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On the automorphism groups of connected bipartite irreducible graphs

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Abstract. Let $G = (V, E)$ be a graph with the vertex-set V and the edge-set E. Let $N(v)$ denote the set of neighbors of the vertex v of G. The graph G is called *irreducible* whenever for every $v, w \in V$ if $v \neq w$, then $N(v) \neq N(w)$. In this paper, we present a method for finding automorphism groups of connected bipartite irreducible graphs. Then, by our method, we determine automorphism groups of some classes of connected bipartite irreducible graphs, including a class of graphs which are derived from Grassmann graphs. Let a_0 be a fixed positive integer. We show that if G is a connected non-bipartite irreducible graph such that $c(v, w) = |N(v) \cap N(w)| = a_0$ when v, w are adjacent, whereas $c(v, w) \neq a_0$, when v, w are not adjacent, then G is a *stable* graph, that is, the automorphism group of the bipartite double cover of G is isomorphic with the group $\mathrm{Aut}(G) \times \mathbb{Z}_2$. Finally, we show that the Johnson graph $J(n, k)$ is a stable graph.

Keywords. Automorphism group; bipartite double cover of a graph; Grassmann graph; stable graph; Johnson graph.

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1. Introduction

In this paper, a graph $G = (V, E)$ is considered as an undirected simple finite graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For the terminology and notation not defined here, we follow [1,2,4,7].

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite graph with parts U and W. It is quite possible that we wish to construct some other graphs which are related to G in some aspects. For instance, there are cases in which we can construct a graph $G_1 = (U, E_1)$ such that we have $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1)$, where $\mathrm{Aut}(X)$ is the automorphism group of the graph X. For example, note the following cases:

(i) Let $n \geq 3$ be an integer and $[n] = \{1, 2, \ldots, n\}$. Let k be an integer such that $1 \leq k < \frac{n}{2}$. The graph $B(n, k)$ introduced in [16] is a graph with the vertex-set $V = \{v \mid v \subset [n], |v| \in \{k, k+1\}\}$ and the edge-set $E = \{\{v, w\} \mid v, w \in V, v \subset w$ or $w \subset v\}$. It is clear that the graph $B(n, k)$ is a bipartite graph with the vertex-set $V = V_1 \cup V_2$, where $V_1 = \{v \subset [n] \mid |v| = k\}$ and $V_2 = \{v \subset [n] \mid |v| = k + 1\}$. This graph has some interesting properties which have been investigated recently

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[11,16,17,20]. Let $G = B(n, k)$ and let $G_1 = (V_1, E_1)$ be the Johnson graph $J(n, k)$ which can be constructed on the vertex-set V_1. It has been proved that if $n \neq 2k + 1$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1)$, and if $n = 2k + 1$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \times \mathbb{Z}_2$ [16].

(ii) Let n and k be integers with $n > 2k, k \geq 1$. Let V be the set of all k-subsets and $(n-k)$-subsets of $[n]$. The *bipartite Kneser graph* $H(n, k)$ has V as its vertex-set, and two vertices v, w are adjacent if and only if $v \subset w$ or $w \subset v$. It is clear that $H(n, k)$ is a bipartite graph. In fact, if $V_1 = \{v \subset [n] \mid |v| = k\}$ and $V_2 = \{v \subset [n] \mid |v| = n-k\}$, then $\{V_1, V_2\}$ is a partition of $V(H(n, k))$ and every edge of $H(n, k)$ has a vertex in V_1 and a vertex in V_2 and $|V_1| = |V_2|$. Let $G = H(n, k)$ and let $G_1 = (V_1, E_1)$ be the Johnson graph $J(n, k)$ which can be constructed on the vertex-set V_1. It has been proved that $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \times \mathbb{Z}_2$ [18].

(iii) Let n, k and l be integers with $0 < k < l < n$. The *set-inclusion graph* $G(n, k, l)$ is the graph whose vertex-set consists of all k-subsets and l-subsets of $[n]$, where two distinct vertices are adjacent if one of them is contained in another. It is clear that the graph $G(n, k, l)$ is a bipartite graph with the vertex-set $V = V_1 \cup V_2$, where $V_1 = \{v \subset [n] \mid |v| = k\}$ and $V_2 = \{v \subset [n] \mid |v| = l\}$. It is easy to show that $G(n, k, l) \cong G(n, n - k, n - l)$, hence we assume that $k + l \leq n$. It is clear that if $l = k + 1$, then $G(n, k, l) = B(n, k)$, where $B(n, k)$ is the graph which is defined in (i). Also, if $l = n - k$, then $G(n, k, l) = H(n, k)$, where $H(n, k)$ is the graph which is introduced in (ii). Let $G = G(n, k, l)$ and let $G_1 = (V_1, E_1)$ be the Johnson graph $J(n, k)$ which can be constructed on the vertex-set V_1. It has been proved that if $n \neq k + l$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1)$, and if $n = k + l$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \times \mathbb{Z}_2$ [9].

