{-1}\|v\|_1$ for each $\omega \in \Lambda_\theta$ and $v \in D(A)$.

(A3) The operator $\gamma(t)\colon D(\gamma(t)) \subseteq Y \to Y$ is linear and closed and $D(A) \subseteq D(\gamma(t))$ for each $t \geq 0$. For any $v \in D(A)$, the function $\gamma(\cdot)v$ is strongly measurable on $(0,\infty)$. There is a $b(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$ such that $\hat{b}(\omega)$ can be obtained for $\mathbf{Re}(\omega) > 0$ and $\|\gamma(t)v\| \leq b(t)\|v\|_1$ for each $t > 0$ and $v \in D(A)$. In addition, the function $\hat{\gamma}\colon \Lambda_{\pi/2} \to \mathcal{L}(D(A), Y)$ has an analytical extension (still denoted by $\hat{\gamma}$) to $\Lambda_\theta$ such that $\|\hat{\gamma}(\omega)v\| \leq \|\hat{\gamma}(\omega)\|\,\|v\|_1$ for each $v \in D(A)$, and $\|\hat{\gamma}(\omega)\| \to 0$ as $|\omega| \to \infty$.

(A4) There is a subspace $E \subseteq D(A)$ which is dense in $D(A)$ and there exist constants $C_i > 0$, $i = 1, 2$, such that $\hat{\gamma}(\omega)(E) \subseteq D(A)$, $\hat{K}(\omega)(E) \subseteq D(A)$, $\|A\hat{\gamma}(\omega)v\| \leq C_1\|v\|$ and $\|\hat{K}(\omega)v\|_1 \leq C_2|\omega|^{-\alpha}\|v\|_1$ for each $v \in E$ and $\omega \in \Lambda_\theta$.

(A5) The operator $\hat{K}(\omega)$ maps $D(A^\alpha)$ to $D(A^\alpha)$ for each $\omega \in \Lambda_\theta$, $0 < \alpha < 1$, and $\frac{\partial \hat{K}(\omega)}{\partial \omega} \to 0$ as $|\omega| \to \infty$ uniformly for $\omega \in \Lambda_\theta$.

In the continuation, for $s > 0$ and $\vartheta \in (\pi\backslash 2, \theta)$, $\Lambda_{s,\vartheta} = \{\omega \in C\backslash\{0\}\colon |\omega| > s, |\arg(\omega)| < \vartheta\}$, $\Gamma_{s,\vartheta}$, $\Gamma^i_{s,\vartheta}$, $i = 1, 2, 3$, are the paths defined as $\Gamma^1_{s,\vartheta} = \{te^{i\vartheta}\colon t \geq s\}$, $\Gamma^2_{s,\vartheta} = \{se^{i\xi}\colon -\vartheta \leq \xi \leq \vartheta\}$, $\Gamma^3_{s,\vartheta} = \{te^{-i\vartheta}\colon t \geq s\}$ and $\Gamma_{s,\vartheta} = \cup_{i=1}^3 \Gamma^i_{s,\vartheta}$ oriented in the positive sense. Let

$$\Omega(G) = \{\omega \in C\colon G(\omega) = (\omega I + \omega\hat{K}(\omega) + A - \hat{\gamma}(\omega))^{-1} \in \mathcal{L}(Y)\}.$$

*Lemma* 2.2 [25, *Lemma* 2.2]. *There is a constant* $s_2 > 0$ *such that* $\Lambda_{s_2,\vartheta} \subset \Omega(G)$ *and the following statements are valid*:

(*i*) *The map* $G$ *from* $\Lambda_{s_2,\vartheta}$ *to* $\mathcal{L}(Y)$ *is analytic, and there exists* $L_1 > 0$ *such that*

$$\|\omega G(\omega)\| \le L_1, \quad \omega \in \Lambda_{s_2,\vartheta}.$$

(*ii*) *The Range(*$G(\omega)$*) is contained in* $D(A)$*, the map* $AG\colon \Lambda_{s_2,\vartheta} \to \mathcal{L}(Y)$ *is analytic, and there are constants* $L_2, L_3 \ge 0$ *such that*

$$\|\omega AG(\omega)v\| \le L_2\|v\|_1, \quad v \in D(A), \quad \omega \in \Lambda_{s_2,\vartheta},$$
$$\|AG(\omega)\| \le L_3, \quad \omega \in \Lambda_{s_2,\vartheta}.$$

If $\mathscr{R}(\cdot)$ is a resolvent operator for the system (2.1), then the Laplace transform of (2.3) gives

$$\hat{\mathscr{R}}(\cdot)(\omega I + \omega\hat{K}(\omega) + A - \hat{\gamma}(\omega))v = v, \quad \forall\, v \in D(A).$$

Using Lemma 2.2 and the inverse Laplace transform, we deduce that $\mathscr{R}(\cdot)$ is the only resolvent operator for the system (2.1). In the rest of this section, let $s > s_2$. Thus the resolvent operator $\{\mathscr{R}(t)\}_{t\ge 0}$ is defined as

$$\mathscr{R}(t) = \begin{cases} \frac{1}{2i\pi}\int_{\Gamma_{s,\vartheta}} \mathrm{e}^{\omega t} G(\omega)\mathrm{d}\omega, & t > 0,\\ I, & t = 0. \end{cases} \tag{2.4}$$

We now introduce some properties of $\mathscr{R}(\cdot)$ that will be used to prove the main results in the next section.

*Lemma* 2.3 [25, *Lemma* 2.3 and *Lemma* 2.5]. *The map* $\mathscr{R} : [0,\infty) \to \mathcal{L}(Y)$ *is strongly continuous and* $\mathscr{R}(\cdot)$ *is exponential bounded in* $\mathcal{L}(Y)$*, i.e.,* $\|\mathscr{R}(t)\| \le C\mathrm{e}^{wt}$*, for some* $C > 0$ *and some* $w > 0$.

*Lemma* 2.4 [25, *Lemma* 2.7]. *The map* $\mathscr{R}\colon (0,\infty) \to \mathcal{L}(Y)$ *has an analytic extension to* $\Lambda_\delta, \delta = \min\{\theta - \frac{\pi}{2}, \frac{\pi}{2} - \theta\}$ *and there is a constant* $M > 0$ *such that* $\sup_{t\in J}\|\mathscr{R}(t)\| \le M$.

**Theorem 2.5 [25, Theorem 2.2].** *Assume that condition (A5) is fulfilled, and* $b(\cdot)$ *is locally bounded on* $(0,\infty)$*. Then for* $\alpha \in (0,1)$ *and* $t > 0$*, the operator* $\mathscr{R}(t) \in \mathcal{L}(D(A^\alpha), D(A))$ *with* $\|\mathscr{R}(t)\| \le m_\alpha t^{-\alpha}$ *for a positive constant* $m_\alpha$*. Furthermore equation* (2.2) *and* (2.3) *hold for* $v \in D(A^\alpha)$ *and* $t > 0$.

