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Laplace transform inversion using Bernstein operational matrix of integration and its application to differential and integral equations

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Abstract. In Rani *et al.* (Numerical inversion of Laplace transform based on Bernstein operational matrix, Mathematical Methods in the Applied Sciences (2018) pp. 1–13), a numerical method is developed to find the inverse Laplace transform of certain functions using Bernstein operational matrix. Here, we describe Bernstein operational matrix of integration and propose an algorithm to solve linear time-varying systems governing differential equations. Apart from discussing error estimate, the method is implemented to linear differential equations on Bessel equation of order zero, damped harmonic oscillator, some higher order differential equations, singular integral equation, Volterra integral and integro-differential equations and nonlinear Volterra integral equations of the first kind. A comparison with some existing methods like Haar operational matrix, block pulse operational matrix and others are discussed. The method is simple and easy to implement on a variety of problems. Relative errors estimate just for 5th or 6th approximation show high applicability of the method.

Keywords. Numerical inverse Laplace transform; orthonormalized Bernstein polynomials; operational matrix of integration.

Mathematics Subject Classification. 44A10, 65R10, 40A25.

1. Introduction

Laplace transform is one of the most popular integral transforms that enables to solve different types of problems like differential and integral equations. Having considerable applications in various fields of science and engineering like design of engineering systems such as electrical circuits and mechanical vibrations, the Laplace transform is the subject of attention for many researchers. Inversion is an ill-posed problem. Hence, to calculate the inverse Laplace transform, numerical techniques are adopted. Several numerical techniques are available in literature to invert the Laplace transform. An extensive survey and comparison of methods were discussed in [11,14,21,41]. Murli and Rizzardi [44] employed Talbot's algorithm, Murli *et al.* [43] proposed numerical approximation of the

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inverse Laplace function based on the Laplace transform eigenfunction expansion of the inverse function, in a real case. Dubner and Abate [23] determined the inverse Laplace transform numerically on the basis of evaluating the inverse Laplace transform integral. There exist a freedom in choosing the contour of integration. They expressed the inverse function as a Fourier cosine series. In [24], an improvement upon Dubner and Abate's method [23] had been presented by Durbin. Rather than expressing the $f(t)$ in Fourier cosine series, he proposed the trapezoidal rule and expressed the function in trigonometric series. Iqbal [28] adopted the Laplace transform and converted into an integral equation of the first kind of convolution type that is an ill-posed problem. Therefore, he used regularization method to solve it. Abate *et al.* [1] investigated the Laguerre method for numerically inverting Laplace transform and developed a new insight into it. Cuomo *et al.* [13] described a numerical technique based on collocation method and Laguerre series expansion of inverse function and discussed the error analysis of this method in [12]. D'Amore *et al.* [16] discussed a method based on the numerical evaluation of the integral which occurs in the Riemann inversion formula. This integral reduces to a Fourier series by using the trapezoidal rule approximation and analyzed the discretization error. A software package for the numerical inversion of a Laplace transform function based on Fourier series expansion was introduced in [17]. The construction of a generalized polynomial smoothing spline for approximating Laplace transform functions only known at a finite set of measurements along the real axis had been studied in [15]. D'Amore and Murli [19] expanded the unknown function in Fourier series and the Fourier coefficients are approximated using the Tikhonov regularization method. D'Amore *et al.* [20] adopted an integral equation of convolution type whose solution is the inverse Laplace transform function. Campagna *et al.* [7] proposed GMRES for the current residuals to the maximum attainable accuracy of the approximate solution. Massouros and Genin [39] presented the algebraic scheme for inversion of Laplace transform, and they expanded $F(s)$ in descending powers of s and used Taylor series approach for the time-domain. Laguerre matrix polynomial series to the numerical inversion of Laplace transform of matrix functions had been developed in [48]. Hwang *et al.* [27] proposed fast Fourier transform based method of numerical inversion of Laplace transform in which indefinite Bromwich integral was evaluated using trapezoidal rule. Matsuura and Saitoh [40] developed a real inversion formula for Laplace transform based on Fredholm integral equation of the second kind and followed Tikhonov regularization and theory of reproducing kernel. Frolov and Kitaev [25] implemented a new technique based on Post–Widder formula. Lee and Sheen [29] developed analytic continuation of Bromwich integral and determined the integral by using quadrature rule. In [18], the authors adopted the Gaver's formula in multi-precision computing environments by determining the expression of machine precision in order to invert Laplace transform.

Historically, many different orthogonal functions and their operational matrices were developed to estimate the solution of some linear and nonlinear differential, integral and integro-differential equations [3–5,8–10,22,26,30,32,35,37,42,47]. The advantage of using operational matrices is to convert the differential or integral equations into a well condition system of algebraic equations which is then easy to solve. A very few papers are there in literature where operational matrices are incorporated with numerical inverse Laplace transform: operational matrix of piecewise constant orthogonal function [5], block pulse function [4], improved operational matrix of block pulse function [37] and operational matrix of Haar wavelet [3,26,55]. For instance, Haar wavelet is the simplest of orthonormal wavelets with compact support. As a shortcoming, Haar wavelets are not continuous functions, thereby direct application of Haar wavelet is not possible in solving the differential or integral equations. For this, either it is to be regularized or some integral method can be used.

In recent years, Bernstein polynomials and its operational matrices are used to solve many differential, integral and integro-differential equations. [6,33,45,52,53]. Bernstein operational matrix of integration has been developed to find the numerical inverse Laplace transform of certain functions in [46]. In this paper, our main focus is to apply the technique for real numerical inverse Laplace transform based on the Bernstein operational matrix of integration on some linear differential, integral, integro-differential equations and nonlinear integral equations. The proposed method expresses the solution of equation in terms of truncated Bernstein expansion and then using its operational matrix of integration, numerical inverse of Laplace transform is obtained. The operational matrix of integration of Bernstein polynomial is easily calculated using a single formula of integration rather than Haar or block pulse function where the order of matrix is taken to be too large, i.e., 8, 16, 64, 128. In our method, we achieve the accuracy using a matrix of order 6 or 7 and even give better result by taking block pulse operational matrix of order 16. The computations are very simple by using low order of the matrix.

The paper is classified as follows: in Section 2, a brief review of Bernstein polynomials and its operational matrix of integration are revealed. Section 3 discusses the proposed approach. In Section 4, the applicability of the technique is tested on some numerical examples and error estimations are presented. The accuracy of the solutions are discussed in Section 5 and the conclusions are given in Section 6.

2. Brief review of Bernstein polynomials and the related operational matrices

The Bernstein basis polynomials of degree n are defined over the interval [0, 1] [6,33,45, 52,53]:

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \ldots, n. \tag{1}$$

It is also defined that $b_{i,n}(t) = 0$, if $i < 0$ or $i > n$. Bernstein polynomials of degree n are orthonormalized using Gram–Schmidt orthonormalization procedure and denoted as $B_{0,n}(t)$, $B_{1,n}(t)\ldots$, $B_{n,n}(t)$.

2.1 *Function approximation*

Given $t \in [0, 1]$, a function $f \in L^2[0, 1]$ can be expressed as

$$f(t) = \lim_{n\to\infty} \sum_{i=0}^{n} c_i B_{i,n}(t), \tag{2}$$

where $c_i = \langle f, B_{i,n}\rangle$, $B_{i,n}$ depends on t, i.e. $B_{i,n}(t)$. If the infinite series is truncated at $n = k$, then the solution is approximated by

$$f_k(t) = \sum_{i=0}^{k} c_i B_{i,k}(t) = C^T B(t), \tag{3}$$

where $C = [c_0\ c_1\ c_2\ \ldots c_k]^T$ and $B(t) = [B_{0,k}(t)\ B_{1,k}(t)\ B_{2,k}(t)\ \ldots B_{k,k}(t)]^T$.