Let $G = (V, E)$ be a graph. The *bipartite double cover* of G which we denote it by $B(G)$ is a graph with the vertex-set $V \times \{0, 1\}$, in which vertices (v, a) and (w, b) are adjacent if and only if $a \neq b$ and $\{v, w\} \in E$. A graph G is called *stable* if and only if $\mathrm{Aut}(B(G)) \cong \mathrm{Aut}(G) \times \mathbb{Z}_2$.

In this paper, we generalize the results of our examples to some other classes of bipartite graphs. In fact, we state some accessible conditions such that if for a bipartite graph $G = (V, E) = (U \cup W, E)$ these conditions hold, then we can determine the automorphism group of the graph G. Also, we determine the automorphism group of a class of graphs which are derived from Grassmann graphs. In particular, we determine automorphism groups of bipartite double covers of some classes of graphs. In fact, we show that if G is a non-bipartite connected irreducible graph, and a_0 is a positive integer such that $c(v, w) = |N(v) \cap N(w)| = a_0$, when v and w are adjacent, whereas $c(v, w) \neq a_0$ when v and w are not adjacent, then the graph G is a stable graph. Finally, we show that Johnson graphs are stable graphs.

2. Preliminaries

The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\alpha : V_1 \longrightarrow V_2$ such that $\{a, b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an *isomorphism*. An *automorphism* of a graph G is an isomorphism of G with itself. The set of automorphisms of Γ with the operation of composition of functions is a group called the *automorphism group* of G and denoted by $\mathrm{Aut}(G)$.

The group of all permutations of a set V is denoted by $\mathrm{Sym}(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* Γ on V is a subgroup of $\mathrm{Sym}(V)$. In this case, we say that Γ *acts* on V. If Γ acts on V we say that Γ is *transitive* on V (or Γ acts *transitively* on V), when there is just one orbit. This means that given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$. If X is a graph with vertex-set V then we can view each automorphism of X as a permutation on V and so $\mathrm{Aut}(X) = \Gamma$ is a permutation group on V.

A graph G is called *vertex-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(\Gamma)$. We say that G is *edge-transitive* if the group $\mathrm{Aut}(G)$ acts transitively on the edge set E, namely, for any $\{x, y\}, \{v, w\} \in E(G)$, there is some π in $\mathrm{Aut}(G)$, such that $\pi(\{x, y\}) = \{v, w\}$. We say that G is *symmetric* (or *arc-transitive*) if for all vertices u, v, x, y of G such that u and v are adjacent, and also, x and y are adjacent, there is an automorphism π in $\mathrm{Aut}(G)$ such that $\pi(u) = x$ and $\pi(v) = y$. We say that G is *distance-transitive* if for all vertices u, v, x, y of G such that $d(u, v) = d(x, y)$, where $d(u, v)$ denotes the distance between the vertices u and v in G, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$.

Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, \ldots, n\}$. The *Johnson graph* $J(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v, w are adjacent if and only if $|v \cap w| = k - 1$. The Johnson graph $J(n, k)$ is a distance-transitive graph [2]. It is easy to show that the set $H = \{f_\theta \mid \theta \in \mathrm{Sym}([n])\}$, $f_\theta(\{x_1, \ldots, x_k\}) = \{\theta(x_1), \ldots, \theta(x_k)\}$ is a subgroup of $\mathrm{Aut}(J(n, k))$. It has been shown that $\mathrm{Aut}(J(n, k)) \cong \mathrm{Sym}([n])$, if $n \neq 2k$, and $\mathrm{Aut}(J(n, k)) \cong \mathrm{Sym}([n]) \times \mathbb{Z}_2$, if $n = 2k$, where $\mathbb{Z}_2$ is the cyclic group of order 2 [10,19].

The group Γ is called a semidirect product of N by Q, denoted by $\Gamma = N \rtimes Q$, if Γ contains subgroups N and Q such that

(i) $N \trianglelefteq \Gamma$ (N is a normal subgroup of Γ);
(ii) $NQ = \Gamma$; and
(iii) $N \cap Q = 1$.

Although in most situations it is difficult to determine the automorphism group of a graph G, there are various papers in the literature dealing with this, and some of the recent works include [5,6,10,14–16,18,19,24].

3. Main results

The proof of the following lemma is easy but its result is necessary for proving the results of our work.

Lemma 3.1. *Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. If f is an automorphism of the graph G, then $f(U) = U$ and $f(W) = W$, or $f(U) = W$ and $f(W) = U$.*

Proof. Automorphisms of G preserve distance between vertices and since two vertices are in the same part if and only if they are at even distance from each other, the result follows. □

We have the following definition due to Sabidussi [22].

DEFINITION 3.2

Let $G = (V, E)$ be a graph with the vertex-set V and the edge-set E. Let $N(v)$ denote the set of neighbors of the vertex v of G. We say that G is an irreducible graph if for every pare of distinct vertices $x, y \in V$ we have $N(x) \neq N(y)$.

From Definition 3.2, it follows that the cycle C_n, $n \neq 4$, is irreducible, but the complete bipartite graph $K_{m,n}$ is not irreducible, when $(m, n) \neq (1, 1)$.

