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On Euler–Ramanujan formula, Dirichlet series and minimal surfaces

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Abstract. In this paper, we rewrite two forms of an Euler–Ramanujan identity in terms of certain Dirichlet series and derive functional equation of the latter. We also use the Weierstrass–Enneper representation of minimal surfaces to obtain some identities involving these Dirichlet series and one complex parameter.

Keywords. Euler–Ramanujan identities; Dirichlet series; minimal surfaces.

Mathematics Subject Classification. 11-XX; 53-XX.

1. Introduction

In this article, we explore two different forms of the Euler–Ramanujan identities and their relationship to certain Dirichlet series and minimal surfaces. The use of Weierstrass–Enneper representation of minimal surfaces in E–R identities first appeared in [2]. When we rewrite the identities in terms of sums of Dirichlet series we get relationships between the latter and, sometimes using the W–E representation of minimal surfaces the relationship is in terms of one-complex parameter. The minimal surfaces which are involved are a family of Scherk's surfaces and helicoids which are minimal surfaces of translation. We devote Section 2, to relationships between sums of Dirichlet series. In Section 3, we study the properties of one of the Dirichlet series and derive a functional equation of the latter.

2. Rewriting a form of the Euler–Ramanujan identity in terms of Dirichlet series

We have Euler–Ramanujan's identity in Example (1), page 38 of [6], where X, A are complex variables and A is not an odd multiple of $\pi/2$:

$$\frac{\cos(X+A)}{\cos A} = \prod_{k=1}^{\infty}\left(1-\frac{X}{(k-\frac{1}{2})\pi - A}\right)\left(1+\frac{X}{(k-\frac{1}{2})\pi + A}\right).$$

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We take complex logarithm on both sides to get

$$\log\left(\frac{\cos(X+A)}{\cos A}\right)=\sum_{k=1}^{\infty}\log\left\{\left(\frac{(k-\frac{1}{2})\pi-(X+A)}{(k-\frac{1}{2})\pi-A}\right)\left(\frac{(k-\frac{1}{2})\pi+(X+A)}{(k-\frac{1}{2})\pi+A}\right)\right\}.$$

Let $c_k=(k-\frac{1}{2})\pi$. Replace $X+A$ by y and A by x in the identity where x (not an odd multiple of $\frac{\pi}{2}$) and y are real parameters. We then have

$$\log\left(\frac{\cos y}{\cos x}\right)=\sum_{k=1}^{\infty}\log\left(\frac{c_k-y}{c_k-x}\right)\left(\frac{c_k+y}{c_k+x}\right) \tag{1}$$

or,

$$\log\left(\frac{\cos y}{\cos x}\right)=\sum_{k=1}^{\infty}\left[\log\left(1+\frac{x^2}{c_k^2-x^2}\right)+\log\left(1-\frac{y^2}{c_k^2}\right)\right].$$

Let us take the special case when $|\frac{x^2}{c_k^2-x^2}|<1$ and $|\frac{y^2}{c_k^2}|<1$. Then

$$\begin{aligned}\log\left(\frac{\cos y}{\cos x}\right)&=\sum_{k=1}^{\infty}\left[\log\left(1+\frac{x^2}{c_k^2-x^2}\right)+\log\left(1-\frac{y^2}{c_k^2}\right)\right]\\&=\sum_{k=1}^{\infty}\left[\left(\frac{x^2}{c_k^2-x^2}-\frac{x^4}{2(c_k^2-x^2)^2}+\frac{x^6}{3(c_k^2-x^2)^3}-\cdots\right)\right.\\&\qquad\left.+\left(-\frac{y^2}{c_k^2}-\frac{y^4}{2c_k^4}-\frac{y^6}{3c_k^6}\cdots\right)\right]\\&=\sum_{k=1}^{\infty}\left[L_k\left(1,\frac{x^2}{c_k^2-x^2}\right)-M_k\left(1,\frac{y^2}{c_k^2}\right)\right],\end{aligned}$$

where $L_k(s,a)=\sum_n((-1)^{n+1}(\frac{x^2}{c_k^2-x^2})^n/n^s)$ is a Dirichlet series with a real paramater $a=\frac{x^2}{c_k^2-x^2}$ and an integer k, $M_k(s,b)=\sum_n(\frac{y^2}{c_k^2})^n/n^s$ is a Dirichlet series with a real parameter $b=\frac{y^2}{c_k^2}$ and an integer k and $L_k(1,\frac{x^2}{c_k^2-x^2})$ and $M_k(1,\frac{y^2}{c_k^2})$ are these Dirichlet series evaluated at $s=1$. We note that $|\frac{x^2}{c_k^2-x^2}|<1$ and $|\frac{y^2}{c_k^2}|<1$ for all k is equivalent to $-\frac{\pi}{2\sqrt{2}}<x<\frac{\pi}{2\sqrt{2}}$ and $-\sqrt{\pi/2}<y<\sqrt{\pi/2}$. Thus we have the following Proposition.