Here, the operational matrix of integration of orthonormal Bernstein polynomials is introduced which depends on the integral property of basis vector, i.e. suppose we have a column vector $\phi(t) = [\phi_0(t), \phi_1(t), \ldots, \phi_k(t)]$, where $\phi_0(t), \phi_1(t), \ldots, \phi_k(t)$ are the basis functions, orthogonal on some interval $[a, b]$, then the property states that

$$\int_a^t \ldots \int_a^t \phi(x)(dx)^m = A_{k+1}^m \phi(t), \tag{4}$$

where A_{k+1}^m is the operational matrix of integration of $\phi(t)$ which is a constant matrix of order $(k+1) \times (k+1)$. Now adopting this property on vector $B(t)$, we get

$$\int_0^t B(x)dx = I_{k+1} B(t), \tag{5}$$

where I_{k+1} is the operational matrix of integration of Bernstein polynomials that is defined as

$$\int_0^t B_{i,k}(x)dx = \alpha_i = \sum_{j=0}^{k} a_{jk}^i B_{j,k}, i = 0, 1, \ldots, k, \quad 0 \le t < 1. \tag{6}$$

Therefore

$$I_{k+1} = (a_{jk}^i) = \langle \alpha_i, B_{j,k} \rangle, \quad i, j = 0, 1, \ldots, k. \tag{7}$$

3. Description of proposed method

In this section, we describe the algorithm proposed in [46] by considering the linear time-varying system

$$f'(t) + \alpha f(t) = u(t), f(0) = 0, \tag{8}$$

where $u(t)$ is the unit step function.
Now converting this differential equation to integral equation, we get

$$f(t) + \alpha \int_0^t f(x)dx = \int_0^t u(x)dx. \tag{9}$$

Performing Laplace transform on the above equation, we obtain

$$F(s) = \frac{1}{s(s+\alpha)}. \tag{10}$$

Rewrite the above equation as

$$F(s) = \frac{\frac{1}{s^2}}{(1 + \frac{\alpha}{s})} = \bar{F}\left(\frac{1}{s}\right). \tag{11}$$

We use here the result of Laplace transform [11] that if $L(f(t)) = F(s)$, then

$$L\left(\int_0^t f(t)dt\right) = \frac{1}{s} F(s) \tag{12}$$

This result can be described as the integration in time-domain which corresponds to multiplication of $1/s$ in s-domain. Now if we look at the definition of operational matrix of integration for Bernstein polynomials, i.e. (5), which states that the integration in time-domain is equivalent to operational matrix I_{k+1} in the matrix-domain. Therefore, $1/s$ can be replaced with operational matrix of integration I_{k+1}.

Also it can be said that there is a mapping from time-domain, i.e. $f(t)$ to s-domain, i.e. $F(s)$ or $\bar{F}\left(\frac{1}{s}\right)$, and then to matrix-domain, i.e. $\tilde{F}(I_{k+1})$. Accordingly (11) becomes

$$\tilde{F}(I_{k+1}) = I_{k+1}^2 (I + \alpha I_{k+1})^{-1}. \tag{13}$$

To solve the integral equation (9) by using function approximation, we express

$$f_k(t) = C^T B(t) \tag{14}$$

and

$$\int_0^t f_k(x)dx = C^T I_{k+1} B(t). \tag{15}$$

Also

$$\int_0^t u(x)dx = d^T I_{k+1} B(t). \tag{16}$$

We define $d = [\, d_0 \; d_1 \; d_2 \; \ldots \; d_k \,]^T$ which can be attained by

$$d_i = \int_0^1 B_{i,k}(t)dt, \quad i = 0, 1, 2 \ldots, k, \quad 0 \le t < 1. \tag{17}$$

Thus, the given integral equation (9) becomes

$$\begin{aligned} &C^T B(t) + \alpha C^T I_{k+1} B(t) = d^T I_{k+1} B(t), \\ &C^T (I + \alpha I_{k+1}) B(t) = d^T I_{k+1} B(t), \\ &C^T = d^T I_{k+1} (I + \alpha I_{k+1}))^{-1} = d^T I_{k+1} I_{k+1}^{-2} [I_{k+1}^2 (I + \alpha I_{k+1})^{-1}], \\ &C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}), \end{aligned} \tag{18}$$

where $\tilde{F}(I_{k+1})$ is taken from (13).

Hence, the solution $f(t)$ is obtained in terms of Bernstein polynomials by substituting the unknown values $c_i's$ in (3).

The above procedure can also be used for second order differential equation given by

$$f''(t) + \alpha f'(t) + \beta f(t) = u(t),\ f(0) = 0,\ f'(0) = 0, \tag{19}$$

where $u(t)$ is the unit step function. Now converting this differential equation into integral equation, we get

$$f(t) + \alpha \int_0^t f(x)dx + \beta \int_0^t \int_0^t f(x)dxdx = \int_0^t \int_0^t u(x)dxdx. \tag{20}$$

Applying Laplace transform to (19), we have

$$F(s) = \frac{1}{s(s^2 + \alpha s + \beta)} = \frac{\frac{1}{s^3}}{\left(1 + \frac{\alpha}{s} + \frac{\beta}{s^2}\right)} = \bar{F}\left(\frac{1}{s}\right). \tag{21}$$

In the argument described above, replace $1/s$ with operational matrix of integration I_{k+1}:

$$\tilde{F}(I_{k+1}) = I_{k+1}^3 (I + \alpha I_{k+1} + \beta I_{k+1}^2)^{-1}. \tag{22}$$

Similar criteria for solving integral equation (20) by using function approximation can be adapted here. Substitute

$$f_k(t) = C^T B(t), \tag{23}$$

$$\int_0^t f_k(x)dx = C^T I_{k+1} B(t), \tag{24}$$

$$\int_0^t \int_0^t f_k(x)dxdx = C^T I_{k+1}^2 B(t) \tag{25}$$

and

$$\int_0^t \int_0^t u(x)dxdx = d^T I_{k+1}^2 B(t). \tag{26}$$

Thus, the given integral equation (20) becomes

$$\begin{aligned} C^T B(t) + \alpha C^T I_{k+1} B(t) + \beta C^T I_{k+1}^2 B(t) &= d^T I_{k+1}^2 B(t), \\ C^T &= d^T I_{k+1}^2 (I + \alpha I_{k+1} + \beta I_{k+1}^2)^{-1} \\ &= d^T I_{k+1}^{-1} [I_{k+1}^3 (I + \alpha I_{k+1} + \beta I_{k+1}^2)^{-1}], \\ C^T &= d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}), \end{aligned} \tag{27}$$

where $\tilde{F}(I_{k+1})$ is taken from (22). Hence, the solution $f(t)$ is obtained in terms of Bernstein polynomials by substituting the unknown values $c_i's$ in (3).
In the same way, we can derive the method for higher order differential equations.

The above procedure for inverse Laplace transform can be summarised as follows:

(1) Calculate $F(s)$ from given differential or integral equation and express in the form of $\bar{F}\left(\frac{1}{s}\right)$.
(2) Replace $1/s$ with I_{k+1} and 1 by I (the identity matrix) in $\bar{F}\left(\frac{1}{s}\right)$.
(3) Compute C^T using $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1})$.
(4) Compute $f_k(t)$ by using (14).

All these computations of algebraic operations of matrices have been computed using MATLAB R2014a on Intel® Core™ i3 processor.

4. Numerical experiments

This section presents the applicability of the proposed method on some linear and nonlinear differential, integral and integro-differential equations. The results achieved by taking $k = 5$ and $k = 6$ in the proposed technique, compared with exact solutions and some existing methods are presented in Tables 1–18. Here the orthonormal Bernstein operational matrix of integration of order 6×6 is obtained [6,33] having condition number as 33.41.

$$I_6 = \begin{bmatrix} 0.152778 & 0.288948 & 0.241533 & 0.20663 & 0.159329 & 0.092228 \\ -0.012563 & 0.125 & 0.242527 & 0.180128 & 0.146743 & 0.082341 \\ 0.002216 & -0.022048 & 0.097222 & 0.191725 & 0.116686 & 0.077868 \\ -0.000624 & 0.006211 & -0.027389 & 0.069444 & 0.134479 & 0.051021 \\ 0.000242 & -0.002406 & 0.010608 & -0.026896 & 0.041667 & 0.065296 \\ -0.00099 & 0.000992 & -0.0043750 & .011092 & -0.017183 & 0.013889 \end{bmatrix}. \tag{28}$$

Also, the orthonormal Bernstein operational matrix of integration of order 7×7 with condition number 45.54 is obtained by

$$I_7 = \begin{bmatrix} 0.132653 & 0.253433 & 0.219333 & 0.195022 & 0.164421 & 0.127498 & 0.073561 \\ -0.009386 & 0.112245 & 0.219980 & 0.175011 & 0.152727 & 0.116644 & 0.067942 \\ 0.001415 & -0.016921 & 0.091836 & 0.184074 & 0.129435 & 0.109257 & 0.059833 \\ -0.000340 & 0.004070 & -0.022088 & 0.071429 & 0.144884 & 0.083131 & 0.0584945 \\ 0.000115 & -0.001376 & 0.007467 & -0.024147 & 0.051020 & 0.100996 & 0.036127 \\ -0.000049 & 0.000592 & -0.003213 & 0.010391 & -0.021956 & 0.030612 & 0.048603 \\ 0.000021 & -0.000256 & 0.001391 & -0.004499 & 0.009507 & -0.013255 & 0.010204 \end{bmatrix}. \tag{29}$$

We also find the value of d from (17) as follows:

For $k = 5$, $d = [\,0.5528\ 0.5000\ 0.4410\ 0.3727\ 0.2887\ 0.1667\,]^T$.
For $k = 6$, $d = [\,0.5151\ 0.4738\ 0.4286\ 0.3780\ 0.3194\ 0.2474\ 0.1429\,]^T$.

4.1 *Applications to linear differential, integral and integro-differential equations*

We have investigated error estimation of the proposed method. The error function $e_k(t)$ of the truncated Bernstein expansion $f_k(t)$ is defined as $e_k(t) = f(t) - f_k(t)$.