Lemma 3.3. *Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite irreducible graph. If f is an automorphism of G such that $f(u) = u$ for every $u \in U$, then f is the identity automorphism of G.*

Proof. Let $w \in W$ be an arbitrary vertex. Since f is an automorphism of the graph G, then for the set $N(w) = \{u | u \in U, u \leftrightarrow w\}$, we have $f(N(w)) = \{f(u) | u \in U, u \leftrightarrow w\} = N(f(w))$. On the other hand, since for every $u \in U$, $f(u) = u$, then we have $f(N(w)) = N(w)$, and therefore $N(f(w)) = N(w)$. Now since G is an irreducible graph we must have $f(w) = w$. Therefore, for every vertex x in $V(G)$ we have $f(x) = x$ and thus f is the identity automorphism of the graph G. □

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite graph. We can construct various graphs on the set U. We show that some of these graphs can help us in finding the automorphism group of the graph G.

DEFINITION 3.4

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite graph. Let $G_1 = (U, E_1)$ be a graph with the vertex-set U such that the following conditions hold:

(i) Every automorphism of the graph G_1 can be uniquely extended to an automorphism of the graph G. In other words, if f is an automorphism of the graph G_1, then there is a unique automorphism e_f in the automorphism group of G such that $(e_f)|_U = f$, where $(e_f)|_U$ is the restriction of the automorphism e_f to the set U.

(ii) If $f \in \mathrm{Aut}(G)$ is such that $f(U) = U$, then the restriction of f to U is an automorphism of the graph G_1. In other words, if $f \in \mathrm{Aut}(G)$ is such that $f(U) = U$ then $f|_U \in \mathrm{Aut}(G_1)$.

When such a graph G_1 exists, then we say that the graph G_1 is a faithful representation of G.

Remark 3.5. Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite irreducible graph, and $G_1 = (U, E_1)$ be a graph. If $f \in \mathrm{Aut}(G_1)$ can be extended to an automorphism g of the graph G, then g is unique. In fact if g and h are extensions of the automorphism $f \in \mathrm{Aut}(G_1)$ to automorphisms of G, then $i = gh^{-1}$ is an automorphism of the graph G such that the restriction of i to the set U is the identity automorphism. Hence by Lemma 3.3, the automorphism i is the identity automorphism of the graph G, and therefore $g = h$. Hence, according to Definition 3.4, the graph G_1 is a faithful representation of the graph G if and only if every automorphism of G_1 can be extended to an automorphism of G and every automorphism of G which fixes U setwise is an automorphism of G_1.

Example 3.6. Let $G = H(n,k) = (V_1 \cup V_2, E)$ be the bipartite Kneser graph which is introduced in (ii) of the Introduction of the present paper. Let $G_1 = (V_1, E_1)$ be the Johnson graph which can be constructed on the vertex V_1. It can be shown that the graph G_1 is a faithful representation of G [18].

In the next theorem, we show that if $G = (U \cup W, E)$, $U \cap W = \emptyset$ is a connected bipartite irreducible graph with $G_1 = (U, E_1)$ as a faithful representation of G, then we can determine the automorphism group of the graph G, provided the automorphism group of the graph G_1 has been determined.

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$, be a connected bipartite irreducible graph such that $G_1 = (U, E_1)$ is a faithful representation of G. If $f \in \mathrm{Aut}(G_1)$ then we let e_f be its unique extension to $\mathrm{Aut}(G)$. It is easy to see that $E_{G_1} = \{e_f | f \in \mathrm{Aut}(G_1)\}$, with the operation of composition, is a group. Moreover, it is easy to see that E_{G_1} and $\mathrm{Aut}(G_1)$ are isomorphic (as abstract groups).

For the bipartite graph $G = (U \cup W, E)$ we let $S(U) = \{f \in \mathrm{Aut}(G) | f(U) = U\}$=$\mathrm{Aut}(G)_U$, the stabilizer subgroup of the set U in the group $\mathrm{Aut}(G)$. The next proposition shows that when $G_1 = (U, E_1)$ is a faithful representation of G, then $S(U)$ is a familiar group.

PROPOSITION 3.7

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite irreducible graph such that $G_1 = (U, E_1)$ is a faithful representation of G. Then $S(U) \cong \mathrm{Aut}(G_1)$, where $S(U) = \{f \in \mathrm{Aut}(G) | f(U) = U\}$.

Proof. Let f be an automorphism of the graph G_1. Then by definition of the graph G_1 we deduce that e_f is an automorphism of the graph G such that $e_f(U) = U$. Hence, we have $E_{G_1} \leq S(U)$, where E_{G_1} is the group which is defined preceding this theorem.