PROPOSITION 2.1

$\log\left(\frac{\cos y}{\cos x}\right) = \sum_{k=1}^{\infty}[L_k(1, \frac{x^2}{c_k^2-x^2}) - M_k(1, \frac{y^2}{c_k^2})]$ *for* $-\frac{\pi}{2\sqrt{2}} < x < \frac{\pi}{2\sqrt{2}}$ *and* $-\sqrt{\pi/2} < y < \sqrt{\pi/2}$.

In [2], using the Weierstrass–Enneper representation of minimal surfaces we derived the following way of writing the equation $z = \log\left(\frac{\cos y}{\cos x}\right)$ (Scherk's minimal surface) in parametric form (in terms of a complex parameter ζ), $x(\zeta, \bar{\zeta}) = 2\mathrm{Re}\tan^{-1}(\zeta)$, $y(\zeta, \bar{\zeta}) = -\mathrm{Im}\log\left(\frac{1+\zeta}{1-\zeta}\right)$ and $z(\zeta, \bar{\zeta}) = \mathrm{Re}\log\left(\frac{1+\zeta^2}{1-\zeta^2}\right)$. This parametrization fails precisely at $\zeta = \pm 1, \pm i$.

Using the log version of the E–R identity, for $\zeta \neq \pm 1, \pm i$ and belonging to a small domain in $\mathbb{C}$, the expression in terms of Dirichlet series will be as follows:

$$\begin{aligned}\mathrm{Re}\log\left(\frac{1+\zeta^2}{1-\zeta^2}\right) &= \sum_{k=1}^{\infty}\left[\log\left(1 + \frac{(2\mathrm{Re}\tan^{-1}(\zeta))^2}{c_k^2 - (2\mathrm{Re}\tan^{-1}(\zeta))^2}\right) + \log\left(1 - \frac{\left(-\mathrm{Im}\log\left(\frac{1+\zeta}{1-\zeta}\right)\right)^2}{c_k^2}\right)\right] \\ &= \sum_{k=1}^{\infty}\left[L_k\left(1, \frac{(2\mathrm{Re}\tan^{-1}(\zeta))^2}{c_k^2 - (2\mathrm{Re}\tan^{-1}(\zeta))^2}\right) - M_k\left(1, \frac{\left(-\mathrm{Im}\log\left(\frac{1+\zeta}{1-\zeta}\right)\right)^2}{c_k^2}\right)\right].\end{aligned}$$

The condition that $|\frac{x^2}{c_k^2-x^2}| < 1$ and $|\frac{y^2}{c_k^2}| < 1$ for all k is satisfied if $|\zeta| < \frac{1}{2}$. This can be seen as follows:

For $\zeta \in \mathbb{C}$ such that $|\zeta| < \frac{1}{2}$, we have $-2 < 2\frac{|\zeta|\cos\theta}{1-|\zeta|^2} < 2$, which implies $-\tan\left(\frac{\pi}{2\sqrt{2}}\right) < 2\frac{|\zeta|\cos\theta}{1-|\zeta|^2} < \tan\left(\frac{\pi}{2\sqrt{2}}\right)$. In other words, $-\tan\left(\frac{\pi}{2\sqrt{2}}\right) < \frac{\zeta+\bar{\zeta}}{1-|\zeta|^2} < \tan\left(\frac{\pi}{2\sqrt{2}}\right)$ and hence $-\frac{\pi}{2\sqrt{2}} < \tan^{-1}(\zeta) + \tan^{-1}(\bar{\zeta}) < \frac{\pi}{2\sqrt{2}}$, or $-\frac{\pi}{2\sqrt{2}} < 2\mathrm{Re}\tan^{-1}(\zeta) < \frac{\pi}{2\sqrt{2}}$. This implies $|\frac{x^2}{c_k^2-x^2}| < 1$, where $x = 2\mathrm{Re}\tan^{-1}(\zeta)$. From this, it follows that $|\frac{x^2}{c_k^2-x^2}| < 1$ for all k, $y^2/c_k^2 < 1$ does not give any additional constraint on ζ.

Thus, we have the following Proposition.

PROPOSITION 2.2

For $\zeta \in \mathbb{C}$ *such that* $|\zeta| < \frac{1}{2}$,

$$\begin{aligned}\mathrm{Re}\log\left(\frac{1+\zeta^2}{1-\zeta^2}\right) = \sum_{k=1}^{\infty}\Bigg[&L_k\left(1, \frac{(2\mathrm{Re}\tan^{-1}(\zeta))^2}{c_k^2 - (2\mathrm{Re}\tan^{-1}(\zeta))^2}\right) \\ &- M_k\left(1, \frac{\left(-\mathrm{Im}\log\left(\frac{1+\zeta}{1-\zeta}\right)\right)^2}{c_k^2}\right)\Bigg].\end{aligned}$$

2.1 *More examples of minimal surfaces and their relation to Dirichlet series*

In this section, we will revisit the *Scherk's* surface [7], and the *helicoid* [4,5]. In fact, we consider a one-parameter family of *Scherk's* type surface. We show that all these surfaces will give rise to a Dirichlet series decomposition. We consider the case of *helicoid* separately.