Example 4.1. Consider the Bessel's differential equation of order zero [3]

$$t\frac{d^2 f(t)}{dt^2} + \frac{df(t)}{dt} + tf(t) = 0, \quad f(0) = 1,\ f'(0) = 0 \tag{30}$$

and $f(t) = J_0(t)$.

Using the algorithm, we first apply the Laplace transform to (30). We get

$$F(s) = \frac{1}{\sqrt{s^2+1}} = \frac{1/s}{\sqrt{1+(1/s)^2}} = \bar{F}\left(\frac{1}{s}\right).$$

Now replacing $1/s$ with operational matrix I_{k+1}, give $\tilde{F}\,(I_{k+1}) = I_{k+1}(I + I_{k+1}^2)^{-1/2}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}) = d^T (I + I_{k+1}^2)^{-1/2}$ for $k = 5$ which follows the result:

$$C^T = \left[0.5479\ 0.4780\ 0.3957\ 0.3120\ 0.2282\ 0.1275\right].$$

Thus the approximate solution $f_k(t)$ using (14) can be found.

In Table 1, the relative errors obtained by our method are presented. Also, Figure 1 presents the relative errors of numerical solution by taking $k = 5$.

Example 4.2. Consider the fourth order differential equation [26]

$$f''''(t) + 2a^2 f''(t) + a^4 f(t) = \cos at \tag{31}$$

with initial conditions $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = 0$ and exact solution $f(t) = \frac{t \sin at - at^2 \cos at}{8a^3}$.

Table 1. Relative errors for Example 4.1.

t	Rel. error at $k = 5$
0	0.49E−04
0.2	0.38E−05
0.4	0.47E−05
0.6	0.51E−05
0.8	0.35E−06
1	0.24E−05

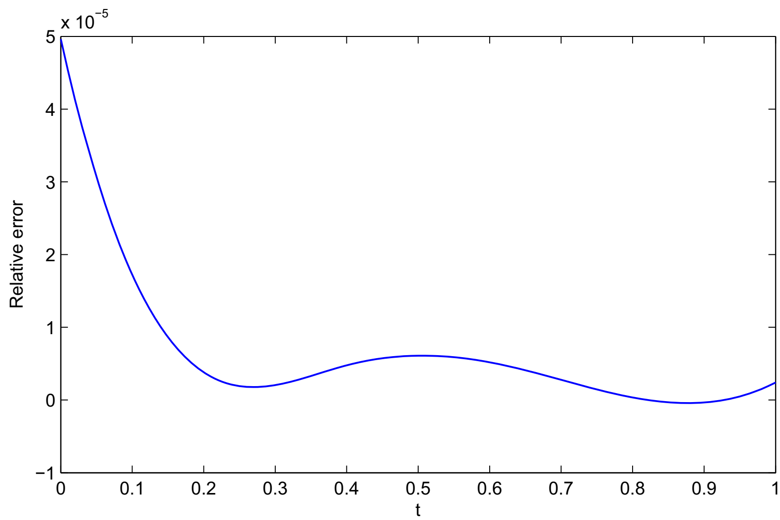

Figure 1. The relative errors for $k = 5$ in Example 4.1.

Using the above described procedure, applying the Laplace transform to (31) (take $a = 1$), we get

$$F(s) = \frac{s}{(s^2+1)^3} = \frac{\frac{1}{s^5}}{(1+1/s^2)^3} = \bar{F}\left(\frac{1}{s}\right).$$

Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}^5(I + I_{k+1}^2)^{-3}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T I_{k+1}^4(I + I_{k+1}^2)^{-3}$ and find the following results:

For $k = 5$,

$$C^T = [7.5885E-05 \quad 8.4243E-04 \quad 3.0668E-03 \quad 6.4197E-03 \quad 8.4970E-03 \quad 6.2676E-03].$$

For $k = 6$,

$$C^T = [6.3799E-05 \quad 5.2367E-04 \quad 1.9585E-03 \quad 4.4677E-03 \quad 7.0024E-03 \quad 7.8427E-03 \quad 5.3782E-03].$$

Table 2. Relative errors for Example 4.2.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0.2	0.78E+00	0.99E−03
0.4	0.44E−01	0.21E−05
0.6	0.91E−02	0.48E−04
0.8	0.29E−02	0.33E−04
1	0.11E−02	0.28E−04

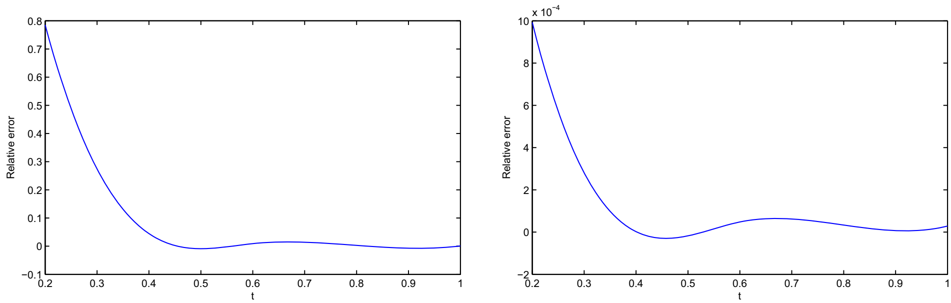

Figure 2. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.2.

Thus the approximate solution $f_k(t)$ using (14) can be found. Table 2 and Figure 2 present the relative errors obtained by proposed method for $k = 5$ and $k = 6$.

Example 4.3. Consider the other critical problem of damped harmonic oscillator [6]

$$2f''(t) + 3f'(t) + f(t) = \cos t, \quad f(0) = 0,\ f'(0) = 0. \tag{32}$$

with exact solution $f(t) = \frac{e^{-t/2} - 2e^{-t/2}(-t/2)}{5} - \frac{\cos t}{10} + \frac{3\sin t}{10}$.

Using the above described procedure, applying the Laplace transform to (32) , we get

$$F(s) = \frac{s}{(s^2+1)(2s^2+3s+1)} = \frac{1}{s^3(1+\frac{1}{s^2})(2+\frac{3}{s}+\frac{1}{s^2})} = \bar{F}\left(\frac{1}{s}\right).$$

Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}\,(I_{k+1}) = I_{k+1}^3(I + I_{k+1}^2)^{-1}(2I + 3I_{k+1} + I_{k+1}^2)^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T I_{k+1}^2(I + I_{k+1}^2)^{-1}(2I + 3I_{k+1} + I_{k+1}^2)^{-1}$ and find the following result:

For $k = 5$,

$$C^T = [4.1282E-03 \quad 1.7441E-02 \quad 3.2795E-02 \quad 4.0055E-02 \quad 3.7317E-02 \quad 2.3293E-02].$$

For $k = 6$,

$$C^T = [3.0641E-03 \quad 1.3191E-02 \quad 2.6178E-02 \quad 3.4889E-02 \quad 3.7175E-02 \quad 3.2684E-02 \quad 1.9963E-02].$$

Table 3. Relative errors for Example 4.3.

t	Rel. error at $k=5$	Rel. error at $k=6$
0.2	0.40E−02	0.72E−04
0.4	0.43E−04	0.28E−04
0.6	0.22E−03	0.24E−04
0.8	0.19E−03	0.35E−04
1	0.15E−03	0.28E−04

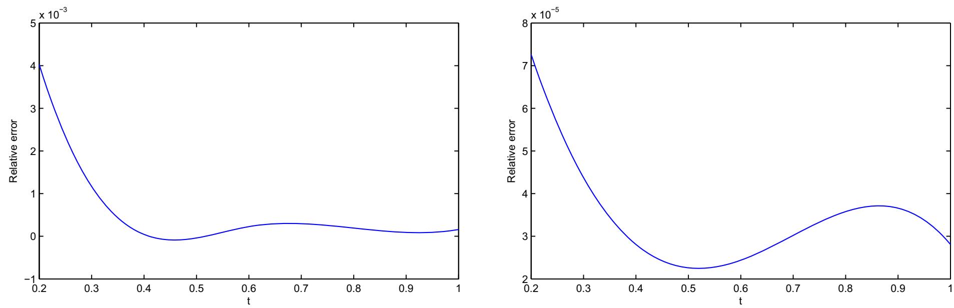

Figure 3. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.3.

Thus the approximate solution $f_k(t)$ using (14) can be found. Table 3 and Figure 3 present the relative errors obtained by proposed method for $k = 5$ and $k = 6$.

Example 4.4. Consider Volterra integral equation of the first kind [4]

$$\int_0^t \cos(t-s) f(s) ds = t \sin t \tag{33}$$

with exact solution $x(t) = 2 \sin t$.