*Lemma* 2.6 [25, *Lemma* 3.11]. *The operator* $\mathscr{R}(t)$ *is compact for all* $t > 0$ *if* $R(\omega_0, -A)$ *is compact for some* $\omega_0 \in \rho(-A)$.

**Theorem 2.7 [22, Theorem 2.3.3].** *Let* $-A$ *be the infinitesimal generator of a* $C_0$ *semigroup* $T(t)$*. The semigroup* $T(t)$ *is compact if and only if* $R(\omega, -A)$ *is compact for* $\omega \in \rho(-A)$ *and* $T(t)$ *is continuous in the uniform operator topology for* $t > 0$.

DEFINITION 2.2

An abstract function $y\colon [-\tau, b] \to Y$ is said to be a (classical) solution of (1.2) if $y(t)$ is continuously differentiable and $y(t) \in D(A)$ for all $t \in [0, b]$, and $y(t)$ satisfies the system (1.2).

Let us derive a variation of parameters formula to represent the solution of (1.2) in the form of resolvent operator theory.

**Theorem 2.8.** *Suppose that $\zeta(0)+\eta(y)(0) \in D(A)$. The functions $f : J \times \mathcal{D} \to Y_\alpha$ and $W : U \to Y$ are continuous, and $y(\cdot)$ is a classical solution of system* (1.2) *on the interval* $[-\tau, b]$. *Then*

$$y(t)=\begin{cases}\mathscr{R}(t)[\zeta(0)+\eta(y)(0)]+\int_0^t \mathscr{R}(t-s)\,[f(s, y_s)+Wu(s)]\mathrm{d}s, & t\in J,\\ \zeta(t)+\eta(y)(t), & t\in[-\tau,0]\end{cases} \tag{2.5}$$

*Proof.* For any arbitrary $\epsilon > 0$, take $t \geq \epsilon$. Define

$$\begin{aligned} H(t) &= \mathscr{R}(\epsilon)y(t-\epsilon) - \mathscr{R}(t)[\zeta(0)+\eta(y)(0)] \\ &\quad - \int_0^{t-\epsilon} \mathscr{R}(t-s)\,[f(s, y_s)+Wu(s)]\mathrm{d}s \\ &= \int_0^{t-\epsilon} \frac{\partial}{\partial s}\,[\mathscr{R}(t-s)y(s)]\mathrm{d}s \\ &\quad - \int_0^{t-\epsilon} \mathscr{R}(t-s)\,[f(s, y_s)+Wu(s)]\mathrm{d}s \\ &= -\int_0^{t-\epsilon} \mathscr{R}'(t-s)y(s)\,\mathrm{d}s + \int_0^{t-\epsilon} \mathscr{R}(t-s)y'(s)\mathrm{d}s \\ &\quad - \int_0^{t-\epsilon} \mathscr{R}(t-s)\,[f(s, y_s)+Wu(s)]\mathrm{d}s \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{2.6}$$

Using (1.2) for $y'(t)$, we get

$$\begin{aligned} I_2 + I_3 = \int_0^{t-\epsilon} \mathscr{R}(t-s)\Bigg[&-\frac{\partial}{\partial s}\left(\int_0^s K(s-r)y(r)\mathrm{d}r\right) \\ &-Ay(s) + \int_0^s \gamma(s-r)y(r)\,\mathrm{d}r\Bigg]\mathrm{d}s. \end{aligned} \tag{2.7}$$

From the equation (2.3), we have

$$\begin{aligned} \mathscr{R}'(t)v &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t \mathscr{R}(t-r)K(r)v\,\mathrm{d}r\right) \\ &\quad - \mathscr{R}(t)Av + \int_0^t \mathscr{R}(t-r)\gamma(r)v\,\mathrm{d}r, \\ &= -\mathscr{R}(0)K(t)v - \int_0^t \mathscr{R}'(t-r)K(r)v\mathrm{d}r - \mathscr{R}(t)Av \\ &\quad + \int_0^t \mathscr{R}(t-r)\gamma(r)v\,\mathrm{d}r. \end{aligned}$$

This implies that

$$\begin{aligned}\mathscr{R}'(t-s)y(s) = &-K(t-s)y(s) - \mathscr{R}(t-s)Ay(s)\\ &-\int_0^{t-s}\mathscr{R}'(t-s-r)K(r)y(s)\mathrm{d}r\\ &+\int_0^{t-s}\mathscr{R}(t-s-r)\gamma(r)y(s)\mathrm{d}r.\end{aligned}\tag{2.8}$$

Thus, in view of (2.7) and (2.8), we get

$$\begin{aligned}H(t) = &\int_0^{t-\epsilon}K(t-s)y(s)\mathrm{d}s + \int_0^{t-\epsilon}\int_0^{t-s}\mathscr{R}'(t-s-r)K(r)y(s)\mathrm{d}r\mathrm{d}s\\ &-\int_0^{t-\epsilon}\int_0^{t-s}\mathscr{R}(t-s-r)\gamma(r)y(s)\mathrm{d}r\mathrm{d}s\\ &-\int_0^{t-\epsilon}\mathscr{R}(t-s)\frac{\partial}{\partial s}\left(\int_0^s K(s-r)y(r)\mathrm{d}r\right)\mathrm{d}s\\ &+\int_0^{t-\epsilon}\mathscr{R}(t-s)\int_0^s\gamma(s-r)y(r)\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

But

$$\begin{aligned}&\int_0^{t-\epsilon}\mathscr{R}(t-s)\frac{\partial}{\partial s}\left(\int_0^s K(s-r)y(r)\,\mathrm{d}r\right)\mathrm{d}s\\ &\quad=\int_0^{t-\epsilon}\int_0^{t-s}\mathscr{R}'(t-s-r)K(r)y(s)\mathrm{d}r\,\mathrm{d}s\\ &\quad\quad-\int_0^{t-\epsilon}\int_{t-s-\epsilon}^{t-s}\mathscr{R}'(t-s-r)K(r)y(s)\mathrm{d}r\,\mathrm{d}s\\ &\quad\quad+\mathscr{R}(\epsilon)\int_0^{t-\epsilon}K(t-r-\epsilon)y(r)\mathrm{d}r\end{aligned}$$

and

$$\begin{aligned}&\int_0^{t-\epsilon}\mathscr{R}(t-s)\int_0^s\gamma(s-r)y(r)\mathrm{d}r\,\mathrm{d}s\\ &\quad=\int_0^{t-\epsilon}\int_0^{t-s}\mathscr{R}(t-s-r)\gamma(r)y(s)\mathrm{d}r\,\mathrm{d}s\\ &\quad\quad-\int_0^{t-\epsilon}\int_{t-s-\epsilon}^{t-s}\mathscr{R}(t-s-r)\gamma(r)y(s)\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

Therefore we obtain that

$$\begin{aligned} H(t) &= \int_0^{t-\epsilon} K(t-s)y(s)\mathrm{d}s - \mathscr{R}(\epsilon)\int_0^{t-\epsilon} K(t-s-\epsilon)y(s)\mathrm{d}s \\ &\quad + \int_0^{t-\epsilon}\int_{t-s-\epsilon}^{t-s} \mathscr{R}'(t-s-r)K(r)y(s)\mathrm{d}r\,\mathrm{d}s \\ &\quad - \int_0^{t-\epsilon}\int_{t-s-\epsilon}^{t-s} \mathscr{R}(t-s-r)\gamma(r)y(s)\mathrm{d}r\,\mathrm{d}s. \end{aligned}$$

It is clear from the above equation that $H(t) \to 0$ as $\epsilon \to 0$. Thus we can easily obtain the formula (2.5) as $\epsilon \to 0$ in (2.6). □

DEFINITION 2.3

A function $y \in C([-\tau, b], Y)$ that satisfies the integral equation (2.5) is called a mild solution of the neutral system (1.2) on $[-\tau, b]$.

