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Supercongruences for sums involving fourth power of some rising factorials

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Abstract. In this paper, we give proof of certain recent conjectural supercongruences on sums involving fourth power of some rising factorials.

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1. Introduction and statement of results

Following [1], we recall the definition of the gamma function given by

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t, \quad \mathrm{Re}(z) > 0.$$

It satisfies the functional equation $\Gamma(1+z) = z\Gamma(z)$, which gives the continuation of $\Gamma(z)$ to a meromorphic function defined for all complex numbers z. Consequently, one can deduce that

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{1.1}$$

where the Pochhammer symbol or the rising factorial $(a)_k$ is defined as

$$(a)_0 := 1 \quad \text{and} \quad (a)_k := a(a+1)\cdots(a+k-1) \text{ for } k \geq 1.$$

Throughout the paper, let p be an odd prime. For $n \in \mathbb{N}$, the p-adic gamma function is defined as

$$\Gamma_p(n) = (-1)^n \prod_{0<j<n,\, p\nmid j} j.$$

Let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ and $v_p(.)$ denote the ring of p-adic integers, the field of p-adic numbers, the completion of the algebraic closure of $\mathbb{Q}_p$, and the p-adic valuation, respectively. Extend Γ_p to all $x \in \mathbb{Z}_p$ by setting

$$\Gamma_p(x) = \lim_{n \to x} \Gamma_p(n),$$

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where n runs through any sequence of positive integers which p-adically approaches x and $\Gamma_p(0) = 1$. For $a \in \mathbb{Q} \cap \mathbb{Z}_p$ and $x \in \mathbb{C}_p$ with $v_p(x) \geq (\frac{1}{p} + \frac{1}{p-1})$, Long and Ramakrishna [14] extended Γ_p by setting

$$\Gamma_p(a+x) = \Gamma_p(a) \sum_{k=0}^{\infty} \frac{G_k(a)}{k!} x^k, \tag{1.2}$$

where $G_k(a) = \frac{\Gamma_p^{(k)}(a)}{\Gamma_p(a)} \in \mathbb{Z}_p$ and $\Gamma_p^{(k)}(x)$ is the k-th derivative of $\Gamma_p(x)$. With these notations, if $A(p)$ denotes the product of the elements in $\{a + x + j \mid j = 0, 1, \ldots, k-1\}$ which are divisible by p, then

$$(a+x)_k = (-1)^k A(p) \frac{\Gamma_p(a+x+k)}{\Gamma_p(a+x)}. \tag{1.3}$$

Supercongruences are congruences those hold for higher powers of primes and are stronger than those predicted by formal group theory. Such supercongruences for sums involving rising factorials arise due to p-adic analogs of complex periods of elliptic curves with complex multiplication computed from the hypergeometric series and gamma function. Van Hamme [20] developed p-adic analogue of some Ramanujan's [17] infinite series representations of $\frac{1}{\pi}$, known as Ramanujan-type supercongruences, which related sums of some rising factorials to the values of p-adic gamma function. Van Hamme himself proved the Ramanujan-type supercongruence [20, (C.2)]

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^4 \equiv p \pmod{p^3}, \tag{1.4}$$

by analyzing a sequence of orthogonal polynomials. Swisher [19] extended (1.4) to modulo p^4 for $p > 3$ by using hypergeometric series identities and evaluation. In [15], Long gave a generalization of (1.4)

$$\sum_{k=0}^{\frac{p^r-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^4 \equiv p^r \pmod{p^{r+3}},$$

which is recently proved by Guo and Wang [8] by establishing a q-analogue, and by Kalita and Jana [12] using hypergeometric series identities and evaluations. In this context, we prove a supercongruence related to (1.4) confirming a recent conjecture of Guo [3, Conjecture 4.3] for $m = 3$.

Theorem 1.1. *For any prime $p > 3$ and positive integer r, we have*

$$\sum_{k=0}^{\frac{p^r-1}{2}} (4k+1)^3 \left(\frac{(\frac{1}{2})_k}{k!} \right)^4 \equiv -p^r \pmod{p^{r+3}}.$$

It is to be noted here that Wang [21] proved Theorem 1.1 for $r = 1$ and $p \equiv 2 \pmod 3$ using the powerful Zeilberger's algorithm [16], and later Liu [13] proved Theorem 1.1 for

$r = 1$ and all odd primes p. Moreover, Guo [4] proved the modulus p^{r+2} case of Theorem 1.1 by the method of creative microscoping introduced in [9] (see also [7]). Very recently, Hou *et al.* [10] reproved Theorem 1.1 for $r = 1$ among other things. Extending Theorem 1.1 to all integers $\ell > 2$, we further prove the following supercongruence.

Theorem 1.2. *Let $\ell > 2$ be a positive integer and $p \geq 5$ a prime number such that $p \equiv -1 \pmod{\ell}$. If r is a positive integer, then*

$$\sum_{k=0}^{\frac{tp^r-1}{\ell}} (2\ell k+1)^3 \frac{(\frac{1}{\ell})_k^4}{k!^4} \equiv \begin{cases} (-1)^{\frac{r}{2}+1} p^{\frac{3r}{2}} \pmod{p^{r+3}}, & \text{if } r \text{ is even and } t=1; \\ (-1)^{\frac{r+1}{2}+\frac{p+1}{\ell}} p^{\frac{3r+1}{2}} \frac{(\ell-1)(2-\ell)}{\ell} \\ \quad \frac{\Gamma_p(1-\frac{2}{\ell})}{\Gamma_p(1-\frac{1}{\ell})^2} \pmod{p^{r+3}}, & \text{if } r \text{ is odd and } t=\ell-1. \end{cases}$$

Recently, Guo and Schlosser [6] proved some supercongruences related to [20, (C.2)] including

$$\sum_{k=0}^{\frac{p^r+1}{2}} (4k-1) \frac{(\frac{-1}{2})_k^4}{k!^4} \equiv 0 \pmod{p^{r+3}} \tag{1.5}$$

as q-analog, by using q-Zeilberger algorithm. In this paper, we prove a generalization of (1.5) in the following theorem using hypergeometric series identities and evaluations.