Using the above described procedure, applying the Laplace transform to (33), we get

$$F(s) = \frac{2}{(s^2+1)} = \frac{\frac{2}{s^2}}{1+\frac{1}{s^2}} = \bar{F}\left(\frac{1}{s}\right)$$

Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = 2I_{k+1}^2(I + I_{k+1}^2)^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}) = 2d^T I_{k+1}(I + I_{k+1}^2)^{-1}$ and find the following result:

For $k = 5$,

$$C^T = [1.5564E-01 \quad 4.1496E-01 \quad 5.4094E-01 \quad 5.5150E-01 \quad 4.6804E-01 \quad 2.8052E-01].$$

For $k = 6$,

$$C^T = [1.2736E-01 \quad 3.4626E-01 \quad 4.7313E-01 \quad 5.1690E-01 \quad 4.9148E-01 \quad 4.0528E-01 \quad 2.4042E-01].$$

Table 4. Relative errors for Example 4.4.

t	Rel. error at $k=5$	Rel. error at $k=6$	Rel. error at $k=128$ by block pulse function [4]
0.2	0.61E−03	0.53E−04	0.26E−02
0.4	0.58E−04	0.10E−04	0.88E−02
0.6	0.11E−03	0.33E−04	0.57E−02
0.8	0.10E−03	0.43E−04	0.10E−02

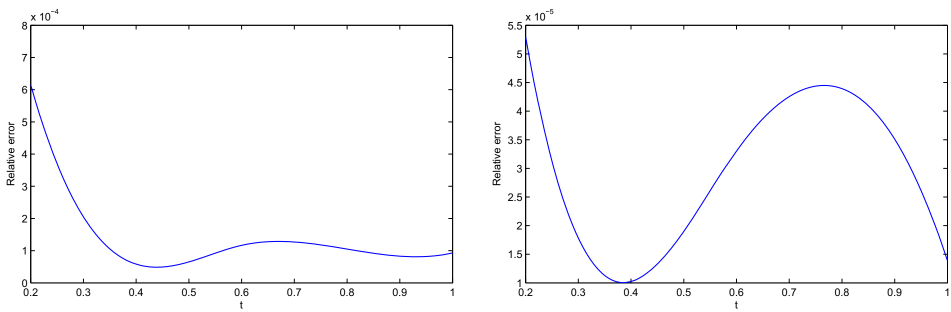

Figure 4. The relative errors for $k=5$ (*left*) and $k=6$ (*right*) in Example 4.4.

Thus the approximate solution $f_k(t)$ using (14) can be found. The comparison among the relative errors obtained by our method and using block pulse function [4] are analysed in Table 4 and Figure 4 present the relative errors obtained by the proposed method for $k=5$ and $k=6$ which reveals that the our results are better than using block pulse function.

Example 4.5. Consider the weakly singular integral equation [5,37]

$$f(t) = t + \frac{4}{3}t^{3/2} - \int_0^t \frac{1}{\sqrt{t-s}} f(s)ds \tag{34}$$

with exact solution $f(t)=t$.

Using the above described procedure, applying the Laplace transform to (34) , we get

$$F(s) = \frac{1}{s^2}$$

Now replacing $1/s$ with operational matrix I_{k+1}, give $\tilde{F}(I_{k+1}) = I_{k+1}^2$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T I_{k+1}$ and find the following results:

For $k=5$,

$$C^T = [0.0788\ 0.2143\ 0.2898\ 0.3088\ 0.2722\ 0.1667].$$

For $k=6$,

$$C^T = [0.0644\ 0.1777\ 0.2500\ 0.2835\ 0.2795\ 0.2371\ 0.1429].$$

Table 5. Relative errors for Example 4.5.

t	Rel. error at $k = 5$	Rel. error at $k = 6$	Rel. error at $k = 16$ by block pulse function [5]
0.0625	0.91E−02	0.78E−05	0.50E+00
0.125	0.32E−02	0.61E−04	0.24E+00
0.1875	0.15E−02	0.56E−04	0.16E+00
0.25	0.78E−03	0.40E−04	0.12E+00
0.3125	0.42E−03	0.23E−04	0.10E+00
0.375	0.23E−03	0.12E−04	0.83E−01
0.4375	0.13E−03	0.10E−04	0.71E−01
0.5	0.74E−04	0.16E−04	0.62E−01
0.5625	0.43E−04	0.26E−04	0.55E−01
0.625	0.26E−04	0.37E−04	0.49E−01
0.6875	0.16E−04	0.45E−04	0.45E−01
0.75	0.11E−04	0.46E−04	0.41E−01
0.8125	0.84E−05	0.41E−04	0.38E−01
0.875	0.62E−05	0.29E−04	0.35E−01
0.9375	0.37E−05	0.17E−04	0.33E−01
1	0.25E−06	0.16E−04	0.31E−01

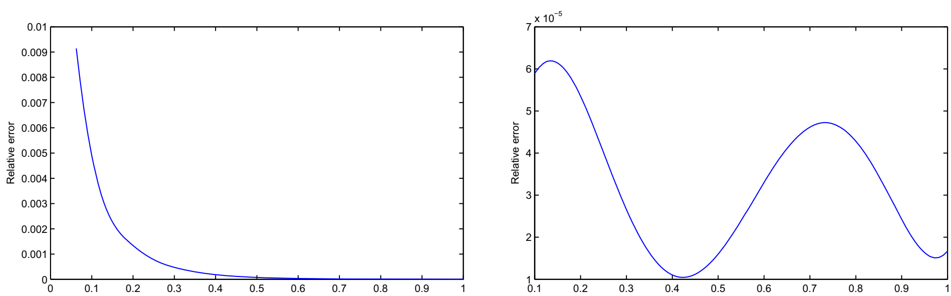

Figure 5. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.5.

Thus the approximate solution $f_k(t)$ using (14) can be found. The comparison among the relative error obtained by our method and using block pulse function [5] are analysed in Table 5 and Figure 5 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$ which reveals that the our results are better than using block pulse function.

Example 4.6. Consider the second order integro-differential equation [5,37]

$$f''(t) = t + \int_0^t (t - s) f(s) ds \tag{35}$$

with exact solution $f(t) = \sinh t$.

Table 6. Relative errors for Example 4.6.

t	Rel. error at $k = 5$	Rel. error at $k = 6$	Rel. error at $k = 16$ by block pulse function [5]
0.0625	0.13E−01	0.45E−04	0.50E+00
0.125	0.49E−02	0.61E−04	0.25E+00
0.1875	0.25E−02	0.56E−04	0.16E+00
0.25	0.14E−02	0.40E−04	0.12E+00
0.3125	0.91E−03	0.24E−04	0.10E+00
0.375	0.63E−03	0.13E−04	0.86E−01
0.4375	0.47E−03	0.11E−04	0.75E−01
0.5	0.37E−03	0.16E−04	0.66E−01
0.5625	0.32E−03	0.26E−04	0.60E−01
0.625	0.28E−03	0.37E−04	0.55E−01
0.6875	0.26E−03	0.44E−04	0.51E−01
0.75	0.25E−03	0.45E−04	0.48E−01
0.8125	0.24E−03	0.40E−04	0.45E−01
0.875	0.23E−03	0.29E−04	0.43E−01
0.9375	0.22E−03	0.19E−04	0.41E−01
1	0.22E−03	0.18E−04	0.40E−01

Using the above described procedure, applying the Laplace transform to (35) , we get

$$F(s) = \frac{1}{s^2 - 1} = \frac{1/s^2}{1 - 1/s^2} = \bar{F}\left(\frac{1}{s}\right)$$

Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}^2(I - I_{k+1}^2)^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T I_{k+1}(I - I_{k+1}^2)^{-1}$ and find the following results:

For $k = 5$,

$$C^T = [\,0.0798\ 0.2213\ 0.3099\ 0.3440\ 0.3137\ 0.1958\,].$$

For $k = 6$,

$$C^T = [\,0.0651\ 0.1823\ 0.2640\ 0.3099\ 0.3159\ 0.2749\ 0.1679\,].$$

Thus the approximate solution $f_k(t)$ using (14) can be found. The comparison among the relative errors obtained by our method and using block pulse function [5] are analysed in Table 6, and Figure 6 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$ which reveals that the our results are better than using block pulse function.

Example 4.7. Consider fourth-order Volterra integro-differential equation [2]

$$f^{(4)}(t) = t(1 + e^t) + 3e^t + f(t) - \int_0^t f(s)ds \tag{36}$$

having conditions $f(0) = 1$, $f(1) = 1 + e$, $f''(0) = 2$, $f''(1) = 3e$ with exact solution $f(t) = 1 + te^t$.

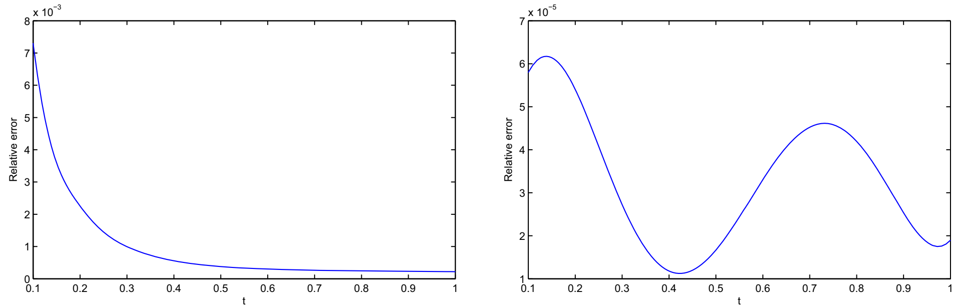

Figure 6. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.6.