We consider a one-parameter family of minimal translation surfaces (*Scherk's* type surfaces),

$$X_\theta(u,v) = \left(u + v\cos\theta, v\sin\theta, \log\frac{\cos v}{\cos u}\right),$$

where $\theta \in \mathbb{R}$. Indeed, for a fixed θ, $X_\theta(u,v) = \alpha(u) + \beta_\theta(v)$, where $\alpha(u) = (u, 0, -\log\cos u)$ and $\beta_\theta(v) = (v\cos\theta, v\sin\theta, \log\cos v)$. Hence it is a minimal surface of translation (see Remark at the end of this section).

Remark 2.3. Observe that $\theta = 0$ corresponds to a *plane* in the above family and $\theta = \frac{\pi}{2}$ corresponds to the classical *Scherk′s* surface, i.e., $X_{\frac{\pi}{2}}(u,v) = (u, v, \log\frac{\cos v}{\cos u})$ (discussed in the previous section).

The θ-family of minimal translation surfaces gives us a family of Euler–Ramanujan's identities. For $\theta \neq \pm\frac{(2n+1)\pi}{2}$, the corresponding identity, we call it a *twisted* Euler-Ramanujan's identity. Indeed, put $u + v\cos\theta = x$ and $v\sin\theta = y$ and then using the Euler–Ramanujan's identity, we obtain

$$\frac{\cos(\frac{y}{\sin\theta})}{\cos(x - y\cot\theta)} = \prod_{k=1}^{\infty}\left(\frac{c_k^2 - \frac{y^2}{\sin^2\theta}}{c_k^2 - (x - y\cot\theta)^2}\right),$$

where $x - y\cot\theta$ is not an odd multiple of $\frac{\pi}{2}$. Taking log on both sides, we get

$$\log\left(\frac{\cos(\frac{y}{\sin\theta})}{\cos(x - y\cot\theta)}\right) = \sum_{k=1}^{\infty}\log\left(\frac{c_k^2 - \frac{y^2}{\sin^2\theta}}{c_k^2 - (x - y\cot\theta)^2}\right).$$

When we take θ as an odd multiple of $\frac{\pi}{2}$ in the above identity, we get back identity (1). Following the same idea (as in Scherk's surface case), we have a Dirichlet series decomposition in this case as well, which is as follows.

PROPOSITION 2.4

$$\log\left(\frac{\cos(\frac{y}{\sin\theta})}{\cos(x - y\cot\theta)}\right) = \sum_{k=1}^{\infty}\left[L_k\left(1, \frac{(x - y\cot\theta)^2}{c_k^2 - (x - y\cot\theta)^2}\right) - M_k\left(1, \left(\frac{y}{c_k\sin\theta}\right)^2\right)\right],$$

where $L_k\left(1, \frac{(x-y\cot\theta)^2}{c_k^2-(x-y\cot\theta)^2}\right)$ *is a Dirichlet series (evaluated at* $s = 1$*) with a real paramater* $a = \frac{(x-y\cot\theta)^2}{c_k^2-(x-y\cot\theta)^2}$ *for a fixed* θ *and an integer* k, $M_k(s, (\frac{y}{c_k\sin\theta})^2)$, *(*$\theta$*-fixed), is a Dirichlet series (evaluated at* $s = 1$*) with a real parameter* $b = (\frac{y}{c_k\sin\theta})^2$ *and an integer* k *as before.*

Now, let us consider the non parametric representation of helicoid which is given by $z = \tan^{-1}\frac{y}{x}$ and we also recall the identity

$$\tan^{-1}\omega = \frac{i}{2}(\log(1 - i\omega) - \log(1 + i\omega)),$$

where ω is a complex number. Now if we put $\omega = \frac{y}{x}$ in the above identity, we get

$$\tan^{-1}\frac{y}{x} = \frac{i}{2}\left\{\log\left(1 - i\frac{y}{x}\right) - \log\left(1 + i\frac{y}{x}\right)\right\}. \tag{2}$$