Theorem 1.3. *Let $\ell \geq 2$ be a positive integer and $p \geq 5$ a prime number such that $p \equiv -1 \pmod{\ell}$. If r is a positive integer, then*

$$\sum_{k=0}^{\frac{tp^r+\ell-1}{\ell}} (2\ell k-\ell+1) \frac{(\frac{1-\ell}{\ell})_k^4}{k!^4} \equiv \begin{cases} 0 \pmod{p^{r+3}}, & \text{if } \ell=2 \text{ and } t=1; \\ (-1)^{\frac{r+2}{2}} 2(\ell-1)(\ell-2) p^{\frac{3r}{2}} \pmod{p^{r+3}}, & \text{if } \ell>2 \text{ with } r \text{ is even and } t=1; \\ (-1)^{\frac{r+1}{2}+\frac{p+1}{\ell}} p^{\frac{3r+1}{2}} \\ \quad \frac{2(\ell-2)^2(\ell-1)^2}{\ell} \frac{\Gamma_p(1-\frac{2}{\ell})}{\Gamma_p(1-\frac{1}{\ell})^2} \pmod{p^{r+3}}, & \text{if } \ell>2 \text{ with } r \text{ is odd and } t=\ell-1. \end{cases}$$

In addition, we finally prove the following supercongruence.

Theorem 1.4. *Let $\ell \geq 2$ be a positive integer and $p \geq 5$ a prime number such that $p \equiv -1 \pmod{\ell}$. If r is any positive integer, then*

$$\sum_{k=0}^{\frac{tp^r+\ell-1}{\ell}} (2\ell k-\ell+1)^3 \frac{(\frac{1-\ell}{\ell})_k^4}{k!^4}$$

$$\equiv \begin{cases} 0 \pmod{p^{r+3}}, & \text{if } \ell = 2 \text{ and } t = 1; \\ (-1)^{\frac{r}{2}} 2(\ell-1)(\ell-2) \\ \quad p^{\frac{3r}{2}} \pmod{p^{r+3}}, & \text{if } \ell > 2 \text{ with } r \text{ is even and } t = 1; \\ (-1)^{\frac{r+3}{2}+\frac{p+1}{\ell}} p^{\frac{3r+1}{2}} \\ \quad \frac{2(\ell-2)^2(\ell-1)^2}{\ell} \frac{\Gamma_p(1-\frac{2}{\ell})}{\Gamma_p(1-\frac{1}{\ell})^2} \pmod{p^{r+3}}, & \text{if } \ell > 2 \text{ with } r \text{ is odd and } t = \ell - 1. \end{cases}$$

For $\ell = 2$ and $t = 1$, Theorem 1.4 confirms a recent conjectural supercongruence of Guo and Liu [5, Conjecture 1.1] for the case $m = 3$, which was already proved by Guo and Liu [5] for $r = 1$ using the software package `Sigma`.

2. Preliminaries

We begin with recalling some elementary properties of p-adic gamma function in the following lemma.

Lemma 2.1 [2, *Section* 11.6]. *Let p be an odd prime and $x \in \mathbb{Z}_p$. Then*

(*i*) $\Gamma_p(1) = -1$.

(*ii*) $\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } \nu_p(x) = 0; \\ -1, & \text{if } \nu_p(x) > 0. \end{cases}$

(*iii*) $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$, *where* $a_0(x) \in \{1, 2, \ldots, p\}$ *satisfies* $a_0(x) \equiv x \pmod{p}$.

Lemma 2.2 [11, *Lemma* 2.3]. *Let p be an odd prime and ζ a primitive n-th root of unity for some positive integer n. Suppose that k and r are positive integers. Assume that $a, b, d \in \mathbb{Q} \cap \mathbb{Z}_p$ and $c \in \mathbb{Q} \cap \mathbb{Z}_p^{\times}$ such that $\nu_p(a+i), \nu_p(c+i) \le r-1$ for each $0 \le i \le k-1$. If $\nu_p((a)_k) - \nu_p((c)_k) \ge 0$, then*

$$\prod_{j=0}^{n-1} \frac{(a + b\zeta^j p^r)_k}{(c + d\zeta^j p^r)_k} \equiv \frac{(a)_k^n}{(c)_k^n} \pmod{p^n}.$$

Moreover for an indeterminate x,

$$\prod_{j=0}^{n-1} \frac{(a + b\zeta^j x^r)_k}{(c + d\zeta^j x^r)_k} \in \mathbb{Z}_p[[x^n]].$$

We use the recurrences relations of some rising factorials listed in the following lemma to prove our main results.