Using the above described procedure, applying the Laplace transform to (36), we get

$$F(s) = \frac{1}{s^5 - s + 1}\left(\frac{1}{s} + \frac{s}{(s-1)^2} + \frac{3s}{s-1} + s^4 + s^3 + 2s^2 + 3s\right)$$
$$= \frac{1}{s}\left(1 - \frac{1}{s^4} + \frac{1}{s^5}\right)^{-1}\left(\frac{1}{s^5} + \frac{1}{s^5\left(1-\frac{1}{s^2}\right)^2} + \frac{3}{s^4\left(1-\frac{1}{s}\right)} + 1 + \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s^3}\right)$$
$$= \bar{F}\left(\frac{1}{s}\right).$$

Now replacing $1/s$ with operational matrix I_{k+1} gives

$$\tilde{F}(I_{k+1}) = I_{k+1}(I - I_{k+1}^4 + I_{k+1}^5)^{-1}(I_{k+1}^5 + I_{k+1}^5(I - I_{k+1})^{-2} + 3I_{k+1}^4(I - I_{k+1})^{-1} + I + I_{k+1} + 2I_{k+1}^2 + 3I_{k+1}^3).$$

Therefore, we compute the coefficient matrix

$$C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T(I - I_{k+1}^4 + I_{k+1}^5)^{-1}(I_{k+1}^5 + I_{k+1}^5(I - I_{k+1})^{-2} + 3I_{k+1}^4(I - I_{k+1})^{-1} + I + I_{k+1} + 2I_{k+1}^2 + 3I_{k+1}^3)$$

and find the following results:

For $k = 5$,

$$C^T = \left[0.6542\ 0.8280\ 0.9908\ 1.0703\ 0.9832\ 0.6194\right].$$

For $k = 6$,

$$C^T = \left[0.5962\ 0.7336\ 0.8721\ 0.9716\ 0.9851\ 0.8640\ 0.5312\right].$$

Thus the approximate solution $f_k(t)$ using (14) can be found. The relative errors obtained by our method besides modified homotopy perturbation method (MHPM) [2] are presented in Table 7 and Figure 7 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$ which shows that the proposed method gives better results as compared to the method in [2].

Table 7. Relative errors for Example 4.7.

t	Rel. error at $k=5$	Rel. error at $k=6$	Rel. error by MHPM [2]
0	0.37E−02	0.10E−03	0.19E−08
0.2	0.14E−02	0.13E−04	0.26E−01
0.4	0.78E−03	0.45E−05	0.33E−01
0.6	0.62E−03	0.77E−04	0.26E−01
0.8	0.58E−03	0.58E−05	0.12E−01
1	0.55E−03	0.99E−04	0.95E−09

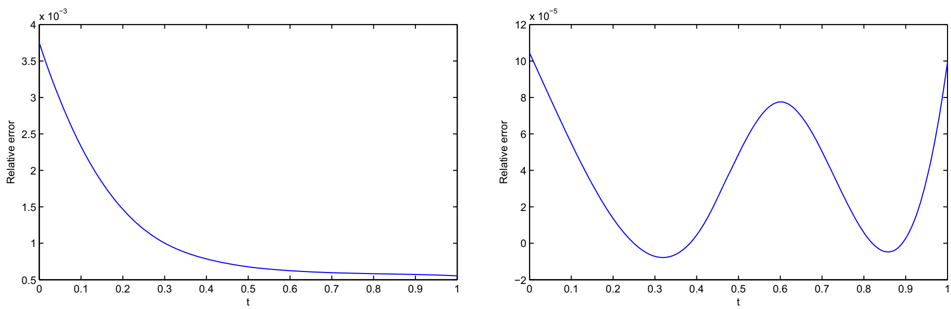

Figure 7. The relative errors for $k=5$ (*left*) and $k=6$ (*right*) in Example 4.7.

4.2 *Applications to nonlinear Volterra integral equations of the first kind*

Now, we consider the nonlinear Volterra integral equations of the first kind with convolution kernel given by

$$x(t)=\lambda\int_0^t K(t-u)N(f(u))du, \tag{37}$$

where $K(t-u)$ is the kernel of integral, $x(t)$ is the known function, $N(f(u))$ is the nonlinear term and $f(t)$ is the unknown function. The nonlinear Volterra integral equations of the first kind arises in a variety of applications in many fields including electrical engineering like modeling of dynamic impulse system and in nonlinear dynamic systems identification [50]. Some methods have been developed to solve such equations in the literature because of the useful applications of these equations. Maleknejad *et al.* [36] solved these kind of equations with wavelet basis, Babolian and Shamloo [5] used operational matrices of piecewise constant orthogonal functions, Babolian and Masouri [4] studied block pulse function and their operational matrix of integration and, Masouri *et al.* [38] proposed the expansion iterative method based on block pulse function. Maleknejad *et al.* [34] determined the unknown function in Volterra integral equations of the first kind with Bernstein approximation, Maleknejad and Nouri [37] improved operational matrix of block pulse functions and, Singh and Kumar [51] proposed Haar wavelet method. Here, we adopt our method to solve these equations. Applying the Laplace transform to both sides of (37), we get

$$L(x(t))=\lambda L(K(t))L(N(f(t))). \tag{38}$$

We can write it as

$$L(N(f(t))) = \frac{L(x(t))}{\lambda L(K(t))}, \tag{39}$$

or

$$N(f(t)) = L^{-1}\left(\frac{L(x(t))}{\lambda L(K(t))}\right). \tag{40}$$

Hence, solution can be obtained by numerical inverse Laplace transform as described above.

4.2.1 *Error function*

Consider

$$r_k(t) = \lambda \int_0^t K(t-u)N(f_k(u))du - x(t), \tag{41}$$

where $f_k(t)$ is the approximate solution and $r_k(t)$ is the perturbation function which depends on $f_k(t)$. Then by subtracting (37) and (41), we get

$$-r_k(t) = \lambda \int_0^t K(t-u)E_k(u)du, \tag{42}$$

where we define the error function $N(f(t)) - N(f_k(t)) = E_k(t)$ (say). Therefore, solving (42) with the same procedure as described in Section 3, gives

$$L[E_k(t)] = \frac{-L[r_k(t)]}{\lambda L[K(t)]}. \tag{43}$$

Adopting inverse Laplace transform and using the value of $r_k(t)$ from (41), we have

$$E_k(t) = L^{-1}\left[\frac{L[x(t)]}{\lambda L[K(t)]}\right] - N(f_k(t)). \tag{44}$$

Consequently from (44), we can simply find the error function of the given integral equation.

Next, some nonlinear Volterra integral equations of the first kind having different nonlinear terms are solved.

Example 4.8. Consider the following integral equation [5,31,37,54]

$$\mathrm{e}^{2t} - \mathrm{e}^{t} = \int_0^t \mathrm{e}^{t-u} f^2(u)du \tag{45}$$

with exact solution $f(t) = e^t$.

Using the above described procedure, applying the Laplace transform to (45), we get $L(f^2(t)) = \frac{1}{s-2}$, i.e., $f^2(t) = L^{-1}\{F(s)\}$, where $F(s) = \frac{1}{s-2} = \frac{1/s}{1-2/s} = \bar{F}\left(\frac{1}{s}\right)$. Take $f^2(t) = X(t)$. Now replacing $1/s$ with the operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}(I - 2I_{k+1})^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}) = d^T(I - 2I_{k+1})^{-1}$ and find the following results:

For $k = 5$,

$$C^T = [0.7580\ 1.1787\ 1.6172\ 1.9237\ 1.8859\ 1.2292].$$

Table 8. Relative errors for Example 4.8.

t	Rel. error at $k=5$	Rel. error at $k=6$	Rel. error at $k=16$ by block pulse function [5]
0.0625	0.32E−02	0.23E−04	0.30E−01
0.125	0.24E−02	0.18E−04	0.30E−01
0.1875	0.18E−02	0.14E−04	0.31E−01
0.25	0.15E−02	0.63E−05	0.31E−01
0.3125	0.12E−02	0.45E−06	0.31E−01
0.375	0.11E−02	0.91E−06	0.31E−01
0.4375	0.10E−02	0.80E−05	0.31E−01
0.5	0.98E−03	0.18E−04	0.31E−01
0.5625	0.95E−03	0.27E−04	0.31E−01
0.625	0.95E−03	0.31E−04	0.31E−01
0.6875	0.95E−03	0.28E−04	0.31E−01
0.75	0.95E−03	0.18E−04	0.31E−01
0.8125	0.94E−03	0.70E−05	0.31E−01
0.875	0.93E−03	0.42E−06	0.31E−01
0.9375	0.92E−03	0.45E−05	0.31E−01
1	0.95E−03	0.30E−04	0.32E−01

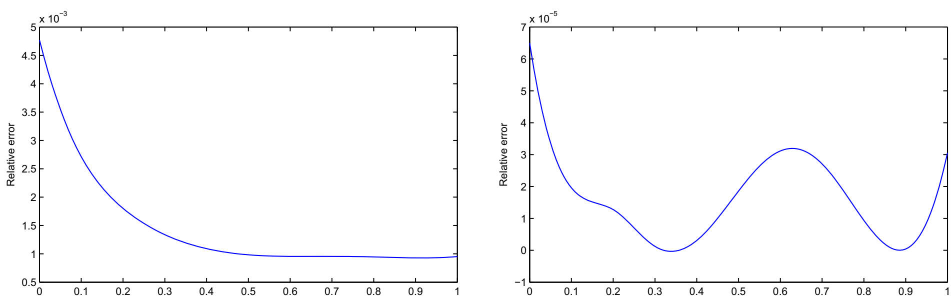

Figure 8. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.8.