Finally, we recall the definition of approximate controllability and its preliminary results.

DEFINITION 2.4

If for a given $\varepsilon > 0$, however small, and any final state $z \in Y_\alpha$, there exists a control $u(\cdot) \in L^2(J, U)$ such that the mild solution $y(\cdot, u)$ of the neutral system (1.2) satisfies that $\|y(b, u) - z\| < \varepsilon$, then the system (1.2) is said to be approximately controllable on $J$.

Consider the associated linear neutral system of (1.2):

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)+\int_0^t K(t-s)y(s)\mathrm{d}s\right) = -Ay(t)+\int_0^t \gamma(t-s)y(s)\mathrm{d}s+Wu(t) \quad t\in J, \\ y(0)=\zeta(0). \end{cases} \tag{2.9}$$

Controllability operator $\Gamma_0^b : Y_\alpha \to Y_\alpha$ and resolvent operator $S(\varepsilon, \Gamma_0^b) : Y_\alpha \to Y_\alpha$ are defined respectively as

$$\begin{aligned} \Gamma_0^b &= \int_0^b \mathscr{R}(b-r)WW^*\mathscr{R}^*(b-r)\mathrm{d}r, \\ S(\varepsilon, \Gamma_0^b) &= (\varepsilon I + \Gamma_0^b)^{-1}, \quad \varepsilon > 0, \end{aligned}$$

where $\mathscr{R}^*$ and $W^*$ represent the adjoint of $\mathscr{R}$ and $W$ respectively.

Since

$$\begin{aligned}\langle \Gamma_0^b z, z\rangle &= \left\langle \int_0^b \mathscr{R}(b-r)WW^*\mathscr{R}^*(b-r)z\,\mathrm{d}r, z\right\rangle\\ &= \int_0^b \langle \mathscr{R}(b-r)WW^*\mathscr{R}^*(b-r)z, z\rangle \mathrm{d}r\\ &= \int_0^b \langle W^*\mathscr{R}^*(b-r)z, W^*\mathscr{R}^*(b-r)z\rangle \mathrm{d}r\\ &= \int_0^b \|W^*\mathscr{R}^*(b-r)z\|\mathrm{d}r\end{aligned}$$

and $R(t)$ is compact for $t > 0$, the operator $\Gamma_0^b$ is a positive and compact self-adjoint operator on $Y$. We obtain from [3, Proposition 7.14 of Chapter II] that the spectrum of $\Gamma_0^b$ has only non-negative real numbers. So $S(\varepsilon, \Gamma_0^b)$ is well defined for all $\varepsilon > 0$. We now make a hypothesis that will be applied in many of our subsequent results:

(H0) $\varepsilon S(\varepsilon, \Gamma_0^b) \to 0$ as $\varepsilon \to 0^+$ in strong operator topology.

*Lemma* 2.9. *The hypothesis* (H0) *holds if and only if the linear system* (2.9) *is approximately controllable on $J$.*

*Proof.* Theorem 4.1.7 of [4] gives that the system (2.9) is approximately controllable on $J$ iff $\langle z, \Gamma_0^b z\rangle > 0$ for any non-zero $z \in Y_\alpha$. Thus, this lemma is a simple result of [20, Theorem 2.3].

*Remark* 2.10. $\langle z, \Gamma_0^b z\rangle = \int_0^b \|W^*\mathscr{R}^*(b-r)z\|^2\, dr > 0$ holds for any non-zero $z \in Y_\alpha$ if and only if $W^*\mathscr{R}^*(b-r)z = 0$ implies $z = 0$.

## 3. Main results

To establish the approximate controllability of (1.2), we make certain assumptions as listed below:

(H1) The function $f : J \times \mathcal{D} \to Y_\alpha$ satisfies that $f(t, \cdot)$ is continuous for all $t \in J$ and $f(\cdot, \psi)$ is strongly measurable for each $\psi \in \mathcal{D}$. Also, there is a $v_k(\cdot) \in L^2([0, b], \mathbb{R}^+)$ for each $k > 0$ such that

$$\sup\{\|f(t,\psi)\|_\alpha : \|\psi\|_{\mathcal{D}} \le k\} \le v_k(t), \quad \text{for a.e. } t \in J$$

and

$$\lim_{k\to\infty}\inf \frac{1}{k}\|v_k\|_{L^2} = \rho < +\infty.$$

(H2) The operator $\eta : \mathcal{C} \to \mathcal{D}$ satisfies Lipschitz condition, i.e., there is a constant $L_\eta > 0$ such that

$$\|\eta(y) - \eta(z)\|_{\mathcal{D}} \le L_\eta \|y - z\|_{\mathcal{C}}, \quad \forall\, y, z \in \mathcal{C}.$$

(H2)$'$ The operator $\eta : \mathcal{C} \to \mathcal{D}$ is completely continuous such that

$$\lim_{\|y\|_{\mathcal{C}}\to\infty} \frac{\|\eta(y)\|_{\mathcal{D}}}{\|y\|_{\mathcal{C}}} = L'_\eta.$$

For any $z \in Y$ and $\varepsilon > 0$, we fix a control $u_\varepsilon(t, y)$ as

$$u_\varepsilon(t, y) = W^*\mathscr{R}^*(b-t)S(\varepsilon, \Gamma_0^b)\Big\{z - \mathscr{R}(b)(\zeta(0) + \eta(y)(0)) - \int_0^b \mathscr{R}(t-s) f(s, y_s)\mathrm{d}s\Big\}. \tag{3.1}$$

Since $\mathscr{R}(t)$ commutes with $A^\alpha$ in many cases (see for e.g., [1, Remark 3.2] and Example 4 of this paper), we assume that $A^\alpha$ commutes with resolvent operator $\mathscr{R}(t)$, for the sake of convenience. Also, we write

$$N_\varepsilon = \frac{\|W\|_\alpha}{\varepsilon} \sup_{0\le t\le b} \|W^*\mathscr{R}^*(b-t)\|.$$

**Theorem 3.1.** *If hypotheses* (H1)–(H2) *are true and* $M(L_\eta + \rho)(1 + C_\alpha N_\varepsilon) < 1$, *then the neutral system* (1.2) *with control* $u_\varepsilon(t, y)$ *has a mild solution for each* $\varepsilon > 0$.