Lemma 2.3 [12, *Lemma* 2.11]. *Let $\ell \ge 2$ be an integer and p be an odd prime with $p \equiv -1 \pmod{\ell}$. If $h \in \mathbb{C}_p$ such that $\nu_p(h) \ge \frac{1}{p} + \frac{1}{p-1} - 1$ and $r > 1$ is a positive integer, then*

$$(a)\ \frac{(\frac{1}{\ell}+1)_{\frac{tp^r-1}{\ell}}}{(\frac{1}{\ell}+1)_{\frac{tp^{r-2}-1}{\ell}}} = \frac{p^{\frac{t(p^{r-1}+p^{r-2})}{\ell}+1}}{(-1)^{t\left(\frac{p^r+p^{r-1}}{\ell}\right)}} \frac{\Gamma_p(1+\frac{tp^r}{\ell})\Gamma_p(1+\frac{tp^{r-1}}{\ell})}{\ell \cdot \Gamma_p(1+\frac{1}{\ell})\Gamma_p(1-\frac{1}{\ell})}.$$

$$(b)\ \frac{(1-\frac{1+hp^r}{\ell})_{\frac{tp^r-1}{\ell}}}{(1-\frac{1+hp^r}{\ell})_{\frac{tp^{r-2}-1}{\ell}}} = \begin{cases} \dfrac{p^{\frac{t(p^{r-1}+p^{r-2})}{\ell}}}{(-1)^{\frac{t(p^r+p^{r-1})}{\ell}}} \dfrac{\Gamma_p(\frac{(\ell-2)+(t-h)p^r}{\ell})\Gamma_p(\frac{2+(t-h)p^{r-1}}{\ell})}{\Gamma_p(1-\frac{1+hp^r}{\ell})\Gamma_p(\frac{1-hp^{r-1}}{\ell})}, & \text{if } \ell > 2; \\ \dfrac{p^{\frac{(p^{r-1}+p^{r-2}-2)}{2}}}{(-1)^{\frac{(p^r+p^{r-1}-2)}{2}}} \dfrac{\Gamma_p(\frac{(1-h)p^r}{2})\Gamma_p(\frac{(1-h)p^{r-1}}{2})}{\Gamma_p(\frac{1-hp^r}{2})\Gamma_p(\frac{1-hp^{r-1}}{2})}, & \text{if } \ell = 2. \end{cases}$$

$$(c)\ \frac{(1+\frac{hp^r}{\ell})_{\frac{tp^r-1}{\ell}}}{(1+\frac{hp^{r-2}}{\ell})_{\frac{tp^{r-2}-1}{\ell}}} = \frac{p^{\left(\frac{t(p^{r-1}+p^{r-2})-\ell}{\ell}\right)}}{(-1)^{\left(\frac{t(p^r+p^{r-1})-\ell}{\ell}\right)}} \frac{\Gamma_p(\frac{(\ell-1)+(h+t)p^r}{\ell})\Gamma_p(\frac{1+(h+t)p^{r-1}}{\ell})}{\Gamma_p(1+\frac{hp^r}{\ell})\Gamma_p(1+\frac{hp^{r-1}}{\ell})}.$$

In the above results, we take $t = \begin{cases} 1, & \textit{if } r \textit{ is even}; \\ \ell-1, & \textit{if } r \textit{ is odd}. \end{cases}$

For complex numbers a_i, b_j and z, with none of the b_j being negative integers or zero, the classical hypergeometric series ${}_{r+1}F_r$ is defined as

$${}_{r+1}F_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}\middle| \lambda\right] := \sum_{k=0}^{\infty} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \frac{\lambda^k}{k!},$$

which converges for $|\lambda| < 1$. When we truncate the above sum at $k = m$, it is known as a truncated hypergeometric series, and we use the following subscript notation to denote truncated hypergeometric series:

$${}_{r+1}F_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}\middle| \lambda\right]_m := \sum_{k=0}^{m} \frac{(a_0)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_r)_k} \frac{\lambda^k}{k!}.$$

Hypergeometric series satisfy many interesting summation and transformation formulas [1,18]. We list here some hypergeometric series identities which will be used to prove our results.

Lemma 2.4 [1, *Theorem* 3.4.4]. *For a positive integer* m,

$${}_7F_6\left(\begin{matrix} a & \frac{a}{2}+1 & b & c & d & e & -m \\ & \frac{a}{2} & 1+a-b & 1+a-c & 1+a-d & 1+a-e & 1+a+m \end{matrix}\middle| 1\right)$$
$$= \frac{(a+1)_m\,(a-d-e+1)_m}{(a-d+1)_m\,(a-e+1)_m} {}_4F_3\left(\begin{matrix} a-b-c+1 & d & e & -m \\ & a-b+1 & a-c+1 & d+e-a-m \end{matrix}\middle| 1\right).$$

Lemma 2.5 [1, *Theorem* 3.4.5].

$${}_7F_6\left(\begin{matrix} a & \frac{a}{2}+1 & b & c & d & e & f \\ & \frac{a}{2} & 1+a-b & 1+a-c & 1+a-d & 1+a-e & 1+a-f \end{matrix}\middle| 1\right)$$
$$= \frac{\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-d-e-f)}{\Gamma(1+a)\Gamma(1+a-e-f)\Gamma(1+a-d-e)\Gamma(1+a-d-f)}$$
$${}_4F_3\left(\begin{matrix} a-b-c+1 & d & e & f \\ & a-b+1 & a-c+1 & d+e-a+f \end{matrix}\middle| 1\right),$$

provided the series on the right-hand side terminates and one of the left converges.