For $k = 6$,

$$C^T = [\,0.6795\ 1.0084\ 1.3667\ 1.6758\ 1.8227\ 1.6760\ 1.0556\,].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \sqrt{X(t)}$ can be obtained. In Table 8, the relative errors obtained by our method with operational matrix using block pulse function [5] is compared which shows that our method gives better results than block pulse operational matrix method. Also, Figure 8 presents the relative errors of numerical solution by taking $k = 5$ and $k = 6$.

Example 4.9. Consider the integral equation [49]

$$te^t = \int_0^t e^{t-u} e^{f(u)} du \tag{46}$$

Table 9. Relative errors for Example 4.9.

t	Rel. error at $k = 5$	Rel. error at $k = 6$	Rel. error at $k = 16$ by block pulse function [49].
0.0625	0.24E−01	0.90E−03	0.47E−00
0.125	0.87E−02	0.28E−03	0.23E−00
0.1875	0.42E−02	0.86E−04	0.15E−00
0.25	0.23E−02	0.84E−05	0.11E−00
0.3125	0.14E−02	0.35E−04	0.95E−01
0.375	0.93E−03	0.11E−04	0.79E−01
0.4375	0.66E−03	0.39E−04	0.68E−01
0.5	0.50E−03	0.94E−04	0.59E−01
0.5625	0.41E−03	0.13E−03	0.53E−01
0.625	0.34E−03	0.13E−03	0.47E−01
0.6875	0.30E−03	0.10E−03	0.43E−01
0.75	0.27E−03	0.51E−04	0.39E−01
0.8125	0.25E−03	0.14E−04	0.36E−01
0.875	0.22E−03	0.55E−04	0.34E−01
0.9375	0.21E−03	0.27E−04	0.31E−01
1	0.19E−03	0.12E−03	0.29E−01

with exact solution $f(t) = t$.

Using the above described procedure, applying the Laplace transform to (46), we get

$$L(\mathrm{e}^{f(t)}) = \frac{1}{s-1},$$

that is, $\mathrm{e}^{f(t)} = L^{-1}\{F(s)\}$, where $F(s) = \frac{1}{s-1} = \frac{1/s}{1-1/s} = \bar{F}\left(\frac{1}{s}\right)$ and take $\mathrm{e}^{f(t)} = X(t)$. Now replacing $1/s$ with the operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}(I - I_{k+1})^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T(I - I_{k+1})^{-1}$ and find the following results:

For $k = 5$,

$$C^T = [0.6424\ 0.7667\ 0.8470\ 0.8501\ 0.7397\ 0.4530].$$

For $k = 6$,

$$C^T = [0.5874\ 0.6896\ 0.7660\ 0.7977\ 0.7646\ 0.6445\ 0.3884].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \log(X(t))$ can be obtained. In Table 9, the relative errors obtained by our method with operational matrix using block pulse function [49] is compared which shows that our method gives better results than block pulse operational matrix method. Also, Figure 9 presents the relative errors of numerical solution by taking $k = 5$ and $k = 6$.

Example 4.10. Consider the integral equation [49]

$$t^{3/2} = \int_0^t \frac{\sqrt{f(u)}}{\sqrt{t-u}}du \tag{47}$$

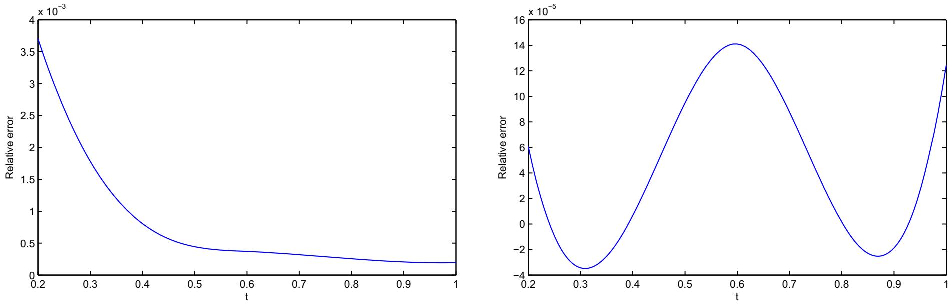

Figure 9. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.9.

Table 10. Relative errors for Example 4.10.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0.2	0.10E−03	0.26E−03
0.4	0.22E−04	0.36E−03
0.6	0.66E−04	0.63E−04
0.8	0.85E−04	0.17E−04
1	0.33E−04	0.50E−06

with exact solution $f(t) = \frac{9}{16}t^2$.

Using the above described procedure, applying the Laplace transform to (47), we get

$$L(\sqrt{f(t)}) = \frac{3}{4s^2},$$

that is, $\sqrt{f(t)} = L^{-1}\{F(s)\}$, where $F(s) = \frac{3}{4s^2} = \frac{3}{4}\left(\frac{1}{s}\right)^2 = \tilde{F}\left(\frac{1}{s}\right)$ and take $\sqrt{f(t)} = X(t)$. Now replacing $1/s$ with the operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = \frac{3}{4}I_{k+1}^2$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = \frac{3}{4}d^T I_{k+1}$ and find the following results:

For $k = 5$,

$$C^T = \left[0.0591\ 0.1607\ 0.2173\ 0.2316\ 0.2041\ 0.1250\right].$$

For $k = 6$,

$$C^T = \left[0.0483\ 0.1333\ 0.1875\ 0.2126\ 0.2096\ 0.1778\ 0.1071\right].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = X(t)^2$ can be obtained. Table 10 and Figure 10 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$.

Example 4.11. Consider the integral equation [54]

$$\frac{1}{4}e^{2t} - \frac{1}{2}t - \frac{1}{4} = \int_0^t (t-u)f^2(u)du \tag{48}$$

with exact solution $f(t) = e^t$.

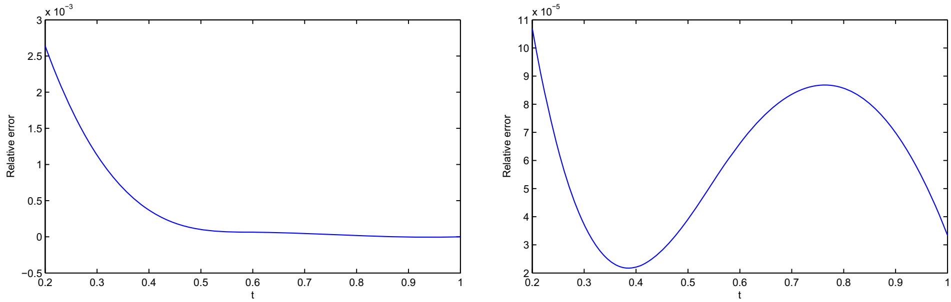

Figure 10. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.10.

Table 11. Relative errors for Example 4.11.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0	0.47E−02	0.64E−04
0.2	0.18E−02	0.12E−04
0.4	0.10E−02	0.30E−05
0.6	0.95E−03	0.31E−04
0.8	0.94E−03	0.93E−05
1	0.95E−03	0.30E−04

Using the above described procedure, applying the Laplace transform to (48), we get $L(f^2(t)) = \frac{1}{s-2}$, i.e., $f^2(t) = L^{-1}\{F(s)\}$, where $F(s) = \frac{1}{s-2} = \frac{1/s}{1-2/s} = \tilde{F}\left(\frac{1}{s}\right)$ and take $f^2(t) = X(t)$. Now replacing $1/s$ with the operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}(I-2I_{k+1})^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T(I - 2I_{k+1})^{-1}$ and find the following results:

For $k = 5$,

$$C^T = [0.7580\ 1.1787\ 1.6172\ 1.9237\ 1.8859\ 1.2292].$$

For $k = 6$,

$$C^T = [0.6795\ 1.0084\ 1.3667\ 1.6758\ 1.8227\ 1.6760\ 1.0556].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \sqrt{X(t)}$ can be obtained. Table 11 and Figure 11 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$.