*Proof.* Let $E_k = \{y \in C([-\tau, b], Y_\alpha) : \|y\|_{\mathcal{C}} \le k\}$, $k = k(\varepsilon) > 0$. Define an operator $\widehat{Q}_\varepsilon : \mathcal{C} \to \mathcal{C}$ by

$$\widehat{Q}_\varepsilon y(t) = Qy(t) + Q_\varepsilon y(t), \quad t \in [-\tau, b], \tag{3.2}$$

where

$$(Qy)(t) = \begin{cases} \mathscr{R}(t)[\zeta(0) + \eta(y)(0)], & t \in J, \\ \zeta(t) + (\eta(y))(t), & t \in [-\tau, 0] \end{cases} \tag{3.3}$$

and

$$(Q_\varepsilon y)(t) = \begin{cases} \int_0^t \mathscr{R}(t-s)\,[f(s, y_s) + Wu_\varepsilon(s, y)]\,ds, & t \in J, \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3.4}$$

It is clear that any fixed point of $\widehat{Q}_\varepsilon = Q + Q_\varepsilon$ is a mild solution of (1.2). For any $y \in E_k$, we obtain from Lemma 2.4 that

$$\begin{aligned}\|u_\varepsilon(t, y)\| \le& \frac{1}{\varepsilon} \sup_{0\le t\le b} \|W^*\mathscr{R}^*(b-t)\| \Big[\|z\| + \|\mathscr{R}(t)(\zeta(0) + \eta(y)(0))\| \\ &+ \int_0^t \|\mathscr{R}(s) f(s, y_s)\|\mathrm{d}s\Big] \\ \le& \frac{1}{\varepsilon} \sup_{0\le t\le b} \|W^*\mathscr{R}^*(b-t)\| \Big[\|z\| + MC_\alpha\{\|\zeta(0)\|_\alpha + L_\eta\|y^{(k)}\|_{\mathcal{C}} \\ &+ \|\eta(0)\|_{\mathcal{D}}\} + MC_\alpha \int_0^b v_k(s)\mathrm{d}s\Big]\end{aligned}$$

$$=L \text{ (say)}. \tag{3.5}$$

We first claim that $\widehat{Q}_\varepsilon(E_k) \subset E_k$ for some $k > 0$. If this is not true, then for each $k > 0$ there would exist $y^{(k)} \in E_k$ such that $\|\widehat{Q}_\varepsilon y^{(k)}(t)\| > k$ for some $t \in J$. Thus, from the hypotheses (H1), (H2) and Lemma 2.4, we get

$$\begin{aligned}
k < \|\widehat{Q}_\varepsilon y^{(k)}(t)\|_\alpha &\le M\{\|\zeta(0)\|_\alpha + \|\eta(y)(0)\|_\alpha\} \\
&\quad + M\int_0^t [\|f(s, y_s)\|_\alpha + \|Wu_\varepsilon(s, y^{(k)})\|_\alpha]\mathrm{d}s \\
&\le M\{\|\zeta(0)\|_\alpha + L_\eta\|y^{(k)}\|_{\mathcal{C}} + \|\eta(0)\|_{\mathcal{D}}\} \\
&\quad + M\int_0^t v_k(s)\,\mathrm{d}s + \frac{\|W\|_\alpha}{\varepsilon}\sup_{0\le t\le b}\|W^*\mathscr{R}^*(b-t)\| \\
&\quad \Big[\|z\| + MC_\alpha\{\|\zeta(0)\|_\alpha + L_\eta\|y^{(k)}\|_{\mathcal{C}} \\
&\quad + \|\eta(0)\|_{\mathcal{D}}\} + MC_\alpha\int_0^b v_k(s)\mathrm{d}s\Big].
\end{aligned} \tag{3.6}$$

Dividing both sides of (3.6) by $k$ and then letting $k \to \infty$, we obtain

$$\begin{aligned}
1 &< ML_\eta + M\rho + N_\varepsilon\{MC_\alpha L_\eta + M\rho C_\alpha\} \\
&= M(L_\eta + \rho)(1 + C_\alpha N_\varepsilon).
\end{aligned}$$

This is a contradiction to the statement of the theorem. Thus we can find a number $k > 0$ such that $\widehat{Q}_\varepsilon$ maps from $E_k$ to $E_k$. Next we show that the operator $Q_\varepsilon : E_k \to E_k$ is completely continuous.

First of all, we show that the map $Q_\varepsilon : E_k \to E_k$ is continuous. Take a sequence $\{w^{(n)}\} \subseteq E_k$ with $w^{(n)} \to w \in E_k$ as $n \to \infty$. Then for each $t \in J$, we have

(i) $f(t, w_t^{(n)}) \to f(t, w_t)$.
(ii) $\eta(w^{(n)}) \to \eta(w)$.
(iii) $\|f(t, w_t^{(n)}) - f(t, w_t)\|_\alpha \le 2v_k(t)$.
(iv) $\|u_\varepsilon(s, w^{(n)}) - u_\varepsilon(s, w)\| \le 2L$.

Therefore we obtain from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\|Q_\varepsilon w^{(n)}(t) - Q_\varepsilon w(t)\| &\le M\int_0^t \|f(s, w_s^{(n)}) - f(s, w_s)\|_\alpha \mathrm{d}s \\
&\quad + M\|W\|_\alpha\sqrt{b}\Big(\int_0^t \|u_\varepsilon(s, w^{(n)}) - u_\varepsilon(s, w)\|^2\mathrm{d}s\Big)^{\frac{1}{2}} \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus the map $Q_\varepsilon : E_k \to E_k$ is continuous. Let $y \in E_k$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Therefore,

$$\begin{aligned}
&\|Q_\varepsilon y(t_2) - Q_\varepsilon y(t_1)\|_\alpha \\
&\quad \le \Big\|\int_0^{t_1} [\mathscr{R}(t_2 - s) - \mathscr{R}(t_1 - s)][f(s, y_s) + Wu_\varepsilon(s, y)]\mathrm{d}s\Big\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \int_{t_1}^{t_2} \mathscr{R}(t_2 - s)[f(s, y_s) + Wu_\varepsilon(s, y)]\mathrm{d}s \right\|_\alpha \\
&\leq \int_0^{t_1} \|[\mathscr{R}(t_2 - s) - \mathscr{R}(t_1 - s)][f(s, y_s) + Wu_\varepsilon(s, y)]\|_\alpha \, \mathrm{d}s \\
&+ M \int_{t_1}^{t_2} [v_k(s) + L\|W\|_\alpha)] \, \mathrm{d}s. \\
&= I_1 + I_2.
\end{aligned}
$$