The above expression helps us to write helicoid as a sum of two Dirichlet series evaluated at a specific value. When $|y| < |x|$, we can express the identity (2) as

$$\begin{aligned}\tan^{-1}\frac{y}{x} &= \frac{i}{2}\left\{\sum_{k=1}^{\infty}(-1)^{2k-1}\frac{i^k}{k}\left(\frac{y}{x}\right)^k - \sum_{k=1}^{\infty}(-1)^{k+1}\frac{i^k}{k}\left(\frac{y}{x}\right)^k\right\}\\ &= \frac{1}{2}\left\{-\sum_{k=1}^{\infty}\frac{i^{k+1}}{k}\left(\frac{y}{x}\right)^k + \sum_{k=1}^{\infty}(-1)^k\frac{i^{k+1}}{k}\left(\frac{y}{x}\right)^k\right\}\\ &= \frac{-1}{2}\left[L\left(1, \frac{y}{x}\right) - M\left(1, -\frac{y}{x}\right)\right],\end{aligned} \tag{3}$$

where $L(s, \frac{y}{x}) = \sum_{k=1}^{\infty}\frac{i^{k+1}}{k^s}(\frac{y}{x})^k$ and $M(s, -\frac{y}{x}) = \sum_{k=1}^{\infty}\frac{i^{k+1}}{k^s}(-\frac{y}{x})^k$ are Dirichlet series which are in turn evaluated at $s = 1$. Thus we have the following proposition.

PROPOSITION 2.5

When $|y| < |x|$, $\tan^{-1}\frac{y}{x} = \frac{1}{2}[L(1, \frac{y}{x}) - M(1, -\frac{y}{x})]$.

The Weierstrass–Enneper representation of the helicoid in terms of the complex parameter is given by (see for instance [2]) $x(\zeta, \bar{\zeta}) = -\frac{1}{2}\mathrm{Im}\left(\zeta + \frac{1}{\zeta}\right)$, $y(\zeta, \bar{\zeta}) = \frac{1}{2}\mathrm{Re}\left(\zeta - \frac{1}{\zeta}\right)$ and $z(\zeta, \bar{\zeta}) = -\frac{\pi}{2} + \mathrm{Im}(\log\zeta)$. The condition $|y| < |x|$ translates to $|\mathrm{Re}(\zeta - \frac{1}{\zeta})| < |\mathrm{Im}(\zeta + \frac{1}{\zeta})|$ which is satisfied if $|\zeta| < 1$. Now it is easy to see that we have as follows.

PROPOSITION 2.6

For $\zeta \in \mathbb{C}$, *such that* $|\zeta| < 1$,

$$\begin{aligned}-\frac{\pi}{2} + \mathrm{Im}(\log\zeta) = \frac{-1}{2}&\left\{L\left(1, \frac{\mathrm{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\mathrm{Im}\left(\zeta + \frac{1}{\zeta}\right)}\right)\right.\\ &\left.-M\left(1, -\frac{\mathrm{Re}\left(\zeta - \frac{1}{\zeta}\right)}{-\mathrm{Im}\left(\zeta + \frac{1}{\zeta}\right)}\right)\right\}.\end{aligned}$$

2.2 *Another Euler–Ramanujan identity*

Next we look at another identity. For X and A real, we have (see Entry 11 in [6])

$$\tan^{-1}(\tanh X \cot A) = \tan^{-1}\left(\frac{X}{A}\right) + \sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{X}{k\pi + A}\right) - \tan^{-1}\left(\frac{X}{k\pi + A}\right)\right). \quad (4)$$

When $A = \frac{\pi}{2}$, the identity (4) reduces to the following identity:

$$\tan^{-1}\left(\frac{2X}{\pi}\right) = \sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{X}{c_k}\right) - \tan^{-1}\left(\frac{X}{d_k}\right)\right), \quad (5)$$

where $c_k = (k - \frac{1}{2})\pi$ and $d_k = (k + \frac{1}{2})\pi$. Next we put $\frac{2X}{\pi} = \frac{y}{x}$ in (5) to obtain

$$\tan^{-1}\left(\frac{y}{x}\right) = \sum_{k=1}^{\infty}\left(\tan^{-1}\left(\frac{e_k y}{x}\right) - \tan^{-1}\left(\frac{f_k y}{x}\right)\right), \quad (6)$$

where $e_k = \frac{\pi}{2c_k}$ and $f_k = \frac{\pi}{2d_k}$. Thus we have as follows.

PROPOSITION 2.7

$$L\Big(1, \frac{y}{x}\Big) - M\Big(1, -\frac{y}{x}\Big) = \sum_{k=1}^{\infty}\left(L_k\Big(1, \frac{e_k y}{x}\Big) - M_k\Big(1, -\frac{e_k y}{x}\Big)\right) - \left(L_k\Big(1, \frac{f_k y}{x}\Big) - M_k\Big(1, -\frac{f_k y}{x}\Big)\right),$$

where $L_k(1, \frac{e_k y}{x}) = L(1, \frac{e_k y}{x})$ *and* $M_k(1, -\frac{e_k y}{x}) = M(1, -\frac{f_k y}{x})$.