On the other hand, if $g \in S(U)$, then $g(U) = U$. Thus by the definition of the graph G_1, the restriction of g to U is an automorphism of the graph G_1. In other words, $h = g|_U \in \mathrm{Aut}(G_1)$. Therefore, by Definition 3.4, there is an automorphism e_h of the graph G such that $e_h(u) = g(u)$ for every $u \in U$. Now by Remark 3.5, we deduce that $g = e_h \in E_{G_1}$. Hence we have $S(U) \leq E_{G_1}$. We now deduce that $S(U) = E_{G_1}$. Now, since $E_{G_1} \cong \mathrm{Aut}(G_1)$, we conclude that $S(U) \cong \mathrm{Aut}(G_1)$. □

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. It is quite possible that $f(U) = U$, for every automorphism of the graph G. For example, if $|U| \neq |W|$, or U contains a vertex of degree d, but W does not contain a vertex of degree d, then we have $f(U) = U$ for every automorphism f of the graph G. In such a case we have $\mathrm{Aut}(G) = S(U)$, and hence by Proposition 3.7, we have the following theorem.

Theorem 3.8. *Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite irreducible graph such that $G_1 = (U, E_1)$ is a faithful representation of G. If $\mathrm{Aut}(G) = S(U)$, then $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1)$.*

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite irreducible graph. Concerning the automorphism group of G, we can say even more if $|U| = |W|$. When $|U| = |W|$ then there is a bijection $\theta : U \to W$. Then $\theta^{-1} \cup \theta = t$ is a permutation on the vertex-set of the graph G such that $t(U) = W$ and $t(W) = U$. In the following theorem, we show that if the graph G has a faithful representation $G_1 = (U, E_1)$, and if such a permutation t is an automorphism of the graph G, then the automorphism group of the graph G is a familiar group.

Theorem 3.9. *Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite irreducible graph such that $G_1 = (U, E_1)$ is a faithful representation of G and $|U| = |W|$. Suppose that there is an automorphism t of the graph G such that $t(U) = W$. Then* $\mathrm{Aut}(G) = \mathrm{Aut}(G_1) \rtimes H$, *where $H = \langle t \rangle$ is the subgroup generated by t in the group* $\mathrm{Aut}(G)$.

Proof. Let $S(U) = \{f \in \mathrm{Aut}(G) | f(U) = U\}$. It is clear that $S(U)$ is a subgroup of $\mathrm{Aut}(G)$. Let $g \in \mathrm{Aut}(G)$ be such that $g(U) \neq U$. Then by Lemma 3.1, we have $g(U) = W$, and hence $tg(U) = t(W) = U$. Therefore, $tg \in S(U)$, and hence there is an element $h \in S(U)$ such that $tg = h$. Thus, $g = t^{-1}h \in \langle t, S(U) \rangle$, where $\langle t, S(U) \rangle = K$ is the subgroup of $\mathrm{Aut}(G)$ which is generated by t and $S(U)$. It follows that $\mathrm{Aut}(G) \leq K$. Since $K \leq \mathrm{Aut}(G)$, we deduce that $K = \mathrm{Aut}(G)$. If f is an arbitrary element in the subgroup $S(U)$ of K, then we have $(t^{-1}ft)(U) = (t^{-1}f)(W) = t^{-1}(f(W)) = (t^{-1})(W) = U$, hence $t^{-1}ft \in S(U)$. We now deduce that $S(U)$ is a normal subgroup of the group K. Therefore, $K = \langle t, S(U) \rangle = S(U) \rtimes \langle t \rangle = S(U) \rtimes H$, where $H = \langle t \rangle$. We have seen in Proposition 3.8 that $S(U) \cong \mathrm{Aut}(G_1)$, and hence we conclude that $K = \mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \rtimes H$. □

In the sequel, we will see how Theorems 3.8 and 3.9 can help us in determining the automorphism groups of some classes of bipartite graphs.

Some applications Let $G = (U \cup W) = G(n, k, l)$ be the bipartite graph which is defined in (iii) of the Introduction of the present paper. Then $U = \{v \subset [n] \mid |v| = k\}$ and $W = \{v \subset [n] \mid |v| = l\}$. It is easy to show that G is connected and irreducible. Let $G_1 = (U, E_1)$ be the Johnson graph which can be constructed on the set U. By a proof exactly similar to what appeared in [16,18] and later [9], it can be shown that G_1 is a faithful representation of G. We know that $\mathrm{Aut}(G_1) = H = \{f_\theta | \theta \in \mathrm{Sym}([n])\}$, where $f_\theta(v) = \{\theta(x) | x \in v\}$ for every $v \in U$, because $k < l$ and $k + l \leq n$ imply that $k < \frac{n}{2}$. When $k+l = n$, then the mapping $t : V(G) \to V(G)$, defined by the rule $t(v) = v^c$, where v^c is the complement of the set v in the set $[n] = \{1, 2, 3, \ldots, n\}$, is an automorphism of G. It is clear that $t(U) = W$ and $t(W) = U$. Moreover, t is of order 2, and hence $\langle t \rangle \cong \mathbb{Z}_2$. It is easy to show that if $f \in H$, then $ft = tf$ [16,19]. Now, from Theorem 3.8 and Theorem 3.9, we obtain the following theorem which has been given in [9].