It is easy to check that $I_2 \to 0$ as $t_1 \to t_2$ independent of $y \in E_k$. Since $\{T(t)\}_{\{t\geq 0\}}$ is a compact and analytic semigroup, we conclude from Theorem 2.7 and Lemma 2.6 that the operator $\mathscr{R}(t)$ is compact for $t > 0$. Therefore, we get

$$
\begin{aligned}
I_1 &\leq \int_0^{t_1-\epsilon} \|[\mathscr{R}(t_2 - s) - \mathscr{R}(t_1 - s)][f(s, y_s) + Wu_\varepsilon(s, y)]\|_\alpha \, \mathrm{d}s \\
&\quad + \int_{t_1-\epsilon}^{t_1} \|[\mathscr{R}(t_2 - s) - \mathscr{R}(t_1 - s)][f(s, y_s) + Wu_\varepsilon(s, y)]\|_\alpha \, \mathrm{d}s \\
&\leq \int_0^{t_1-\epsilon} [v_k(s) + L\|W\|_\alpha)] \, \mathrm{d}s \cdot \sup_{s\in[0,t_1-\epsilon]} \|\mathscr{R}(t_2 - s) - \mathscr{R}(t_1 - s)\| \\
&\quad + 2M \int_{t_1-\epsilon}^{t_1} [v_k(s) + L\|W\|_\alpha)]\mathrm{d}s.
\end{aligned}
$$

We can now say that $I_1 \to 0$ as $t_1 \to t_2$ independent of $y \in E_k$. Thus $Q_\varepsilon(E_k)$ is equicontinuous on $[-\tau, b]$.

Further, we show the relatively compactness of the set $G(t) = \{(Q_\varepsilon y)(t) : \ y \in E_k\}$, $t \in J$ in $Y_\alpha$ for each $\varepsilon > 0$. Let $0 < \alpha < \beta < 1$ with $0 < \beta - \alpha < \frac{1}{2}$. Then

$$
\begin{aligned}
\|A^\beta Q_{\varepsilon,2} y(t)\| &\leq \int_0^t \|A^{\beta-\alpha}\mathscr{R}(t - s)\| \, [\|f(s, y_s)\|_\alpha + \|Wu_\varepsilon(s, y)\|_\alpha]\mathrm{d}s \\
&\leq m_{\beta-\alpha} \int_0^t (t - s)^{-(\beta-\alpha)}[v_k(s) + \|W\|_\alpha\|u_\varepsilon(s, y)\|]\mathrm{d}s \\
&\leq m_{\beta-\alpha} \frac{b^{1-2(\beta-\alpha)}}{1 - 2(\beta - \alpha)}[\|v_k\|_{L^2} + \|W\|_\alpha\|u\|_{L^2}]. \qquad (3.7)
\end{aligned}
$$

So $\{A^\beta Q_\varepsilon y(t) : \ y \in E_k\}$ is bounded in $Y$. By the compactness of the embedding $Y_\beta \hookrightarrow Y_\alpha$, the set $\{Q_\varepsilon y(t) : \ y \in E_k\}$ is compact in $Y_\alpha$ for each $t \in (0, b]$. Since $G(t) = \{0\}$ for $t \in [-\tau, 0]$, the set $G(t)$ is relatively compact in $Y_\alpha$ for each $t \in [-\tau, b]$.

We can now conclude from Ascoli–Arzela theorem that the set $\{Q_\varepsilon y : y \in E_k\}$ is relatively compact in space $C([-\tau, b] : Y_\alpha)$. Therefore, the map $Q_\varepsilon \colon E_k \to E_k$ is completely continuous. On the other hand, we obtain

$$
\begin{aligned}
\|Qy(t) - Qz(t)\|_\alpha &\leq \|\mathscr{R}(t)[\eta(y)(0) - \eta(z)(0)]\|_\alpha \\
&\leq ML_\eta\|y - z\|_{\mathcal{C}}, \quad t \in J
\end{aligned}
$$

and

$$\|Qy(t) - Qz(t)\|_\alpha \leq L_\eta \|y - z\|_{\mathcal{C}}, \quad t \in [-\tau, 0].$$

Therefore,

$$\|Qy - Qz\|_{\mathcal{C}} \leq ML_\eta \|z - y\|_{\mathcal{C}}.$$

Since $0 < ML_\eta < 1$, $Q\colon E_k \to E_k$ is a contraction. Hence, by Krasnoselskii's fixed point theorem, the operator $\widehat{Q}_\varepsilon y = Qy + Q_\varepsilon y$ has a fixed point on $E_k$ for each $\varepsilon > 0$. That is, the neutral system (1.2) has a mild solution $y$ in $C([-\tau, b], Y_\alpha)$. □

In the next theorem, we again show the existence of a mild solution of the system (1.2) when $\eta$ satisfies the condition (H2)$'$ instead of (H2).

**Theorem 3.2.** *If hypotheses* (H1) *and* (H2)$'$ *are true and* $M(L'_\eta + \rho)(1 + C_\alpha N_\varepsilon) < 1$, *then the neutral system* (1.2) *with control* $u_\varepsilon(t, y)$ *has a mild solution for each* $\varepsilon > 0$.

*Proof.* Let $E_k = \{y \in C([-\tau, b], Y_\alpha) : \|y\|_{\mathcal{C}} \leq k\}, k = k(\varepsilon) > 0$. Consider the operators $\widehat{Q}_\varepsilon$, $Qy(t)$ and $Q_\varepsilon$ on $\mathcal{C}$ which are defined by (3.2), (3.3) and (3.4) respectively. For any $y \in E_k$, we obtain from Lemma 2.4 that

$$\begin{aligned}\|u_\varepsilon(t, y)\| &\leq \frac{1}{\varepsilon} \sup_{0\leq t\leq b} \|W^*\mathscr{R}^*(b - t)\| \Big[ \|z\| + MC_\alpha \{\|\zeta(0)\|_\alpha + \|\eta(y)\|_{\mathcal{D}}\} \\ &\quad + MC_\alpha \int_0^b v_k(s)\,\mathrm{d}s \Big] \\ &= L_1 \text{ (say)}. \end{aligned} \tag{3.8}$$

We wish to show that $\widehat{Q}_\varepsilon(E_k) \subset E_k$ for some $k > 0$. On the contrary, suppose that this is wrong. Thus for each $k > 0$ there would be an element $y^{(k)} \in E_k$ such that $\|\widehat{Q}_\varepsilon y^{(k)}(t)\| > k$ for some $t \in J$. So we obtain from the hypotheses (H1), (H2)$'$ and Lemma 2.4 that

$$\begin{aligned} k < \|\widehat{Q}_\varepsilon y^{(k)}(t)\|_\alpha \leq M \{\|\zeta(0)\|_\alpha + \|\eta(y)\|_{\mathcal{D}}\} \\ + M \int_0^t v_k(s)\mathrm{d}s + N_\varepsilon \Big[ \|z\| + MC_\alpha\{\|\zeta(0)\|_\alpha \\ + \|\eta(y)\|_{\mathcal{D}}\} + MC_\alpha \int_0^b v_k(s)\mathrm{d}s \Big]. \end{aligned} \tag{3.9}$$

Dividing both sides of (3.9) by $k$ and letting $k \to \infty$, we obtain

$$1 < M(L'_\eta + \rho)(1 + C_\alpha N_\varepsilon).$$

This is a contradiction to the statement of the theorem. Therefore, there is a number $k > 0$ for which $\widehat{Q}_\varepsilon$ maps from $E_k$ to $E_k$.