Remark 2.8. Our object of interest here happen to be minimal surfaces of translation. In the past, there has been a good amount of interest in knowing what are all the minimal translation surfaces in $\mathbb{R}^3$ (see for instance, [3]). It would be interesting to see if number-theoretical identities like the E–R identities are available for all minimal surfaces of translation.

3. Properties of the Dirichlet series

We consider the general form of the Dirichlet series defined in the previous section

$$L_k(s, a) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{a^n}{n^s}, \quad \text{where } a = \frac{x^2}{c_k^2 - x^2}, 0 < a < 1,\ s \in \mathbb{C},$$

$$M_k(s, a) = \sum_{n=1}^{\infty}(-1)^{n}\frac{a^n}{n^s}, \quad \text{where } a = \frac{y^2}{c_k^2}, 0 < a < 1,\ s \in \mathbb{C}.$$

3.1 *Convergence of the series*

We show that the series is convergent for all $s \in \mathbb{C}$ by showing its absolute convergence. Thus

$$L_k(s,a) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}, \quad 0 < a < 1, \; s = \sigma + ib, \; \sigma, b \in \mathbb{R}.$$

Now, $|n^s| = |n^\sigma||n^{ib}| = n^\sigma |e^{ib \ln n}| = n^\sigma$.

The absolute series then is

$$\sum_{n=1}^{\infty} \frac{|a|^n}{n^\sigma} = \sum_{n=1}^{\infty} \frac{1}{|c|^n n^\sigma}, \quad a = 1/c, c > 1.$$

Then the ratio test gives

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|c|^n n^\sigma}{|c|^{n+1}(n+1)^\sigma} = \lim_{n\to\infty} \frac{1}{|c|} \frac{1}{(1+1/n)^\sigma} = \frac{1}{c} < 1.$$

Thus $L_k(s,a)$ is convergent on $0 < a < 1$ for each k. Similar is the case for $M_k(s,a)$.

3.2 *Functional equation*

In this section, we follow a technique of Hardy, expounded in [1].

Our general Dirichlet series is $\gamma(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}$, where, $0 < a < 1$, $s \in \mathbb{C}$. Then, defining $F(x) = \sum_{n \le x} (-1)^{n-1} = \frac{1-(-1)^m}{2}$, $m < x < m+1$, we have

$$\begin{aligned}
\gamma(s) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s} \\
&= \sum_{n=1}^{\infty} [F(n) - F(n-1)] \frac{a^n}{n^s} \\
&= \sum_{n=1}^{\infty} F(n) \frac{a^n}{n^s} - \sum_{n=1}^{\infty} F(n) \frac{a^{n+1}}{(n+1)^s} - F(0) \cdot a \\
&= \sum_{n=1}^{\infty} F(n) \int_n^{n+1} \Big(\frac{a^x \ln a}{x^s} - \frac{s a^x}{x^{s+1}} \Big) \mathrm{d}x \\
&= \ln a \sum_{n=1}^{\infty} \int_n^{n+1} \frac{a^x F(x)}{x^s} - s \sum_{n=1}^{\infty} \int_n^{n+1} \frac{a^x F(x)}{x^{s+1}} \mathrm{d}x \\
&= \ln a \int_1^{\infty} \frac{a^x F(x)}{x^s} - s \int_1^{\infty} \frac{a^x F(x)}{x^{s+1}} \mathrm{d}x.
\end{aligned}$$

Let

$$\begin{aligned} f(x) &= \ln a \int_1^\infty \frac{a^x F(x)}{x^s} \\ &= \ln a \int_1^\infty \frac{a^x (F(x) - 1/2)}{x^s} \mathrm{d}x + \frac{\ln a}{2} \int_1^\infty \frac{a^x}{x^s} \mathrm{d}x \\ &= I_1(s) + I_2(s) \text{ (say)}, \end{aligned}$$

and

$$\begin{aligned} g(x) &= s \int_1^\infty \frac{a^x F(x)}{x^{s+1}} \mathrm{d}x \\ &= s \int_1^\infty \frac{a^x (F(x) - 1/2)}{x^{s+1}} \mathrm{d}x + \frac{s}{2} \int_1^\infty \frac{a^x}{x^{s+1}} \mathrm{d}x \\ &= I_3(s) + I_4(s) \text{ (say)}. \end{aligned}$$

The integrals can all be seen to be convergent for all s.

Let $F(x) - \frac{1}{2}$ be bounded, say by M. Then, taking $a = \frac{1}{c}, c > 1$ we can see that $\int_1^\infty |\frac{a^x (F(x) - \frac{1}{2})}{x^s}| \mathrm{d}x = \int_1^\infty |\frac{(F(x)-\frac{1}{2})}{|c|^x x^\sigma}| \mathrm{d}x \leq M \int_1^\infty \frac{1}{c^x x^\sigma} \mathrm{d}x$ (for $\sigma > 0$), which is convergent. Then I_1 and similarly I_2 are thus absolutely convergent since c^x is exponential growth and x^σ has only polynomial growth. I_3 and I_4 can similarly be concluded to be convergent.