Theorem 3.10. *Let n, k and l be integers with $1 \leq k < l \leq n - 1$ and $G = G(n, k, l)$. If $n \neq k + l$, then* $\mathrm{Aut}(G) \cong \mathrm{Sym}([n])$, *and if $n = k + l$, then* $\mathrm{Aut}(G) = H \rtimes \langle t \rangle \cong H \times \langle t \rangle \cong \mathrm{Sym}([n]) \times \mathbb{Z}_2$, *where H and t are the group and automorphism which are defined preceding this theorem.*

We now consider a class of graphs which are in some combinatorial aspects similar to Johnson graphs.

DEFINITION 3.11

Let p be a positive prime integer and $q = p^m$, where m is a positive integer. Let n, k be positive integers with $k < n$. Let $V(q, n)$ be a vector space of dimension n over the finite field $\mathbb{F}_q$. Let V_k be the family of all subspaces of $V(q, n)$ of dimension k. Every element of V_k is also called a k-subspace. The Grassmann graph $G(q, n, k)$ is the graph with the vertex-set V_k, in which two vertices u and w are adjacent if and only if $\dim(u \cap w) = k - 1$.

Note that if $k = 1$, we have a complete graph, so we shall assume that $k > 1$. It is clear that the number of vertices of the Grassmann graph $G(q, n, k)$, that is, $|V_k|$, is the Gaussian binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \frac{(q^n - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1) \cdots (q - 1)}.$$

Noting that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$, it follows that $|V_k| = |V_{n-k}|$. It is easy to show that if $1 \leq i < j \leq \frac{n}{2}$, then $|V_i| < |V_j|$. Let $(,)$ be any nondegenerate symmetric bilinear form on $V(q, n)$. For each $X \subset V(q, n)$, we let $X^{\perp} = \{w \in V(q, n) | (x, w) = 0$, for every $x \in X\}$. It can be shown that if v is a subspace of $V(q, n)$, then $v^{\perp}$ is also a subspace of $V(q, n)$ and $\dim(v^{\perp}) = n - \dim(v)$. It can be shown that $G(n, q, k) \cong G(n, q, n - k)$ [2], and hence in the sequel we assume that $k \leq \frac{n}{2}$.

It is easy to see that the distance between two vertices v and w in this graph is $k - \dim(v \cap w)$. The Grassmann graph is a distance-regular graph of diameter k [2]. Let K be a field and $V(n)$ be a vector space of dimension n over the field K. Let $\tau : K \longrightarrow K$ be a field automorphism. A semilinear operator on $V(n)$ is a mapping $f : V(n) \longrightarrow V(n)$ such that

$$f(c_1 v_1 + c_2 v_2) = \tau(c_1) f(v_1) + \tau(c_2) f(v_2) \quad (c_1, c_2 \in K \quad \text{and} \quad v_1, v_2 \in V(n)).$$

A semilinear operator $f : V(n) \longrightarrow V(n)$ is a semilinear automorphism if it is a bijection. Let $\Gamma L_n(K)$ be the group of semilinear automorphisms on $V(n)$. Note that this group contains $A(V(n))$, where $A(V(n))$ is the group of non-singular linear mappings on the space $V(n)$. Also, this group contains a normal subgroup isomorphic to K^*, namely, the group $Z = \{k I_{V(n)} | k \in K\}$, where $I_{V(n)}$ is the identity mapping on $V(n)$. We denote the quotient group $\frac{\Gamma L_n(K)}{Z}$ by $P\Gamma L_n(K)$.

Note that if $(a + Z) \in P\Gamma L_n(K)$ and x is an m-subspace of $V(n)$, then $(a + Z)(x) = \{a(u) | u \in x\}$ is an m-subspace of $V(n)$. In the sequel, we also denote $(a + Z) \in P\Gamma L_n(K)$ by a. Now, if $a \in P\Gamma L_n(\mathbb{F}_q)$, it is easy to see that the mapping $f_a : V_k \longrightarrow V_k$, defined by the rule $f_a(v) = a(v)$, is an automorphism of the Grassmann graph $G = G(q, n, k)$. Therefore, if we let

$$A = \{f_a | a \in P\Gamma L_n(\mathbb{F}_q)\}, \tag{1}$$

then A is a group isomorphic to the group $P\Gamma L_n(\mathbb{F}_q))$ (as abstract groups), and we have $A \leq \mathrm{Aut}(G)$.

When $n = 2k$, then the Grassmann graph $G = G(q, n, k)$ has some other automorphisms. In fact if $n = 2k$, then the mapping $\theta : V_k \longrightarrow V_k$, which is defined by this rule $\theta(v) = v^{\perp}$, for every k-subspace of $V(2k)$, is an automorphism of the graph $G = G(q, 2k, k)$. Hence $M = \langle A, \theta\rangle \leq \mathrm{Aut}(G)$. It can be shown that A is a normal subgroup of the group M. Therefore $M = A \rtimes \langle\theta\rangle$. Note that the order of θ is 2 and hence $\langle\theta\rangle \cong \mathbb{Z}_2$. Concerning the automorphism groups of Grassmann graphs, from a known fact which appeared in [3], we have the following result [2].