Since $\zeta \in \mathcal{D}$, we can easily conclude from the assumption (H2)$'$ and Theorem 3.1 that the operators $Q$ and $Q_\varepsilon$ are completely continuous from $E_k$ to $E_k$. Thus the operator $\widehat{Q}_\varepsilon$

is completely continuous from closed bounded and convex subset $E_k$ to $E_k$. Hence, from Schauder's fixed point theorem, the operator $\widehat{Q}_\varepsilon$ has a fixed point on $E_k$ for each $\varepsilon > 0$. That is, the neutral system (1.2) has a mild solution $y$ in $C([-\tau, b], Y_\alpha)$. □

In the next theorem, we establish the approximate controllability of the system (1.2).

**Theorem 3.3.** *Assume that the hypotheses* (H0), (H1) *and* (H2) (*or* (H2)′) *are satisfied, and the functions* $f: J \times \mathcal{D} \to Y_\alpha$ *and* $\eta: \mathcal{C} \to \mathcal{D}$ *are uniformly bounded. Then the neutral integro-differential system* (1.2) *is approximately controllable on* $J$ *if* $ML_\eta < 1$ (*or* $ML'_\eta < 1$).

*Proof.* Since all hypotheses of Theorem 3.1 (or Theorem 3.2) are satisfied, we can say that for each $\varepsilon > 0$, the system (1.2) has a mild solution $y^{(\varepsilon)}$ in some $E_k \subset C([-\tau, b], Y_\alpha)$ using the control

$$u_\varepsilon(t, y_\varepsilon) = W^* \mathscr{R}^*(b-t) S(\varepsilon, \Gamma_0^b) Z(y^{(\varepsilon)}),$$

where

$$Z(y^{(\varepsilon)}) = z - \mathscr{R}(b)(\zeta(0) + \eta(y^{(\varepsilon)}))(0)) + \int_0^b \mathscr{R}(b-s) f(s, y_s^{(\varepsilon)}) \mathrm{d}s,$$

and $z$ is an arbitrary element of $Y$. It is easy to check that

$$\begin{aligned} y^{(\varepsilon)}(b) =& \mathscr{R}(b)(\zeta(0) + \eta(y^{(\varepsilon)})(0)) \\ &+ \int_0^b \mathscr{R}(b-s)[f(s, y_s^{(\varepsilon)}) + W u_\varepsilon(s, y^{(\varepsilon)})] \mathrm{d}s \\ =& z - \varepsilon S(\varepsilon, \Gamma_0^b) Z(y^{(\varepsilon)}). \end{aligned} \tag{3.10}$$

Let $0 < \alpha < \beta < 1$. In view of (H1) and (3.7), the set $\left\{\int_0^b \mathscr{R}(b-s) f(s, y_s^{(\varepsilon)}) \mathrm{d}s\right\}$ is bounded in $Y_\beta$. Since the operator $A^{-\beta}: Y \to Y_\alpha$ is compact, the set $\left\{\int_0^b \mathscr{R}(b-s) f(s, y_s^{(\varepsilon)}) \mathrm{d}s\right\}$ is relatively compact in $Y_\alpha$. Therefore, there is a subsequence denoted by $\left\{\int_0^b \mathscr{R}(b-s) f(s, y_s^{(\varepsilon)}) \mathrm{d}s\right\}$ that converges to $\overline{f} \in Y_\alpha$. Since the operator $A^\alpha \eta(\cdot)$ is uniformly bounded in $C([-\tau, b], Y)$, we conclude from Lemma 2.6 and Theorem 2.7 that the set $\left\{\mathscr{R}(b) A^\alpha \eta(y^{(\varepsilon)})(0)\right\}$ is relatively compact in $Y$. Therefore, there exists a subsequence denoted by itself, that converges to $\overline{\eta}$ in $Y$. Let $l = z - \mathscr{R}(b)\zeta(0) - A^{-\alpha}\overline{\eta} - \overline{f}$, then

$$\begin{aligned} \|Z(y^{(\varepsilon)}) - l\|_\alpha \le& \|\mathscr{R}(b) A^\alpha \eta(y^{(\varepsilon)})(0) - \overline{\eta}\| \\ &+ \left\| \int_0^b \mathscr{R}(b-s) f(s, y_s^{(\varepsilon)}) \mathrm{d}s - \overline{f} \right\|_\alpha \\ &\to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{3.11}$$

From equation (3.10) and inequality (3.11), we obtain that

$$\begin{aligned}\|y^{(\varepsilon)}(b)-z\|_\alpha &\le \|\varepsilon S(\varepsilon,\Gamma_0^b)(l)\|_\alpha + \|\varepsilon S(\varepsilon,\Gamma_0^b)\|\|Z(y^{(\varepsilon)})-l\|_\alpha \\ &\le \|\varepsilon S(\varepsilon,\Gamma_0^b)(l)\|_\alpha + \|Z(y_\varepsilon)-l\|_\alpha \\ &\to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

This proves the approximate controllability of abstract neutral integro-differential equations (1.2). □

## 4. Example

Consider the following neutral integro-differential equations with nonlocal condition that arise in theory of heat flow in materials with fading memory:

$$\begin{cases}\frac{\partial}{\partial t}\Big[y(t,\varrho)+\int_0^t (t-s)^m \mathrm{e}^{-l(t-s)} y(s,\varrho)\,\mathrm{d}s\Big] \\ \qquad = \frac{\partial^2 y(t,\varrho)}{\partial\varrho^2} + \int_0^t \mathrm{e}^{-v(t-s)}\frac{\partial^2 y(s,\varrho)}{\partial\varrho^2}\,\mathrm{d}s + f(t,\varrho,y(t-\tau,\varrho)) \\ \qquad +u(t,\varrho), \quad t\in[0,b],\ \varrho\in[0,\pi], \\ y(t,0)=y(t,\pi)=0, \quad t\in[0,b], \\ y(\nu,\varrho)=\zeta(\nu,\varrho)+\sum_{i=1}^r\int_0^\pi k(\varrho,p,\nu)\cos(y(t_i,p))\mathrm{d}p, \quad -\tau\le\nu\le 0,\end{cases} \tag{4.1}$$

where $f$ is a given continuous function, $k\in C([0,\pi]\times[0,\pi]\times[-\tau,0],\mathbb{R})$ and $\zeta\in C([-\tau,0]\times[0,\pi],\mathbb{R})$.