We try to evaluate I_4 first. Let $x \ln a = u \Rightarrow \mathrm{d}u = \ln a \, \mathrm{d}x$. Then u ranges from $\ln a$ to $-\infty$ as x ranges from 1 to ∞.

Then

$$\begin{aligned} I_4(s) &= \frac{s}{2} \int_1^\infty \frac{a^x}{x^{s+1}} \mathrm{d}x \\ &= \frac{s}{2} \int_1^\infty x^{-s-1} \mathrm{e}^{x \ln a} \mathrm{d}x \\ &= \frac{s}{2 \ln a} \int_{\ln a}^{-\infty} \left(\frac{u}{\ln a}\right)^{-s-1} \mathrm{e}^u \mathrm{d}u \\ &= \frac{s (\ln a)^s}{2} \int_{\ln a}^{-\infty} u^{-s-1} \mathrm{e}^u \mathrm{d}u \\ &= \frac{-s (\ln a)^s}{2} \int_{-\ln a}^{\infty} (-1)^{-s-1} u^{-s-1} \mathrm{e}^{-u} \mathrm{d}u \\ &= \frac{s}{2} (-\ln a)^s \, \Gamma(-s, -\ln a). \end{aligned}$$

Similar to $I_4(s)$, $I_2(s)$ will then be $-\frac{(-\ln(a))^s}{2} \Gamma(-(s-1), -\ln a)$, where Γ is the incomplete gamma function.

We now calculate I_3. Since the function $F(x)$ is piecewise continuous of period 2, we calculate its Fourier expansion. Then $F(x) = \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L})$, where

$$a_0 = \frac{1}{L} \int_0^{2L} F(x) \cos \frac{0\pi x}{L} \mathrm{d}x = \int_0^2 F(x) \mathrm{d}x = \int_1^2 \mathrm{d}x = 1,$$

$$a_m = \frac{1}{L}\int_0^{2L} F(x)\cos\frac{m\pi x}{L}\mathrm{d}x = 0,$$
$$b_n = \frac{1}{L}\int_0^{2L} F(x)\sin\frac{n\pi x}{L}\mathrm{d}x = \frac{-2}{n\pi},\ \text{for } n \text{ odd.}$$

Therefore, $F(x) = \frac{1}{2}+\sum_{k=0}^{\infty}\dfrac{-2\sin(2k\pi x+\pi x)}{(2k+1)\pi}$. Substituting $F(x)-\frac{1}{2}$ by its Fourier expansion, we have

$$\begin{aligned} I_3(s) &= s\int_1^{\infty}\frac{a^x(F(x)-1/2)}{x^{s+1}}\mathrm{d}x \\ &= \frac{-2s}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^{\infty}\frac{a^x\sin(2k\pi x+\pi x)}{x^{s+1}}\mathrm{d}x \\ &= \frac{-2s}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^{\infty}\frac{a^{\frac{t}{\pi(2k+1)}}\sin(t)}{\left(\frac{t}{\pi(2k+1)}\right)^{s+1}}\frac{\mathrm{d}t}{(2k+1)\pi} \\ &\qquad\qquad\qquad\qquad (\text{taking } 2k\pi x+\pi x = t) \\ &= \frac{-2s}{\pi}\sum_{k=0}^{\infty}\frac{\pi^s}{(2k+1)^{1-s}}\int_0^{\infty}\frac{a^{\frac{t}{\pi(2k+1)}}\sin(t)}{t^{s+1}}\mathrm{d}t. \end{aligned}$$

We now try to evaluate $I_{5k}(s) = \int_0^{\infty}\frac{a^{\frac{t}{\pi(2k+1)}}\sin(t)}{t^{s+1}}\mathrm{d}t$. Let $\mathrm{Re}(s) < 0$. Then I_{5k} can be evaluated in terms of a complex gamma function as follows:

$$\begin{aligned} I_{5k}(s) &= \int_0^{\infty} t^{-s-1}a^{\frac{t}{\pi(2k+1)}}\sin(t) = \int_0^{\infty} t^{-s-1}a^{\frac{t}{\pi(2k+1)}}(\mathrm{e}^{it}-\mathrm{e}^{-it})/2i\,\mathrm{d}t \\ &= \int_0^{\infty} t^{-s-1}(\mathrm{e}^{C_+t}-e^{C_-t})/2i\,\mathrm{d}t, \end{aligned}$$

where $C_\pm = \frac{ln(a)}{\pi(2k+1)} \pm i$.