Theorem 3.12. *Let G be the Grassmann graph $G = G(q, n, k)$, where $n > 3$ and $k \leq \frac{n}{2}$. If $n \neq 2k$, then we have $\mathrm{Aut}(G) = A \cong P\Gamma L_n(\mathbb{F}_q)$, and if $n = 2k$, then we have $\mathrm{Aut}(G) = \langle A, \theta\rangle \cong A \rtimes \langle\theta\rangle \cong P\Gamma L_n(\mathbb{F}_q) \rtimes \mathbb{Z}_2$, where A is the group which is defined in* (1) *and θ is the mapping which is defined preceding this theorem.*

We now proceed to determine the automorphism group of a class of bipartite graphs which are similar in some aspects to the graphs $B(n, k)$

DEFINITION 3.13

Let n, k be positive integers such that $n \geq 3$, $k \leq n - 1$. Let q be a power of a prime and $\mathbb{F}_q$ be the finite field of order q. Let $V(q, n)$ be a vector space of dimension n over $\mathbb{F}_q$. We define the graph $S(q, n, k)$ as a graph with the vertex-set $V = V_k \cup V_{k+1}$, in which two vertices v and w are adjacent whenever v is a subspace of w or w is a subspace of v, where V_k and V_{k+1} are the sets of subspaces in $V(q, n)$ of dimension k and $k + 1$, respectively.

When $n = 2k+1$, then the graph $S(q, n, k)$ is known as a doubled Grassmann graph [2]. Noting that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$, it is easy to show that $S(n, q, k) \cong S(n, q, n - k - 1)$. Hence in the sequel we assume $k < \frac{n}{2}$. It can be shown that the graph $S(q, n, k)$ is a connected bipartite irreducible graph. We formally state and prove these facts.

PROPOSITION 3.14

The graph $G = S(q, n, k)$ which is defined in Definition 3.13, is a connected bipartite irreducible graph.

Proof. It is clear that the graph $G = S(q, n, k)$ is a bipartite graph with partition $V_k \cup V_{k+1}$. It is easy to show that G is an irreducible graph. We now show that G is a connected graph. It is sufficient to show that if v_1, v_2 are two vertices in V_k, then there is a path in G between v_1 and v_2. Let $\dim(v_1 \cap v_2) = k - j$, $1 \leq j \leq k$. We prove our assertion by induction on j. If $j = 1$, then $u = v_1 + v_2$ is a subspace of $V(n, q)$ of dimension $k + k - (k - 1) = k + 1$, which contains both of v_1 and v_2. Hence, $u \in V_{k+1}$ is adjacent to both of the vertices v_1 and v_2. Thus, if $j = 1$, then there is a path between v_1 and v_2 in the graph G. Assume when $j = i$, $0 < i < k$, then there is a path in G between v_1 and v_2. We now assume $j = i + 1$. Let $v_1 \cap v_2 = w$, and let $B = \{b_1, \ldots, b_{k-i-1}\}$ be a basis for the subspace w in the space $V(q, n)$. We can extend B to bases B_1 and B_2 for the subspaces v_1 and v_2, respectively. Let $B_1 = \{b_1, \ldots, b_{k-i-1}, c_1, \ldots, c_{i+1}\}$ be a basis for v_1 and $B_2 = \{b_1, \ldots, b_{k-i-1}, d_1, \ldots, d_{i+1}\}$ be a basis for v_2. Consider the subspace $s = \langle b_1, \ldots, b_{k-i-1}, c_1, d_2, \ldots, d_{i+1}\rangle$. Then s is a k-subspace of the space $V(q, n)$ such that $\dim(s \cap v_2) = k - 1$ and $\dim(s \cap v_1) = k - i$. Hence by the induction assumption, there is a path P_1 between vertices v_2 and s, and a path P_2 between vertices s and v_1. We now conclude that there is a path in the graph G between vertices v_1 and v_2. □

Theorem 3.15. *Let $G = S(q, n, k)$ be the graph which is defined in Definition 3.13. If $n \neq 2k + 1$, then we have $\mathrm{Aut}(G) \cong P\Gamma L_n(\mathbb{F}_q)$. If $n = 2k + 1$, then $\mathrm{Aut}(G) \cong P\Gamma L_n(\mathbb{F}_q) \rtimes \mathbb{Z}_2$.*

Proof. From Proposition 3.14, it follows that the graph $G = S(q, n, k)$ is connected, bipartite and irreducible with the vertex-set $V_k \cup V_{k+1}$, $V_k \cap V_{k+1} = \emptyset$. Let $G_1 = G(q, n, k) = (V_k, E)$ be the Grassmann graph with the vertex-set V_k when $k > 1$ and the vertex-set V_2, when $k = 1$. We show that G_1 is a faithful representation of the graph G.

Firstly, the condition (i) of Definition 3.4 holds because $k < \frac{n}{2}$ and every automorphism of the Grassmann graph $G(q, n, r)$ is of the form f_a, $a \in P\Gamma L_n(\mathbb{F}_q)$, and is an automorphism of the graph $G(q, n, s)$ when $r, s < \frac{n}{2}$. Also, note that if X, Y are subspaces of $V(q, n)$ such that $X \leq Y$, then $f_a(X) \leq f_a(Y)$.