Lemma 2.6 [1, *Corollary* 3.4.3]. *If m is a positive integer, then*

$$
{}_5F_4\left(\begin{matrix} a, & \frac{a}{2}+1, & c, & d, & -m \\ & \frac{a}{2}, & a-c+1, & a-d+1, & a+m+1 \end{matrix}\,\Big|\,1\right)_m = \frac{(a+1)_m\,(a-c-d+1)_m}{(a-c+1)_m(a-d+1)_m}.
$$

Lemma 2.7 [1, *Corollary* 3.5.2].

$$
{}_5F_4\left(\begin{matrix} a, & \frac{a}{2}+1, & c, & d, & e \\ & \frac{a}{2}, & a-c+1, & a-d+1, & a-e+1 \end{matrix}\,\Big|\,1\right) = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-d-e-c)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-d-c)}.
$$

3. Proof of the results

We first deduce certain congruence relations between sums involving fourth power of some rising factorials and truncated hypergeometric series.

Lemma 3.1. *Let $\ell \geq 2$ be an integer and p be an odd prime such that $p \equiv -1 \pmod{\ell}$. If r is a positive integer and $w \neq 1$ is a cubic root of unity, then the following hold for modulo p^{3+r}:*

$$
(a)\ \sum_{k=0}^{\frac{tp^r-1}{\ell}} (2\ell k+1)^3 \frac{(\frac{1}{\ell})_k^4}{k!^4} \equiv {}_7F_6\left(\begin{matrix} \frac{1}{\ell}, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1-wtp^r}{\ell}, & \frac{1-w^2tp^r}{\ell}, & \frac{1-tp^r}{\ell} \\ & \frac{1}{2\ell}, & \frac{1}{2\ell}, & \frac{1}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell}, \end{matrix}\,\Big|\,1\right)_{\frac{tp^r-1}{\ell}},
$$

$$
(b)\ \sum_{k=0}^{\frac{tp^r+\ell-1}{\ell}} \left(\frac{2\ell k}{1-\ell}+1\right)^3 \frac{(\frac{1-\ell}{\ell})_k^4}{k!^4} \equiv {}_7F_6\left(\begin{matrix} \frac{1-\ell}{\ell}, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell-wtp^r}{\ell}, & \frac{1-\ell-w^2tp^r}{\ell}, & \frac{1-\ell-tp^r}{\ell} \\ & \frac{1-\ell}{2\ell}, & \frac{1-\ell}{2\ell}, & \frac{1-\ell}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell}, \end{matrix}\,\Big|\,1\right)_{\frac{tp^r+\ell-1}{\ell}},
$$

$$
(c)\ \sum_{k=0}^{\frac{tp^r+\ell-1}{\ell}} \left(\frac{2\ell k}{1-\ell}+1\right) \frac{(\frac{1-\ell}{\ell})_k^4}{k!^4} \equiv {}_5F_4\left(\begin{matrix} \frac{1-\ell}{\ell}, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell-wtp^r}{\ell}, & \frac{1-\ell-w^2tp^r}{\ell}, & \frac{1-\ell-tp^r}{\ell} \\ & \frac{1-\ell}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell}, \end{matrix}\,\Big|\,1\right)_{\frac{tp^r+\ell-1}{\ell}}.
$$

Proof. The proofs of all these three congruences are identical. We use Lemma 2.4 and Lemma 2.5 for the proof of (a) and (b), whereas Lemmas 2.6 and 2.7 shall be used to prove (c). We give here the proof of (a).

Let p be any odd prime and $\ell \geq 2$ an integer with $p \equiv -1 \pmod{\ell}$. One can easily see that $\nu_p((1)_k) \leq \nu_p((\frac{1}{\ell})_k)$. Therefore, Lemma 2.2 yields

$$
\begin{aligned}
&{}_7F_6\left(\begin{matrix} \frac{1}{\ell}, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1-wtp^r}{\ell}, & \frac{1-w^2tp^r}{\ell}, & \frac{1-tp^r}{\ell} \\ & \frac{1}{2\ell}, & \frac{1}{2\ell}, & \frac{1}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell}, \end{matrix} \;\middle|\; 1\right)_{\frac{tp^r-1}{\ell}} \\
&= \sum_{k=0}^{\frac{tp^r-1}{\ell}} \frac{\left(\frac{1}{\ell}\right)_k}{(1)_k} \frac{\left(\frac{1}{2\ell}+1\right)_k^3}{(\frac{1}{2\ell})_k^3} \frac{(\frac{1-wtp^r}{\ell})_k(\frac{1-w^2tp^r}{\ell})_k(\frac{1-tp^r}{\ell})_k}{(1+\frac{wtp^r}{\ell})_k(1+\frac{w^2tp^r}{\ell})_k(1+\frac{tp^r}{\ell})_k} \\
&\equiv \sum_{k=0}^{\frac{tp^r-1}{\ell}} (2\ell k+1)^3 \frac{(\frac{1}{\ell})_k^4}{k!^4} \pmod{p^3}.
\end{aligned}
$$

Changing p to an indeterminate x in the above hypergeometric series and truncating at $\frac{tp^r-1}{\ell}$, we consider

$$
F(x^r) := {}_7F_6\left(\begin{matrix} \frac{1}{\ell}, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1-wtx^r}{\ell}, & \frac{1-w^2tx^r}{\ell}, & \frac{1-tx^r}{\ell} \\ & \frac{1}{2\ell}, & \frac{1}{2\ell}, & \frac{1}{2\ell}, & 1+\frac{wtx^r}{\ell}, & 1+\frac{w^2tx^r}{\ell}, & 1+\frac{tx^r}{\ell}, \end{matrix} \;\middle|\; 1\right)_{\frac{tp^r-1}{\ell}}.
$$