Let $u = C_+t$ and $w = C_-t$. Then making change to these complex variables and multiplying by $(-1)^{-s-1}$ when needed, we get

$$I_{5k}(s) = \frac{1}{2iC_+}\int_{\gamma_1} u^{-s-1}\mathrm{e}^{-u}\mathrm{d}u - \frac{1}{2iC_-}\int_{\gamma_2} w^{-s-1}\mathrm{e}^{-w}\mathrm{d}w,$$

where γ_1 is from 0 to $C_+\infty$ along the line $y = \frac{(2k+1)\pi}{ln(a)}x$ and γ_2 is from 0 to $C_-\infty$ along the line $y = \frac{-(2k+1)\pi}{ln(a)}x$. Take contours shown in the figures (with $\gamma_1 = e$ in Figure 1 and $\gamma_2 = e$ in Figure 2) and let the radius of the circular arc grow bigger. It can be shown that the first integral evaluates to $\int_{\infty}^{0} u^{-s-1}\mathrm{e}^{-u}\mathrm{d}u = -\Gamma(-s)$ and so does the second one, if $\mathrm{Re}(s) < 0$. Here $\Gamma(-s) = \int_0^{\infty} u^{-s-1}\mathrm{e}^{-u}\mathrm{d}u$ and has poles along non-negative integers for $\mathrm{Re}(s) < 0$. It is easy to check that $I_{5k}(s) = -\Gamma(-s)/\big(\big(\frac{\ln(a)}{(2k+1)\pi}\big)^2+1\big)$.

In this case, $I_3(s) = \frac{-2s}{\pi}\sum_{k=0}^{\infty}\frac{\pi^s}{(2k+1)^{1-s}}\,I_{5k}(s) = \frac{2s}{\pi}\sum_{k=0}^{\infty}\frac{\pi^s}{(2k+1)^{1-s}((\frac{\ln(a)}{(2k+1)\pi})^2+1)}\Gamma(-s)$.

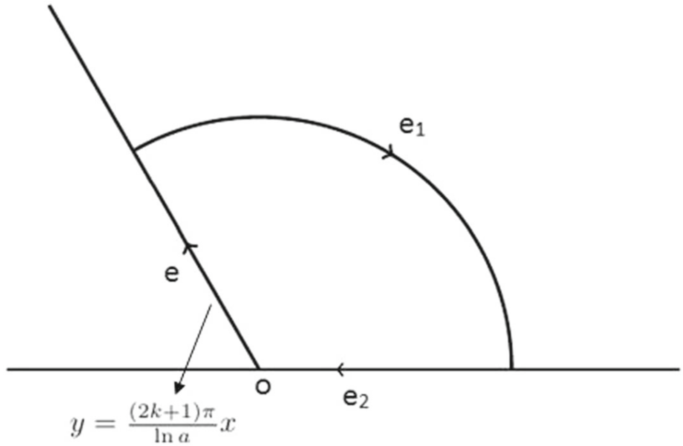

Figure 1. Contour for the first integral.

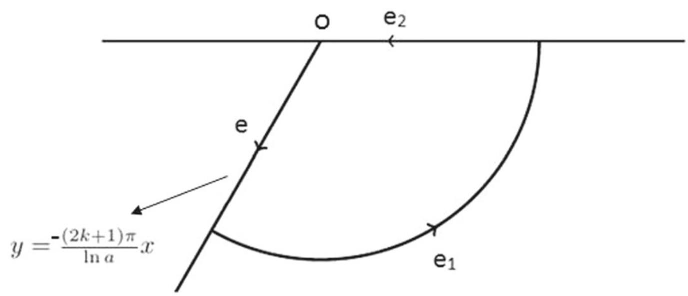

Figure 2. Contour for the second integral.

$I_1(s) = \frac{\ln(a)}{s-1} I_3(s-1) = \frac{2\ln(a)}{\pi} \sum_{k=0}^{\infty} \frac{\pi^{s-1}}{(2k+1)^{2-s}((\frac{\ln(a)}{(2k+1)\pi})^2+1)} \Gamma(-s+1)$. Thus

$$\begin{aligned}
\gamma(s) &= I_1(s) + I_2(s) + I_3(s) + I_4(s) \\
&= \frac{2\ln(a)}{\pi} \sum_{k=0}^{\infty} \frac{\pi^{s-1}}{(2k+1)^{2-s}((\frac{\ln(a)}{(2k+1)\pi})^2 + 1)} \Gamma(-s+1) \\
&\quad - \frac{(-\ln(a))^s}{2}\, \Gamma(-(s-1), -\ln a) \\
&\quad + \frac{2s}{\pi} \sum_{k=0}^{\infty} \frac{\pi^s}{(2k+1)^{1-s}((\frac{\ln(a)}{(2k+1)\pi})^2 + 1)} \Gamma(-s) \\
&\quad + \frac{s}{2}(-\ln a)^s\, \Gamma(-s, -\ln a).
\end{aligned}$$