Now, suppose that f is an automorphism of the graph G such that $f(V_k) = V_k$. We show that the restriction of f to the set V_k, namely $g = f|_{V_k}$, is an automorphism of the graph G_1. It is trivial that g is a permutation of the vertex-set V_k. Let v and w be adjacent vertices in the graph G_1. We show that $g(v)$ and $g(w)$ are adjacent in the graph G_1. We assert that there is exactly one vertex u in the graph G such that u is adjacent to both of the vertices v and w. If the vertex u is adjacent to both of the vertices v and w, then v and w are k-subspaces of the $(k + 1)$-space u. Hence u contains the space $v + w$. Since $\dim(v + w)=\dim(v) + \dim(w) - \dim(v \cap w) = k + k - (k - 1) = k + 1$, we have $u = v + w$. In other words, the vertex $u = v + w$ is the unique vertex in the graph G such that u is adjacent to both of the vertices v and w. Also, note that our discussion shows that if $x, y \in V_k$ are such that $\dim(x \cap y) \neq (k - 1)$, then x and y have no common neighbor in the graph G.

Now since the vertices v and w have exactly 1 common neighbor in the graph G, therefore, $f(v) = g(v)$ and $f(w) = g(w)$ have exactly 1 common neighbor in the graph G. It follows that $\dim(g(v) \cap g(w)) = k - 1$, and hence $g(v)$ and $g(w)$ are adjacent vertices in the Grassmann graph G_1.

We now conclude that the graph G_1 is a faithful representation of the graph G. There are two possible cases, namely, (1) $2k + 1 \neq n$ or (2) $2k + 1 = n$.

(1) Let $2k + 1 \neq n$. Noting that $\begin{bmatrix} n \\ k \end{bmatrix}_q < \begin{bmatrix} n \\ k+1 \end{bmatrix}_q$, it follows that $|V_k| \neq |V_{k+1}|$. Therefore by Theorems 3.8, and 3.12, we have $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \cong P\Gamma L_n(\mathbb{F}_q)$.

(2) If $2k + 1 = n$, since $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k+1 \end{bmatrix}_q$, then $|V_k| = |V_{k+1}|$. Hence, the mapping $\theta : V(G) \longrightarrow V(G)$ defined by the rule $\theta(v) = v^{\perp}$ is an automorphism of the graph G of order 2 such that $\theta(V_k) = V_{k+1}$. Hence, by Theorems 3.9 and 3.12, we have $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \rtimes \langle\theta\rangle \cong P\Gamma L_n(\mathbb{F}_q) \rtimes \mathbb{Z}_2$.

□

We now show another application of Theorem 3.9, in determining the automorphism groups of some classes of graphs which are important in algebraic graph theory.

If G_1, G_2 are graphs, then their direct product (or tensor product) is the graph $G_1 \times G_2$ with vertex set $\{(v_1, v_2) \mid v_1 \in G_1, v_2 \in G_2\}$, and for which vertices (v_1, v_2) and (w_1, w_2)

are adjacent precisely if v_1 is adjacent to w_1 in G_1 and v_2 is adjacent to w_2 in G_2. It can be shown that the direct product is commutative and associative [8]. The following theorem, first was proved by Weichsel (1962), characterizes connectedness in direct products of two factors.

Theorem 3.16 **[8]**. *Suppose G_1 and G_2 are connected nontrivial graphs. If at least one of G_1 or G_2 has an odd cycle, then $G_1 \times G_2$ is connected. If both G_1 and G_2 are bipartite, then $G_1 \times G_2$ has exactly two components.*

Thus, if one of the graphs G_1 or G_2 is a connected non-bipartite graph, then the graph $G_1 \times G_2$ is a connected graph. If K_2 is the complete graph on the set $\{0, 1\}$, then the direct product $B(G) = G \times K_2$ is a bipartite graph, and is called the bipartite double cover of G (or the *bipartite double* of G). Then

$$V(B(G)) = \{(v, i) | v \in V(G), i \in \{0, 1\}\},$$

and two vertices (x, a) and (y, b) are adjacent in the graph $B(G)$ if and only if $a \neq b$ and x is adjacent to y in the graph G. The notion of the bipartite double cover of G has many applications in algebraic graph theory [2].

Consider the bipartite double cover of G, namely, the graph $B(G) = G \times K_2$. It is easy to see that the group $\mathrm{Aut}(B(G))$ contains the group $\mathrm{Aut}(G) \times \mathbb{Z}_2$ as a subgroup. In fact, if for $g \in \mathrm{Aut}(G)$, we define the mapping e_g by the rule $e_g(v, i) = (g(v), i)$, $i \in \{0, 1\}$, $v \in V(G)$, then $e_g \in \mathrm{Aut}(B(G))$. It is easy to see that $H = \{e_g | g \in \mathrm{Aut}(G)\} \cong \mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}(B(G))$. Let t be the mapping defined on $V(B(G))$ by the rule $t(v, i) = (v, i^c)$, where $i^c = 1$ if $i = 0$ and $i^c = 0$ if $i = 1$. It is clear that t is an automorphism of the graph $B(G)$. Hence, $\langle H, t\rangle \leq \mathrm{Aut}(B(G))$. Noting that for every $e_g \in H$ we have $e_g t = t e_g$, we deduce that $\langle H, t\rangle \cong H \times \langle t\rangle$. We now conclude that $\mathrm{Aut}(G) \times \mathbb{Z}_2 \cong H \times \mathbb{Z}_2 \leq \mathrm{Aut}(B(G))$.