By Lemma 2.2, we have $F(x^r) \in \mathbb{Z}_p[[x^3]]$, and thus $F(x^r) = \sum_{i\geq 0} f_{3i}x^{3i}$ with

$$
f_0 = \sum_{k=0}^{\frac{tp^r-1}{\ell}} (2\ell k+1)^3 \frac{(\frac{1}{\ell})_k^4}{k!^4}.
$$

We now show that $p^r \mid f_3$ to obtain the desired congruence. For this reason, we see that

$$
G(x^r) := {}_7F_6\left(\begin{matrix} \frac{1-tp^r}{\ell}, & \frac{1-tp^r}{2\ell}+1, & \frac{1-tp^r}{2\ell}+1, & \frac{1-tp^r}{2\ell}+1, & \frac{1-tp^r-wtx^r}{\ell}, & \frac{1-tp^r-w^2tx^r}{\ell}, & \frac{1-tp^r-tx^r}{\ell} \\ & \frac{1-tp^r}{2\ell}, & \frac{1-tp^r}{2\ell}, & \frac{1-tp^r}{2\ell}, & 1+\frac{wtx^r}{\ell}, & 1+\frac{w^2tx^r}{\ell}, & 1+\frac{tx^r}{\ell}, \end{matrix} \;\middle|\; 1\right)
$$

truncates at $\frac{tp^r-1}{\ell}$, and is a rational function in $Z_p[[x^3]]$ having similar coefficients as $F(x^r)$ in $Z_p[[x^3]]$ modulo p^r. Note that Lemma 2.5 gives

$$
\begin{aligned}
G(x^r) &:= \frac{\Gamma(1+\frac{wtx^r}{\ell})\Gamma(1+\frac{w^2tx^r}{\ell})\Gamma(1+\frac{tx^r}{\ell})\Gamma(\frac{\ell-2+2tp^r}{\ell})}{\Gamma(1+\frac{1-tp^r}{\ell})\Gamma(1-\frac{1-tp^r-wtx^r}{\ell})\Gamma(1-\frac{1-tp^r-w^2tx^r}{\ell})\Gamma(1-\frac{1-tp^r-tx^r}{\ell})} \\
&\quad {}_4F_3\left(\begin{matrix} -1 & \frac{1-tp^r-wtx^r}{\ell}, & \frac{1-tp^r-w^2tx^r}{\ell}, & \frac{1-tp^r-tx^r}{\ell} \\ & \frac{1-tp^r}{2\ell}, & \frac{1-tp^r}{2\ell}, & \frac{2(1-tp^r)}{\ell} \end{matrix} \;\middle|\; 1\right)_1 \\
&= 0
\end{aligned}
$$

as $1+\frac{1-tp^r}{\ell}$ is a negative integer. Since

$$G(x^r):=\sum_{k=0}^{\frac{tp^r-1}{\ell}}(2\ell k+1-tp^r)^3(1+tp^r+\cdots)^3 \frac{\left(\frac{1-tp^r}{\ell}\right)_k\left(\frac{1-tp^r-wtx^r}{\ell}\right)_k\left(\frac{1-tp^r-w^2tx^r}{\ell}\right)_k\left(\frac{1-tp^r-tx^r}{\ell}\right)_k}{(1)_k\left(1+\frac{wtx^r}{\ell}\right)_k\left(1+\frac{w^2tx^r}{\ell}\right)_k\left(1+\frac{tx^r}{\ell}\right)_k},$$

we have $F(x^r)\equiv G(x^r)=0 \pmod{p^r}$. As a result, $p^r\,|\,f_i$ for each i, and hence we complete the proof. □

Proof of Theorem 1.1. Let $w=\exp(\frac{2\pi i}{3})$ be a cubic primitive root of unity. Putting $a=\frac{1}{2}, b=\frac{5}{4}, c=\frac{5}{4}, d=\frac{1-wp^r}{2}, e=\frac{1-w^2p^r}{2}$, and $m=\frac{p^r-1}{2}$ in Lemma 2.4, we obtain

$$\begin{aligned}
&{}_7F_6\left(\begin{matrix}\frac{1}{2} & \frac{5}{4} & \frac{5}{4} & \frac{5}{4} & \frac{1-wp^r}{2} & \frac{1-w^2p^r}{2} & \frac{1-p^r}{2}\\ & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1+\frac{wp^r}{2} & 1+\frac{w^2p^r}{2} & 1+\frac{p^r}{2}\end{matrix}\ \Big|\ 1\right)_{\frac{p^r-1}{2}}\\
&=\frac{\left(\frac{3}{2}\right)_{\frac{p^r-1}{2}}\left(\frac{1-p^r}{2}\right)_{\frac{p^r-1}{2}}}{\left(1+\frac{wp^r}{2}\right)_{\frac{p^r-1}{2}}\left(1+\frac{w^2p^r}{2}\right)_{\frac{p^r-1}{2}}}\,{}_4F_3\left(\begin{matrix}-1 & \frac{1-wp^r}{2} & \frac{1-w^2p^r}{2} & \frac{1-p^r}{2}\\ & 1 & \frac{1}{4} & \frac{1}{4}\end{matrix}\ \Big|\ 1\right)_1\\
&=\frac{\left(\frac{3}{2}\right)_{\frac{p^r-1}{2}}\left(\frac{1-p^r}{2}\right)_{\frac{p^r-1}{2}}}{\left(1+\frac{wp^r}{2}\right)_{\frac{p^r-1}{2}}\left(1+\frac{w^2p^r}{2}\right)_{\frac{p^r-1}{2}}}\{1-2(1-p^{3r})\}. \qquad (3.1)
\end{aligned}$$