Let $Y=L^2([0,\pi],\mathbb{R})$. Define an operator $A$ on $Y$ which is defined as $Ay=-y''$ with

$$D(A)=\{y\in Y:\ y,y' \text{ are absolutely continuous, } y''\in Y \text{ and } y(0)=y(\pi)=0\}.$$

It is well known that the operator $-A$ generates a compact and analytic semigroup $\{T(t), t\ge 0\}$ which is self-adjoint in $Y$. Moreover, the operator $A$ and semigroup $\{T(t)\}$ are given respectively by

$$Av=\sum_{n=1}^\infty n^2\langle v,e_n\rangle e_n, \quad v\in D(A)$$

and

$$T(t)v=\sum_{n=1}^\infty \exp(-n^2 t)\langle v,e_n\rangle e_n, \quad v\in Y,$$

where $e_n(\varrho)=\sqrt{\frac{2}{\pi}}\sin(n\varrho)$, $n\in\mathbb{N}$. Obviously, the set $\{e_n: n\in\mathbb{N}\}$ is an orthonormal basis for $Y$. For $\alpha=\frac{1}{2}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}v=\sum_{n=1}^\infty n\langle v,e_n\rangle e_n, \quad v\in D(A^{\frac{1}{2}}),$$

where $D(A^{\frac{1}{2}})=\{v\in Y:\ \sum_{n=1}^\infty n\langle v,e_n\rangle e_n\in Y\}$. In particular, $\|A^{-\frac{1}{2}}\|=1$.

Let $Y_{\frac{1}{2}} = (D(A^{\frac{1}{2}}), \|\cdot\|_{\frac{1}{2}})$, $W = I$ and $U = Y_{\frac{1}{2}}$, where $\|y\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}y\|$ for $y \in D(A^{\frac{1}{2}})$. We define operators $\gamma(t)\colon D(A) \subset Y \to Y$ and $K(t)\colon Y \to Y$ respectively by $\gamma(t)y = \mathrm{e}^{-vt}Ay$ for $y \in D(A)$ and $K(t)y = t^m\mathrm{e}^{-lt}y$ for $y \in Y$. Moreover, we define $y(t)(\varrho) = y(t, \varrho)$, $f(t, y_t)(\varrho) = f(t, \varrho, y(t-\tau, \varrho))$, $u(t)(\varrho) = u(t, \varrho)$, $\eta(y)(\nu)(\varrho) = \sum_{i=1}^{r}\int_0^{\pi} k(\varrho, p, \nu)\cos(y(t_i, p))\mathrm{d}p$ and $\zeta(\nu)(\varrho) = \zeta(\nu, \varrho)$.

By the above notations and conditions, the system (4.1) can be represented in the abstract form (1.2). It is easy to see that the conditions (A1)–(A5) are satisfied with $\hat{K}(\omega) = \frac{\Gamma(m+1)}{(\omega+l)^{m+1}}I$, $\hat{\gamma}(\omega) = \frac{1}{\omega+v}A$ and $E = C_0^{\infty}([0,\pi])$, where $C_0^{\infty}([0,\pi])$ is the space of infinitely differential functions vanishing at 0 and $\pi$. Then the linear system of (4.1) has a resolvent operator $\mathscr{R}(\cdot)\colon [0,\infty) \to \mathcal{L}(Y)$ that is defined as

$$\mathscr{R}(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\vartheta}} \mathrm{e}^{\omega t}(\omega I + \omega\hat{K}(\omega) + A - \hat{\gamma}(\omega))^{-1}\,\mathrm{d}\omega, & t > 0, \\ I, & t = 0. \end{cases} \tag{4.2}$$

We now assume the following conditions:

(1) The function $f\colon [0,b]\times[0,\pi]\times\mathbb{R} \to \mathbb{R}$ satisfies that

  (i) $f(t,\varrho,\cdot)$ is continuous for each $(t,\varrho) \in [0,b]\times[0,\pi]$ and $f(\cdot,\cdot,\kappa)$ is strongly measurable for each $\kappa \in \mathbb{R}$.

  (ii) $f(t,\cdot,\kappa)$ is differentiable for $t \in [0,b]$, $\kappa \in \mathbb{R}$ such that $\frac{\partial}{\partial\varrho}f(t,\varrho,\kappa) \in Y$ and $f(\cdot,0,\cdot) = f(\cdot,\pi,\cdot) = 0$, and

$$\left|\frac{\partial}{\partial\varrho}f(t,\varrho,\kappa)\right| \le C,\ \forall\,(t,\varrho,\kappa) \in [0,b]\times[0,\pi] \times\mathbb{R},\ \text{and for some constant } C > 0.$$

(2) The function $k\colon [0,\pi]\times[0,\pi]\times[-\tau,0] \to \mathbb{R}$ is continuously differentiable with $k(0,\cdot,\cdot) = k(\pi,\cdot,\cdot) = 0$ and

$$N = \sup_{\nu\in[-\tau,0]}\int_0^{\pi}\int_0^{\pi}\left|\frac{\partial^2 k(\varrho,p,\nu)}{\partial\varrho^2}\right|^2\mathrm{d}p\,\mathrm{d}\varrho < \infty.$$

For any $y \in C([-\tau,b],Y_{\frac{1}{2}})$, $t \in [0,b]$, we have

$$\begin{aligned} \|A^{\frac{1}{2}}f(t,y_t)\|^2 &= \sum_{n=1}^{\infty} n^2|\langle f(t,y_t)(\varrho), e_n(\varrho)\rangle|^2 \\ &= \sum_{n=1}^{\infty}\left|\left\langle\frac{\partial}{\partial\varrho}f(t,\varrho,y(t-\tau,\varrho)), \sqrt{\frac{2}{\pi}}\cos(n\varrho)\right\rangle\right|^2 \\ &\le \int_0^{\pi}\left(\frac{\partial}{\partial\varrho}f(t,\varrho,y(t-\tau,\varrho))\right)^2\mathrm{d}\varrho, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \|A^{\frac{1}{2}}\eta(y)(\nu)\|^2 &= \sum_{n=1}^{\infty} n^2|\langle\eta(y)(\nu)(\varrho), e_n(\varrho)\rangle|^2 \\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\left|\left\langle\frac{\partial^2}{\partial\varrho^2}\eta(y)(\nu)(\varrho), e_n(\varrho)\right\rangle\right|^2 \end{aligned}$$

$$\leq \frac{\pi^2}{6}\int_0^{\pi}\left(\frac{\partial^2}{\partial \varrho^2}\eta(y)(\nu)(\varrho)\right)^2 \mathrm{d}\varrho$$
$$\leq \frac{\pi^2}{6}\int_0^{\pi}\left(\frac{\partial^2}{\partial \varrho^2}\sum_{n=1}^{r}\int_0^{\pi} k(\varrho, p, \nu)\cos(y(t_n, p))\,\mathrm{d}p\right)^2 \mathrm{d}\varrho$$
$$\leq \frac{r^2\pi^2}{6}\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial \varrho^2}k(\varrho, p, \nu)\right)^2 \mathrm{d}p\,\mathrm{d}\varrho \tag{4.4}$$

and

$$\|A^{\frac{1}{2}}[\eta(y)(\nu) - \eta(z)(\nu)]\|^2 \leq \frac{\pi^2 r^2}{6}\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial \varrho^2}k(\varrho, p, \nu)\right)^2 \mathrm{d}p\,\mathrm{d}\varrho \sup_{t\in[-\tau,b]}\|y(t,\cdot) - z(t,\cdot)\|_Y^2. \tag{4.5}$$