Example 4.12. Consider the integral equation [54]

$$\frac{1}{2}\sin t - \frac{1}{2}t\cos 2t = \int_0^t \sin(t-u)\sin(f(u))du \tag{49}$$

with exact solution $f(t) = t$.

Using the above described procedure, applying the Laplace transform to (49), we get $L(\sin(f(t))) = \frac{1}{s^2+1}$, i.e., $\sin(f(t)) = L^{-1}\{F(s)\}$, where $F(s) = \frac{1}{s^2+1} = \frac{1/s^2}{1+1/s^2} =$

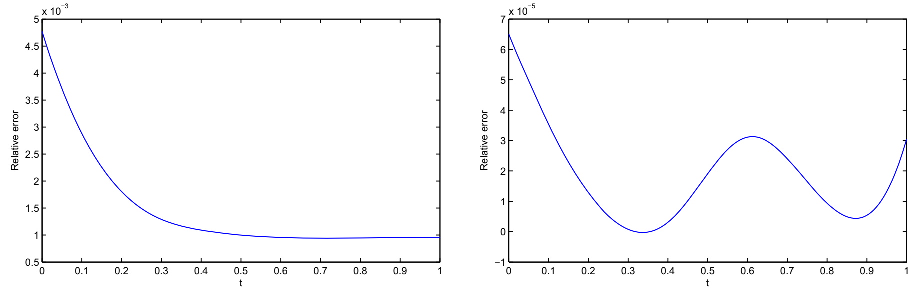

Figure 11. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.11.

Table 12. Relative errors for Example 4.12.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0.2	0.62E−03	0.53E−04
0.4	0.61E−04	0.10E−04
0.6	0.13E−03	0.37E−04
0.8	0.13E−03	0.56E−04
1	0.14E−03	0.21E−04

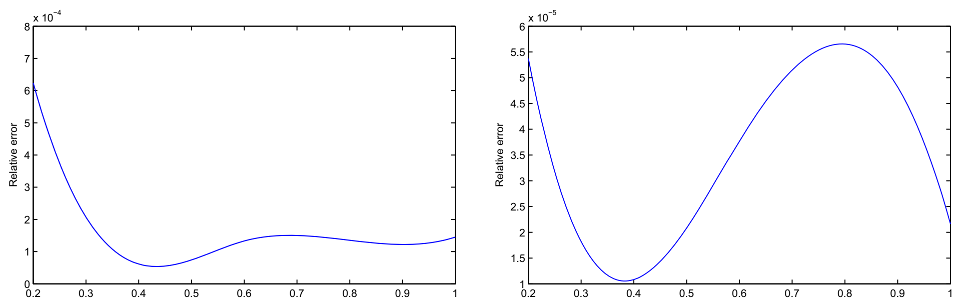

Figure 12. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.12.

$\bar{F}\left(\frac{1}{s}\right)$ and take $\sin(f(t)) = X(t)$. Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}^2(I + I_{k+1}^2)^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}) = d^T I_{k+1}(I + I_{k+1}^2)^{-1}$ and find the following results:

For $k = 5$,

$$C^T = [\,0.0778\ 0.2075\ 0.2705\ 0.2758\ 0.2340\ 0.1403\,].$$

For $k = 6$,

$$C^T = [\,0.0637\ 0.1731\ 0.2366\ 0.2585\ 0.2457\ 0.2026\ 0.1202\,].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \arcsin X(t)$ can be obtained. Table 12 and Figure 12 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$.

Table 13. Relative errors for Example 4.13.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0.2	0.16E−02	0.57E−04
0.4	0.35E−02	0.11E−04
0.6	0.19E−02	0.15E−04
0.8	0.14E−02	0.16E−04
1	0.12E−02	0.13E−04

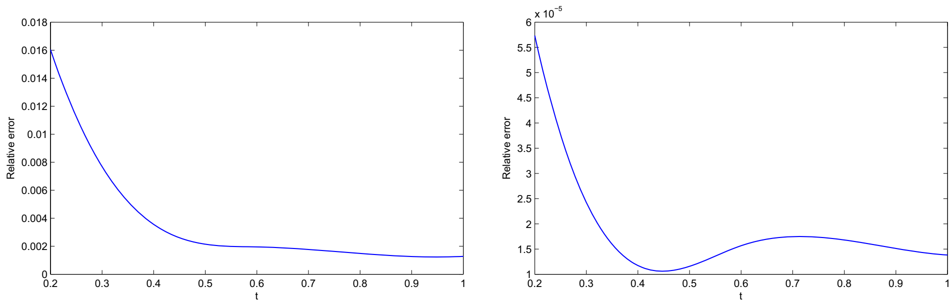

Figure 13. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.13.

Example 4.13. Consider the integral equation [54]

$$\frac{1}{2} + \frac{1}{6}\cosh 2t - \frac{2}{3}\cosh t = \int_0^t \sinh(t-u) f^2(u)du \tag{50}$$

with exact solution $f(t) = \sinh t$.

Using the above described procedure, applying the Laplace transform to (50), we get $L(f^2(t)) = -\frac{1}{2s} + \frac{s}{2(s^2-4)}$, i.e. $f^2(t) = L^{-1}\{F(s)\}$, where $F(s) = -\frac{1}{2s} + \frac{s}{2(s^2-4)} = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{\frac{1}{s}}{2(1-4/s^2)} = \bar{F}\left(\frac{1}{s}\right)$ and take $f^2(t) = X(t)$. Now replacing $1/s$ with the operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = \frac{1}{2} I_{k+1}(-I + (I - 4I_{k+1}^2)^{-1})$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}) = 0.5 d^T(-I + (I - 4I_{k+1}^2)^{-1})$ and find the following results:

For $k = 5$,

$$C^T = [0.0199\ 0.0962\ 0.2127\ 0.3120\ 0.3380\ 0.2296].$$

For $k = 6$,

$$C^T = [0.0148\ 0.0703\ 0.1601\ 0.2507\ 0.3097\ 0.3044\ 0.1973].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \sqrt{X(t)}$ can be obtained. Table 13 and Figure 13 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$.

Table 14. Relative errors for Example 4.14.

t	Rel. error at $k=5$	Rel. error at $k=6$
0	0.15E−02	0.52E−04
0.2	0.40E−03	0.93E−05
0.4	0.11E−03	0.26E−05
0.6	0.46E−04	0.34E−04
0.8	0.29E−04	0.59E−05
1	0.20E−04	0.45E−04

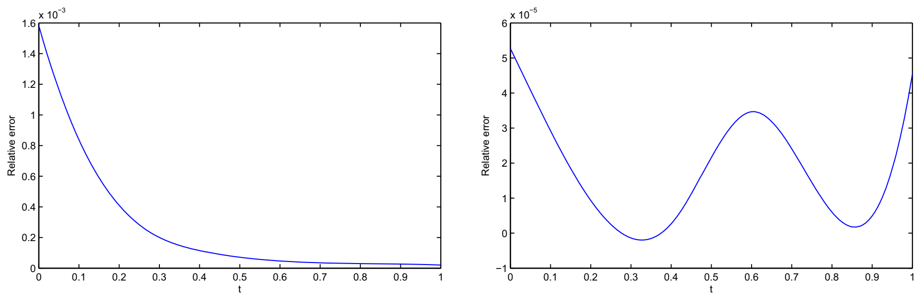

Figure 14. The relative errors for $k=5$ (*left*) and $k=6$ (*right*) in Example 4.14.

Example 4.14. Consider the integral equation [54]

$$\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{1}{12}t^4=\int_0^t (t-u)f^2(u)du \tag{51}$$

with the exact solution $f(t)=1+t$.

Using the above described procedure, applying the Laplace transform to (51), we get $L(f^2(t))=\frac{1}{s}+\frac{2}{s^2}+\frac{2}{s^3}$ i.e. $f^2(t)=L^{-1}\{F(s)\}$, where $F(s)=\frac{1}{s}+\frac{2}{s^2}+\frac{2}{s^3}=\bar{F}\left(\frac{1}{s}\right)$ and take $f^2(t)=X(t)$. Now replacing $1/s$ with the operational matrix I_{k+1} gives $\tilde{F}(I_{k+1})=I_{k+1}+2I_{k+1}^2+2I_{k+1}^3$. Therefore, we compute the coefficient matrix $C^T=d^T I_{k+1}^{-1}\tilde{F}(I_{k+1})=d^T(I+2I_{k+1}+2I_{k+1}^2)$ and find the following results:

For $k=5$,

$$C^T=[0.7298\ 1.0178\ 1.2063\ 1.2431\ 1.0887\ 0.6666].$$

For $k=6$,

$$C^T=[0.6582\ 0.8950\ 1.0715\ 1.1550\ 1.1216\ 0.9485\ 0.5715].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t)=\sqrt{X(t)}$ can be obtained. Table 14 and Figure 14 present the relative errors obtained by the proposed method for $k=5$ and $k=6$.