Using Lemma 2.3(a) for $\ell=2, h=1$, Lemma 2.3(b) for $\ell=2, h=1$ and Lemma 2.3(c) for $\ell=2$ with $h=w$ and w^2, we obtain

$$\begin{aligned}
&\frac{\left(\frac{3}{2}\right)_{\frac{p^r-1}{2}}\left(\frac{1-p^r}{2}\right)_{\frac{p^r-1}{2}}}{\left(1+\frac{wp^r}{2}\right)_{\frac{p^r-1}{2}}\left(1+\frac{w^2p^r}{2}\right)_{\frac{p^r-1}{2}}}\\
&=\left\{\frac{-p^2\Gamma_p(0)^2}{2\cdot\Gamma_p\left(\frac{3}{2}\right)\Gamma_p\left(\frac{1}{2}\right)}\right\}\prod_{u=r-1}^{r}\left\{\prod_{i=1}^{3}\left\{\frac{\Gamma_p\left(1+\frac{w^ip^u}{2}\right)}{\Gamma_p(\frac{1-w^ip^u}{2})}\right\}\right\}\\
&\quad\frac{\left(\frac{3}{2}\right)_{\frac{p^{r-2}-1}{2}}\left(\frac{1-p^{r-2}}{2}\right)_{\frac{p^{r-2}-1}{2}}}{\left(1+\frac{wp^{r-2}}{2}\right)_{\frac{p^{r-2}-1}{2}}\left(1+\frac{w^2p^{r-2}}{2}\right)_{\frac{p^{r-2}-1}{2}}}. \qquad (3.2)
\end{aligned}$$

Case I: Let r be an even positive integer. In this case, we use (3.2) repeatedly to deduce that

$$\frac{\left(\frac{3}{2}\right)_{\frac{p^r-1}{2}}\left(\frac{1-p^r}{2}\right)_{\frac{p^r-1}{2}}}{\left(1+\frac{wp^r}{2}\right)_{\frac{p^r-1}{2}}\left(1+\frac{w^2p^r}{2}\right)_{\frac{p^r-1}{2}}} = (2p^{3r}-1)\left\{\frac{-p^2\Gamma_p(0)^2}{2\cdot\Gamma_p\left(\frac{3}{2}\right)\Gamma_p\left(\frac{1}{2}\right)}\right\}^{\frac{r}{2}}$$
$$\prod_{u=1}^{r}\left\{\prod_{i=1}^{3}\left\{\frac{\Gamma_p\left(1+\frac{w^ip^u}{2}\right)}{\Gamma_p(\frac{1-w^ip^u}{2})}\right\}\right\}$$
$$\equiv -p^r \pmod{p^{3+r}},$$

where the last congruence follows from Lemma 2.1. Thus we complete the proof of this case from (3.1) and Lemma 3.1(a) with $\ell = 2$.

Case II: Let r be an odd positive integer. Following the proof of Case I, we repeatedly use (3.2) to deduce

$$\frac{\left(\frac{3}{2}\right)_{\frac{p^r-1}{2}}\left(\frac{1-p^r}{2}\right)_{\frac{p^r-1}{2}}}{\left(1+\frac{wp^r}{2}\right)_{\frac{p^r-1}{2}}\left(1+\frac{w^2p^r}{2}\right)_{\frac{p^r-1}{2}}}$$
$$= \left\{\frac{-p^2\Gamma_p(0)^2}{2\cdot\Gamma_p\left(\frac{3}{2}\right)\Gamma_p\left(\frac{1}{2}\right)}\right\}^{\frac{r-1}{2}}\prod_{u=2}^{r}\left\{\prod_{i=1}^{3}\left\{\frac{\Gamma_p\left(1+\frac{w^ip^u}{2}\right)}{\Gamma_p(\frac{1-w^ip^u}{2})}\right\}\right\}$$
$$\frac{\left(\frac{3}{2}\right)_{\frac{p-1}{2}}\left(\frac{1-p}{2}\right)_{\frac{p-1}{2}}}{\left(1+\frac{wp}{2}\right)_{\frac{p-1}{2}}\left(1+\frac{w^2p}{2}\right)_{\frac{p-1}{2}}}. \tag{3.3}$$

Noting $\left(\frac{3}{2}\right)_{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}\frac{\Gamma_p(1+\frac{p}{2})}{\Gamma_p\left(\frac{3}{2}\right)}\frac{p}{2}$, $\left(\frac{1-p}{2}\right)_{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}\frac{\Gamma_p(0)}{\Gamma_p\left(\frac{1-p}{2}\right)}$, $\left(1+\frac{wp}{2}\right)_{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}\frac{\Gamma_p\left(\frac{1-w^2p}{2}\right)}{\Gamma_p(1+\frac{wp}{2})}$ and $\left(1+\frac{w^2p}{2}\right)_{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}\frac{\Gamma_p\left(\frac{1-wp}{2}\right)}{\Gamma_p\left(1+\frac{w^2p}{2}\right)}$, we put this in (3.1) to obtain

$${}_7F_6\left(\begin{matrix}\frac{1}{2} & \frac{5}{4} & \frac{5}{4} & \frac{5}{4} & \frac{1-wp^r}{2} & \frac{1-w^2p^r}{2} & \frac{1-p^r}{2}\\ & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1+\frac{wp^r}{2} & 1+\frac{w^2p^r}{2} & 1+\frac{p^r}{2}\end{matrix}\ \middle|\ 1\right)_{\frac{p^r-1}{2}} \equiv -p^r \pmod{p^{3+r}}.$$