We obtain from (4.3), (4.4) and (4.5) that the functions $f$ and $g$ satisfy the assumptions (H1) and (H2) respectively, and both functions are uniformly bounded. Since the semigroup $T(t)$ is compact, it is clear from Lemma 2.6 and Theorem 2.7 that the resolvent operator $\mathscr{R}(t)$ is compact for all $t > 0$. Therefore, the associated linear system of (4.1) can not be exactly controllable but it may be approximately controllable. For any $y \in Y$, we get

$$\begin{aligned}\langle W^*\mathscr{R}^*(t)y, v\rangle &= \langle \mathscr{R}^*(t)y, v\rangle \\ &= \langle y, \mathscr{R}(t)v\rangle, \quad \forall\, v \in U \text{ and } \forall\, t \in J.\end{aligned}$$

Let $W^*\mathscr{R}^*(t)y = 0$. Then

$$\langle y, \mathscr{R}(t)v\rangle = 0, \quad \forall\, v \in U \text{ and } \forall\, t \in J.$$

Since $\mathscr{R}(0) = I$, we get for $t = 0$,

$$\langle y, v\rangle = 0, \quad \forall\, v \in U.$$

This implies that $y = 0$ as $U$ is dense in $Y$. By Lemma 2.9 and Remark 2.10, the linear system of (4.1) is approximately controllable, that is, the condition (H0) holds. Therefore, in view of Theorems 3.1 and 3.3, the neutral integro-differential system (4.1) is approximately controllable on $[0, b]$.

## Acknowledgements

The author would like to express his gratitude to Prof. N Sukavanam, IIT Roorkee and the reviewers for their valuable suggestions. He would also like to acknowledge the Science and Engineering Research Board, a statutory body of the Department of Science and Technology (DST), Government of India for supporting this research work under Grant PDF/2016/003875.

## References

[1] Andrade B and Santos J, Existence of solutions for a fractional neutral integro-differential equation with unbounded delay, *Electron. J. Differ. Equ.* **2012(90)** (2012) 1–13

[2] Balasubramaniam P and Tamilalagan P, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Appl. Math. Comput.* **256** (2015) 232–246

[3] Conway J B, A Course in Functional Analysis, Graduate Texts in Mathematics (1997) (New York: Springer-Verlag)

[4] Curtain R F and Zwart H, An Introduction to Infinite-Dimensional Linear Systems Theory, Texts in Applied Mathematics, vol. 21 (1995) (New York: Springer-Verlag)

[5] Ezzinbi K, Toure H and Zabsonre I, Local existence and regularity of solutions for some partial functional integrodifferential equations with infinite delay in Banach spaces, *Nonlinear Anal.* **70(9)** (2009) 3378–3389

[6] Farahi S and Guendouzi T, Approximate controllability of fractional neutral stochastic evolution equations with nonlocal conditions, *Results Math.* **65(3–4)** (2014) 501–521

[7] Fu X, Lu J and You Y, Approximate controllability of semilinear neutral evolution systems with delay, *Internat. J. Control* **87(4)** (2014) 665–681

[8] George R K, Approximate controllability of nonautonomous semilinear systems, *Nonlinear Anal.: Theory, Methods Appl.* **24**(9) (1995) 1377–1393

[9] Grimmer R, Resolvent operators for integral equations in a Banach space, *Trans. Amer. Math. Soc.* **273(1)** (1982) 333–349

[10] Grimmer R and Pritchard A, Analytic resolvent operators for integral equations in Banach space, *J. Differ. Equ.* **50(2)** (1983) 234–259

[11] Grimmer R and Kappel F, Series expansions for resolvents of Volterra integro-differential equations in Banach space, *SIAM J. Math. Anal.* **15(3)** (1984) 595–604

[12] Gripenberg G, Londen S and Staffans O, Volterra Integral and Functional Equations, Encyclopedia of Mathematics and its Applications (1990) (Cambridge: Cambridge University Press)

[13] Henríquez H and Santos J, Differentiability of solutions of abstract neutral integro-differential equations, *J. Integral Equ. Appl.* **25(1)** (2013) 47–77

[14] Hernández E, Henríquez H and Santos J, Existence results for abstract partial neutral integro-differential equation with unbounded delay, *Electron. J. Qual. Theory Differ. Equ.* **2009(29)** (2009) 1–23

[15] Kamal Jeet and Bahuguna D, Approximate controllability of nonlocal neutral fractional integro-differential equations with finite delay, *J. Dynamical Control Systems* (2015) pp. 1–20 (Online published)

[16] Kumar R, Nonlocal Cauchy problem for analytic resolvent integrodifferential equations in Banach spaces, *Appl. Math. Comput.* **204(1)** (2008) 352–362

[17] Kumar S and Sukavanam N, Approximate controllability of fractional order semilinear system with bounded delay, *J. Differ. Equ.* **252(11)** (2012) 6163–6174

[18] Machado J A, Ravichandran C, Rivero M and Trujillo J, Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions, *Fixed Point Theory Appl.* **2013(66)** (2013) 1–16

[19] Mahmudov N I and Zorlu S, On the approximate controllability of fractional evolution equations with compact analytic semigroup, *J. Computational Appl. Math.* **259**, part A (2014) 194–204

[20] Mahmudov N I, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, *SIAM J. on Control and Optimization* **42(5)** (2003) 1604–1622

[21] Mokkedem F and Fu X, Approximate controllability of semi-linear neutral integro-differential systems with finite delay, *Appl. Math. Comput.* **242** (2014) 202–215

[22] Pazy A, Semigroup of Linear Operators and Applications to Partial Differential Equations (1983) (New York: Springer-Verlag)

[23] Prüss J, Evolutionary Integral Equations and Applications, Monographs in Mathematics (1993) (Basel: Birkhäuser Verlag)

[24] Sakthivel R, Ganesh R, Ren Y and Anthoni S M, Approximate controllability of nonlinear fractional dynamical systems, *Comm. Nonlinear Sci. Numerical Simulation* **18(1@)** (2013) 3498–3508

[25] Santos J, Henríquez H and Hernández E, Existence results for neutral integro-differential equations with unbounded delay, *J. Integral Equ. Appl.* **23(2)** (2011) 289–330
[26] Tai Z and Lun S, On controllability of fractional impulsive neutral infinite delay evolution integrodifferential systems in Banach spaces, *Appl. Math. Lett.* **25(2)** (2012) 104–110
[27] Wang J, Fan Z and Zhou Y, Nonlocal controllability of semilinear dynamic system with fractional derivative in Banach spaces, *J. Optimization Theory Appl.* **154(1)** (2012) 292–302
[28] Wang W and Zhou Y, Complete controllability of fractional evolution systems, *Comm. Nonlinear Sci. Numerical Simulation* **17(11)** (2012) 4346–4355
[29] Yan Z and Jia X, Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay, *Collect. Math.* **66(1)** (2015) 93–124
[30] Yan Z and Lu F, On approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay, *Appl. Anal.* **94(6)** (2015) 1235–1258
[31] Zhou H X, Approximate controllability for a class of semilinear abstract equations, *SIAM J. Control and Optimization* **21(4)** (1983) 551–565

COMMUNICATING EDITOR: Parameswaran Sankaran