Table 15. Relative errors for Example 4.15.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0	0.79E−03	0.24E−05
0.2	0.26E−03	0.10E−04
0.4	0.73E−04	0.44E−05
0.6	0.19E−04	0.19E−04
0.8	0.71E−05	0.34E−04
1	0.25E−06	0.16E−04

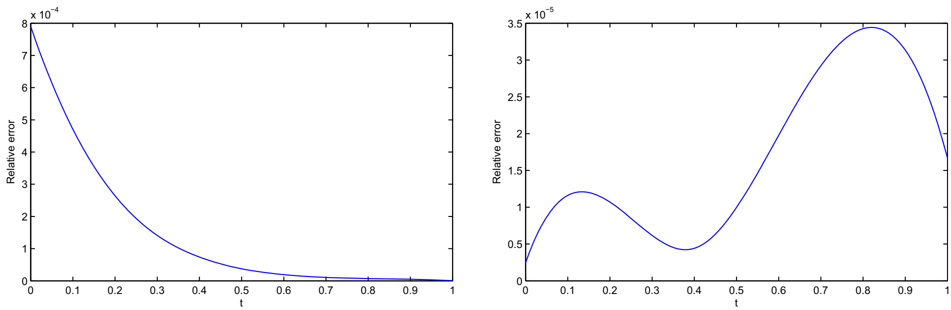

Figure 15. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.15.

Example 4.15. Consider the integral equation [31]

$$\int_0^t \mathrm{e}^{(t-u)} \log(f(u))du = \mathrm{e}^t - t - 1 \tag{52}$$

with the exact solution $f(t) = \mathrm{e}^t$.

Using the above described procedure, applying the Laplace transform to (52), we get $L(\log(f(t))) = \frac{1}{s^2}$, i.e., $\log(f(t)) = L^{-1}\{F(s)\}$, where $F(s) = \frac{1}{s^2} = \bar{F}\left(\frac{1}{s}\right)$ and take $\log(f(t)) = X(t)$. Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}^2$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1} \tilde{F}(I_{k+1}) = d^T I_{k+1}$ and find the following results:

For $k = 5$,

$$C^T = [0.0788\ 0.2143\ 0.2898\ 0.3088\ 0.2722\ 0.1667].$$

For $k = 6$,

$$C^T = [0.0644\ 0.1777\ 0.2500\ 0.2835\ 0.2795\ 0.2371\ 0.1429].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \exp X(t)$ can be obtained. Table 15 and Figure 15 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$.

Table 16. Relative errors for Example 4.16.

t	Rel. error at $k = 5$	Rel. error at $k = 6$
0.2	0.23E−02	0.21E−04
0.4	0.20E−03	0.36E−04
0.6	0.18E−04	0.36E−03
0.8	0.19E−04	0.10E−03
1	0.23E−04	0.34E−03

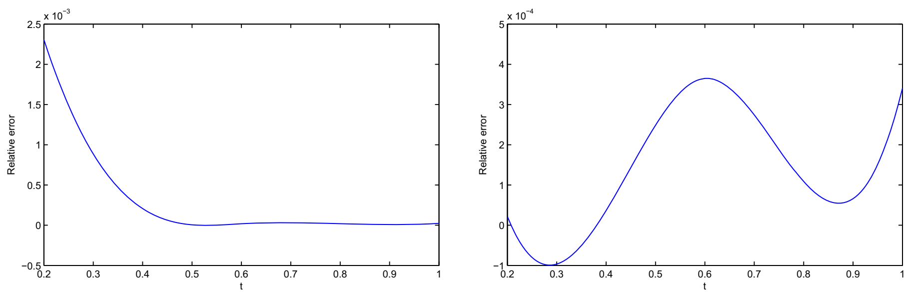

Figure 16. The relative errors for $k = 5$ (*left*) and $k = 6$ (*right*) in Example 4.16.

Example 4.16. Consider the integral equation [31]

$$\int_0^t (\sin(t-u)+1)\cos(f(u))du = \frac{t\sin t}{2} + \sin t \tag{53}$$

with the exact solution $f(t) = t$.

Using the above described procedure, applying the Laplace transform to (53), we get $L(\cos(f(t))) = \frac{s}{s^2+1}$, i.e., $\cos(f(t)) = L^{-1}\{F(s)\}$, where $F(s) = \frac{1}{s}\left(\frac{1}{1+1/s^2}\right) = \bar{F}\left(\frac{1}{s}\right)$ and take $\cos(f(t)) = X(t)$ Now replacing $1/s$ with operational matrix I_{k+1} gives $\tilde{F}(I_{k+1}) = I_{k+1}(I + I_{k+1}^2)^{-1}$. Therefore, we compute the coefficient matrix $C^T = d^T I_{k+1}^{-1}\tilde{F}(I_{k+1}) = d^T(I + I_{k+1}^2)^{-1}$ and find the following results:

For $k = 5$,

$$C^T = [\,0.5431\ 0.4563\ 0.3512\ 0.2530\ 0.1698\ 0.0900\,].$$

For $k = 6$,

$$C^T = [\,0.5080\ 0.4414\ 0.3592\ 0.2776\ 0.2052\ 0.1423\ 0.0772\,].$$

Here, we can find simply $X(t)$ using (14) and then the approximate solution $f_k(t) = \arccos X(t)$ can be obtained. Table 16 and Figure 16 present the relative errors obtained by the proposed method for $k = 5$ and $k = 6$.

5. Accuracy of the solution

To evaluate the accuracy of the proposed method, L_∞, L_2 errors and root mean square error (RMS) are also measured based on the following formulae:

Table 17. Computed L_∞-error, L_2-error and RMS by taking interval [0, 1] and $\triangle t = 0.2$ for different examples.

Examples	L_∞-error	L_2-error	RMS
Example 4.1	0.50E−04	0.54E−04	0.16E−04
Example 4.2	0.10E−05	0.15E−05	0.47E−06
Example 4.3	0.42E−03	0.77E−05	0.23E−05
Example 4.4	0.23E−04	0.11E−03	0.33E−04
Example 4.5	0.34E−04	0.61E−04	0.18E−04
Example 4.6	0.37E−04	0.67E−04	0.20E−04
Example 4.7	0.37E−03	0.47E−03	0.14E−03
Example 4.8	0.83E−04	0.12E−03	0.50E−04
Example 4.9	0.12E−03	0.18E−03	0.74E−05
Example 4.10	0.30E−04	0.38E−04	0.15E−04
Example 4.11	0.83E−04	0.12E−03	0.51E−04
Example 4.12	0.45E−04	0.56E−04	0.22E−04
Example 4.13	0.16E−04	0.27E−04	0.12E−04
Example 4.14	0.90E−04	0.11E−03	0.48E−04
Example 4.15	0.76E−04	0.96E−04	0.39E−04
Example 4.16	0.34E−03	0.41E−03	0.18E−03

Table 18. Comparison of L_∞-error.

Examples	Presented method, $k = 6$	DE Sinc Nystrom method, $k = 8$ [31]	SE Sinc Nystrom method, $k = 8$ [31]
Example 4.8	0.83E−04	0.13E−002	0.23E−003
Example 4.15	0.76E−04	0.23E−002	0.46E−003
Example 4.16	0.34E−03	0.39E−002	0.44E−002

$$L_\infty = \max_{0\le j\le N} |f_{\text{exact}}(t_j) - f_{\text{num}}(t_j)|, \tag{54}$$

$$L_2 = \sqrt{\sum_{j=0}^{N} |f_{\text{exact}}(t_j) - f_{\text{num}}(t_j)|^2}, \tag{55}$$

$$\text{RMS} = \sqrt{\frac{1}{N}\sum_{j=0}^{N} |f_{\text{exact}}(t_j) - f_{\text{num}}(t_j)|^2}. \tag{56}$$

Now, we estimate the maximum absolute error, L_2 error and root mean square error (RMS) for all the above examples and present them in Table 17. In Table 18, the maximum error occurred in Examples 4.8, 4.15 and 4.16 by the proposed method together with DE Sinc Nystrom method and SE Sinc Nystrom method in [31] is compared, which shows that our method gives more accurate results than the method adopted in [31].

6. Conclusion

For calculating inverse Laplace transform, orthonormal Bernstein operational matrix of integration is derived to solve the problems relating differential, integral and integro-differential equations. The method is simple to use and effective compared to other methods. The advantage of the method is that only small size operational matrix is required to provide the solution of good accuracy. A variety of functions can be inverted using this method. The proposed technique is also applicable to solve nonlinear Volterra integral equations of the first kind having the advantage that there is no need of any separate procedure for solving nonlinear term. This one simple procedure can solve the nonlinear function involved in integral sign. The proposed approach gives results without selecting any collocation points. The accuracy of the method is illustrated including the error estimation when compared to the exact solutions and with some existing methods. Furthermore, the inverse function is obtained up to $t = 1$, which could be a strong limitation. Finally, it is concluded that our method is applicable to some extent to have a small accuracy and the inverse function for t varying from 0 to 1.

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