This, together with Lemma 3.1(a) yield the desired result. □

Proof of Theorem 1.2. Putting $a = \frac{1}{\ell}$, $b = \frac{1}{2\ell}+1$, $c = \frac{1}{2\ell}+1$, $d = \frac{1-twp^r}{\ell}$, $e = \frac{1-w^2tp^r}{\ell}$ and $m = \frac{tp^r-1}{\ell}$ in Lemma 2.4, we have

$${}_7F_6\left(\begin{matrix}\frac{1}{\ell}, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1}{2\ell}+1, & \frac{1-wtp^r}{\ell}, & \frac{1-w^2tp^r}{\ell}, & \frac{1-tp^r}{\ell}\\ & \frac{1}{2\ell}, & \frac{1}{2\ell}, & \frac{1}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell},\end{matrix}\ \middle|\ 1\right)_{\frac{tp^r-1}{\ell}}$$

$$= \frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}}\left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}\, {}_4F_3\left(\begin{matrix} -1 & \frac{1-wtp^r}{\ell} & \frac{1-tw^2p^r}{\ell} & \frac{1-tp^r}{\ell} \\ & \frac{2}{\ell} & \frac{1}{2\ell} & \frac{1}{2\ell}\end{matrix} \,\Big|\, 1\right)_1$$

$$= \frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}}\left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}\{1-2(1-t^3p^{3r})\}. \tag{3.4}$$

We use Lemma 2.3(a) and (b) for $h = 1$ and Lemma 2.3(c) for $h = w$ and w^2 to obtain

$$\frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}}\left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}} = \left\{\frac{p^3\Gamma_p\left(1-\frac{2}{\ell}\right)\Gamma_p\left(\frac{2}{\ell}\right)}{\ell\cdot\Gamma_p\left(1+\frac{1}{\ell}\right)\Gamma_p\left(1-\frac{1}{\ell}\right)}\right\}$$
$$\prod_{i=1}^{3}\left\{\frac{\Gamma_p\left(1+\frac{w^ip^r}{\ell}\right)\Gamma_p\left(1+\frac{w^ip^{r-1}}{\ell}\right)}{\Gamma_p\left(\frac{\ell-1-w^ip^r}{\ell}\right)\Gamma_p\left(\frac{1-w^ip^{r-1}}{\ell}\right)}\right\}$$
$$\frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^{r-2}-1}{\ell}}\left(\frac{\ell-1-tp^{r-2}}{\ell}\right)_{\frac{tp^{r-2}-1}{\ell}}}{\left(1+\frac{wtp^{r-2}}{\ell}\right)_{\frac{tp^{r-2}-1}{\ell}}\left(1+\frac{w^2tp^{r-2}}{\ell}\right)_{\frac{tp^{r-2}-1}{\ell}}}. \tag{3.5}$$

Case I: Let r be an even positive integer. We use (3.5) repeatedly to deduce that

$$\frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}}\left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}$$
$$= \left\{2t^3p^{3r}-1\right)\Big\}\left\{\frac{p^3\Gamma_p\left(1-\frac{2}{\ell}\right)\Gamma_p\left(\frac{2}{\ell}\right)}{\ell\cdot\Gamma_p\left(1+\frac{1}{\ell}\right)\Gamma_p\left(1-\frac{1}{\ell}\right)}\right\}^{\frac{r}{2}}$$
$$\prod_{j=1}^{\frac{r}{2}}\left\{\prod_{i=1}^{3}\left\{\frac{\Gamma_p\left(1+\frac{w^ip^{2j}}{\ell}\right)\Gamma_p\left(1+\frac{w^ip^{2j-1}}{\ell}\right)}{\Gamma_p\left(\frac{\ell-1-w^ip^{2j}}{\ell}\right)\Gamma_p\left(\frac{1-w^ip^{2j-1}}{\ell}\right)}\right\}\right\},$$
$$\equiv (-1)^{\frac{r}{2}+1}p^{\frac{3r}{2}} \pmod{p^{3+\frac{3r}{2}}},$$

where the last congruence follows from Lemma 2.1. Putting this in (3.4), the result is obtained because of Lemma 3.1(a).

Case II: Let r be an odd positive integers. Noting that

$$\left(\frac{1}{\ell}+1\right)_{\frac{tp-1}{\ell}} = (-1)^{\frac{tp-1}{\ell}}\frac{\Gamma_p\left(1+\frac{tp}{\ell}\right)}{\Gamma_p\left(1+\frac{1}{\ell}\right)}\frac{(\ell-1)p}{\ell},$$
$$\left(1-\frac{1+tp}{\ell}\right)_{\frac{tp-1}{\ell}} = (-1)^{\frac{tp-1}{\ell}}\frac{\Gamma_p\left(1-\frac{2}{\ell}\right)}{\Gamma_p\left(\frac{(\ell-1)-tp}{\ell}\right)}\frac{(2-\ell)p}{\ell},$$

$$\left(1+\frac{wtp}{\ell}\right)_{\frac{tp-1}{\ell}} = (-1)^{\frac{tp-1}{\ell}} \frac{\Gamma_p\left(\frac{(\ell-1)-w^2tp}{\ell}\right)}{\Gamma_p\left(1+\frac{wtp}{\ell}\right)} \quad \text{and}$$

$$\left(1+\frac{w^2tp}{\ell}\right)_{\frac{tp-1}{\ell}} = (-1)^{\frac{tp-1}{\ell}} \frac{\Gamma_p\left(\frac{(\ell-1)-wtp}{\ell}\right)}{\Gamma_p\left(1+\frac{w^2tp}{\ell}\right)},$$