Let $N_k = (2k+1)$ and $A = \frac{-\ln(a)}{\pi}$. Let us restrict ourselves to $\mathrm{e}^{-\pi} < a < 1$ such that $0 < \frac{A}{N_k} < 1$ for all k. Let $\tilde{\zeta}(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s}$. Notice that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2-s}\left(\left(\frac{\ln(a)}{(2k+1)\pi}\right)^2 + 1\right)} = \sum \frac{1}{N_k^{2-s}\left(\frac{A^2}{N_k^2} + 1\right)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{N_k^{2-s}} \left[1 - \frac{A^2}{N_k^2} + \frac{A^4}{N_k^4} - \frac{A^6}{N_k^6} \cdots \right]$$
$$= \tilde{\zeta}(2-s) - A^2 \tilde{\zeta}(4-s) + \cdots$$
$$= \sum_{n=0}^{\infty} \tilde{\zeta}((2n+2-s)(-1)^n A^{2n}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{1-s}\left(\left(\frac{\ln(a)}{(2k+1)\pi}\right)^2 + 1\right)} = \sum \frac{1}{N_k^{1-s}\left(\frac{A^2}{N_k^2} + 1\right)}$$
$$= \sum_{k=0}^{\infty} \frac{1}{N_k^{1-s}} \left[1 - \frac{A^2}{N_k^2} + \frac{A^6}{N_k^6} - \frac{A^8}{N_k^8} \cdots \right]$$
$$= \tilde{\zeta}(1-s) - A^2 \tilde{\zeta}(3-s) + \cdots$$
$$= \sum_{n=0}^{\infty} \tilde{\zeta}(2n+1-s)(-1)^n A^{2n}.$$

Thus, the following proposition gives us a functional relationship between the Dirichlet series and $\tilde{\zeta}$ function.

PROPOSITION 3.1

Let $A = \frac{-\ln(a)}{\pi}$ *and* $\tilde{\zeta}(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s}$. *For* $\mathrm{Re}(s) < 0$ *and* $\mathrm{e}^{-\pi} < a < 1$, *we have the following functional equation*:

$$\gamma(s) = \frac{2\ln(a)\pi^{s-1}\Gamma(-s+1)}{\pi} \sum_{n=0}^{\infty} \tilde{\zeta}((2n+2-s)(-1)^n A^{2n}$$
$$- \frac{(-\ln(a))^s}{2} \Gamma(-(s-1), -\ln a)$$
$$+ \frac{2s\pi^s}{\pi} \Gamma(-s) \sum_{n=0}^{\infty} \tilde{\zeta}((2n+1-s)(-1)^n A^{2n}$$
$$+ \frac{s}{2}(-\ln a)^s \Gamma(-s, -\ln a)$$

The infinite sums converge since $0 < A < 1$.

3.3 *Essential singularity at* ∞

We consider the limit of our function as $s \to \infty$ along two directions. Let $L_k(s,a) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s}$, where $a = \frac{x^2}{c_k^2 - x^2}$, $0 < a < 1$, $s = \sigma + it \in \mathbb{C}$. When $\mathrm{Im}(s)$ is fixed and $\mathrm{Re}(s) \to \infty$,

$$\left| \sum_{n=2}^{\infty} (-1)^{n-1} \frac{a^n}{n^s} \right| \le \sum_{n=2}^{\infty} \frac{1}{n^{\sigma}} = \sum_{n=2}^{\infty} \frac{1}{n^{\sigma-c+c}} = \sum_{n=2}^{\infty} \frac{1}{n^{\sigma-c}} \frac{1}{n^c}$$

$$\leq \frac{1}{2^{\sigma-c}} \sum_{n=2}^{\infty} \frac{1}{n^c} \to 0 \text{ as } \sigma \to \infty$$

Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^s} \to a$ as $\sigma \to \infty$.

Let $\mathrm{Re}(s)$ is fixed and $\mathrm{Im}(s) \to \infty$. If possible, let $L_k(\frac{1}{2}+it, a) \to a$ as $t \to \infty$. Then given $\epsilon > 0$, $\exists\, t_0 \in \mathbb{R}$ such that

$$\left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^{\frac{1}{2}+it}} - a \right| \leq \epsilon \quad \forall\, t \geq t_0$$

or

$$\left| \left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^{\frac{1}{2}+it}} \right| - |a| \right| \leq \epsilon \quad \forall\, t \geq t_0$$

or

$$|a| - \epsilon \leq \left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{a^n}{n^{\frac{1}{2}+it}} \right| \quad \forall\, t \geq t_0$$

or

$$|a| - \epsilon \leq \zeta\left(\frac{1}{2}\right)$$

or

$$|a| \leq -1.46 + \epsilon$$

which is not true for sufficiently small ϵ.

Thus we arrive at a contradiction. The limit of $L_k(s, a)$ as $s \to \infty$ does not exist and the series has an essential singularity at infinity.

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