we use (3.5) repeatedly to obtain

$$\begin{aligned}
&\frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}}\left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}} \\
&\quad = \left\{\frac{p^3\Gamma_p\left(1-\frac{2}{\ell}\right)\Gamma_p\left(\frac{2}{\ell}\right)}{\ell\cdot\Gamma_p\left(1+\frac{1}{\ell}\right)\Gamma_p\left(1-\frac{1}{\ell}\right)}\right\}^{\frac{r-1}{2}} \\
&\qquad \prod_{j=1}^{\frac{r-1}{2}}\left\{\prod_{i=1}^{3}\left\{\frac{\Gamma_p\left(1+\frac{w^i p^{2j+1}}{\ell}\right)\Gamma_p\left(1+\frac{w^i p^{2j}}{\ell}\right)}{\Gamma_p\left(\frac{\ell-1-w^i p^{2j+1}}{\ell}\right)\Gamma_p\left(\frac{1-w^i p^{2j}}{\ell}\right)}\right\}\right\} \\
&\qquad \frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp-1}{\ell}}\left(\frac{\ell-1-tp}{\ell}\right)_{\frac{tp-1}{\ell}}}{\left(1+\frac{wtp}{\ell}\right)_{\frac{tp-1}{\ell}}\left(1+\frac{w^2tp}{\ell}\right)_{\frac{tp-1}{\ell}}} \\
&\quad \equiv (-1)^{\frac{r+1}{2}+\frac{p+1}{\ell}} p^{\frac{3r+1}{2}} \frac{(\ell-1)(2-\ell)}{\ell} \frac{\Gamma_p(1-\frac{2}{\ell})}{\Gamma_p(1-\frac{1}{\ell})^2} \pmod{p^{\frac{3r+7}{2}}}.
\end{aligned}$$

where the last congruence follows from Lemma 2.1. Using this in (3.4), Lemma 3.1(a) completes the proof of the theorem. □

Proof of Theorem 1.3. From Lemma 2.6, we have

$$\begin{aligned}
&{}_5F_4\left(\begin{matrix}\frac{1-\ell}{\ell}, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell-wtp^r}{\ell}, & \frac{1-\ell-w^2tp^r}{\ell}, & \frac{1-\ell-tp^r}{\ell} \\ & \frac{1-\ell}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell},\end{matrix} \middle| \, 1\right)_{\frac{tp^r+\ell-1}{\ell}} \\
&\quad = \frac{\left(\frac{1}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}}\left(\frac{2\ell-1-tp^r}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}}} \\
&\quad = \frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}}\left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}\left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}} \\
&\qquad \left\{\frac{\frac{1}{\ell}\left(1-\frac{2}{\ell}\right)\left(2-\frac{2}{\ell}\right)}{\left(\frac{\ell-1-tp^r}{\ell}\right)\left(\frac{\ell-1-wtp^r}{\ell}\right)\left(\frac{\ell-1-w^2tp^r}{\ell}\right)}\right\}. \qquad (3.6)
\end{aligned}$$

For $\ell = 2$, clearly the right-hand side of (3.6) is zero. If $\ell > 2$, we use (3.5) appropriately in (3.6), and Lemma 3.1(c) completes our proof. □

Proof of Theorem 1.4. Using Lemma 2.4, we have

$$
\begin{aligned}
&{}_7F_6\left(\begin{matrix} \frac{1-\ell}{\ell}, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell}{2\ell}+1, & \frac{1-\ell-wtp^r}{\ell}, & \frac{1-\ell-w^2tp^r}{\ell}, & \frac{1-\ell-tp^r}{\ell} \\ & \frac{1-\ell}{2\ell}, & \frac{1-\ell}{2\ell}, & \frac{1-\ell}{2\ell}, & 1+\frac{wtp^r}{\ell}, & 1+\frac{w^2tp^r}{\ell}, & 1+\frac{tp^r}{\ell}, \end{matrix} \;\middle|\; 1\right)_{\frac{tp^r+\ell-1}{\ell}} \\
&= \frac{\left(\frac{1}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}} \left(\frac{2\ell-1-tp^r}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}} \left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r+\ell-1}{\ell}}} {}_4F_3\left(\begin{matrix} -1 & \frac{1-\ell-wtp^r}{\ell} & \frac{1-\ell-tw^2p^r}{\ell} & \frac{1-\ell-tp^r}{\ell} \\ \frac{1-\ell}{2\ell} & & \frac{1-\ell}{2\ell} & \frac{2(1-\ell)}{\ell} \end{matrix} \;\middle|\; 1\right)_1 \\
&= \frac{\left(\frac{1}{\ell}+1\right)_{\frac{tp^r-1}{\ell}} \left(\frac{\ell-1-tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}}{\left(1+\frac{wtp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}} \left(1+\frac{w^2tp^r}{\ell}\right)_{\frac{tp^r-1}{\ell}}} \left\{\frac{\frac{1}{\ell}\left(1-\frac{2}{\ell}\right)\left(2-\frac{2}{\ell}\right)}{\left(\frac{\ell-1-tp^r}{\ell}\right)\left(\frac{\ell-1-wtp^r}{\ell}\right)\left(\frac{\ell-1-w^2tp^r}{\ell}\right)}\right\} \\
&\quad \left\{1-2\left(1-\frac{t^3p^{3r}}{(1-\ell)^3}\right)\right\}.
\end{aligned} \tag{3.7}
$$

Following the proof of Theorem 1.3 and using Lemma 3.1(b), we easily obtain the desired result. □

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