Let G be a graph. G is called a *stable* graph when we have $\mathrm{Aut}(B(G)) \cong \mathrm{Aut}(G) \times \mathbb{Z}_2$. Concerning the notion and some properties of stable graphs, see [12,13,21,23].

Let $n, k \in \mathbb{N}$ with $k < \frac{n}{2}$ and let $[n] = \{1, \ldots, n\}$. The *Kneser graph* $K(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v,w are adjacent if and only if $|v \cap w|$=0. It is easy to see that if $H(n, k)$ is a bipartite Kneser graph, then $H(n, k) \cong K(n, k) \times K_2$. Now, it follows from Theorem 3.9 (or [18]) that Kneser graphs are stable graphs.

The next theorem provides a sufficient condition such that when a connected non-bipartite irreducible graph G satisfies this condition, then G is a stable graph.

Theorem 3.17. *Let $G = (V, E)$ be a connected non-bipartite irreducible graph. Let $v, w \in V$ be arbitrary. Let $c(v, w)$ be the number of common neighbors of v and w in the graph G. Let $a_0 > 0$ be a fixed integer. If $c(v, w) = a_0$, when v and w are adjacent and $c(v, w) \neq a_0$ when v and w are non-adjacent, then we have*

$$\mathrm{Aut}(G \times K_2) = \mathrm{Aut}(B(G)) \cong \mathrm{Aut}(G) \times \mathbb{Z}_2.$$

In other words, G is a stable graph.

Proof. Note that the graph $G \times K_2$ is a bipartite graph with the vertex set $V = U \cup W$, where $U = \{(v, 0)|v \in V(G)\}$ and $W = \{(v, 1)|v \in V(G)\}$. Since G is an irreducible graph, then the graph $G \times K_2$ is an irreducible graph. In fact if the vertices $x, y \in V$ are such that $N(x) = N(y)$, then $x, y \in U$ or $x, y \in W$. Without loss of generality, we can assume that $x, y \in U$. Let $x = (u_1, 0)$ and $y = (u_2, 0)$. Let $N(x) = \{(v_1, 1), (v_2, 1), \ldots, (v_m, 1)\}$ and $N(y) = \{(t_1, 1), (t_2, 1), \ldots, (t_p, 1)\}$, where $v_i s$ and $t_j s$ are in $V(G)$. Thus $m = p$ and $N(u_1) = \{u_1, \ldots, u_m\}$=$\{t_1, \ldots, t_m\} = N(u_2)$. Now since G is an irreducible graph, it follows that $u_1 = u_2$ and therefore $x = y$.

Let $G_1 = (U, E_1)$ be the graph with vertex-set U in which two vertices $(v, 0)$ and $(w, 0)$ are adjacent if and only if v_1 and v_2 are adjacent in the graph G. It is clear that $G_1 \cong G$. Therefore we have $\mathrm{Aut}(G_1) \cong \mathrm{Aut}(G)$. For every $f \in \mathrm{Aut}(G)$, we let

$$d_f : U \to U,\ d_f(v, 0) = (f(v), 0), \quad \text{for every } (v, 0) \in U.$$

Then d_f is an automorphism of the graph G_1. If we let $A = \{d_f | f \in \mathrm{Aut}(G)\}$, then A with the operation of composition is a group, and it is easy to see that $A \cong \mathrm{Aut}(G_1)$ (as abstract groups). We now assert that the graph G_1 is a faithful representation of the bipartite graph $B = G \times K_2$. Let $g \in \mathrm{Aut}(B)$ be such that $g(U) = U$. We assert that $h = g|_U$, the restriction of g to U, is an automorphism of the graph G_1. It is clear that h is a permutation of U. Let $(v, 0)$ and $(w, 0)$ be adjacent vertices in G_1. Then v, w are adjacent in the graph G. Hence there are vertices $u_1, \ldots, u_{a_0}$ in the graph G such that the set of common neighbor(s) of v and w in G is $\{u_1, \ldots, u_{a_0}\}$. Noting that $(x, 1)$ is a common neighbor of $(v, 0)$ and $(w, 0)$ in the graph B if and only if x is a common neighbor of v, w in the graph G, we deduce that the set $\{(u_1, 1), \ldots, (u_{a_0}, 1)\}$ is the set of common neighbor(s) of $(v, 0)$ and $(w, 0)$ in the graph B. Since g is an automorphism of the graph B, $g(v, 0)$ and $g(w, 0)$ have a_0 common neighbor(s) in the graph B. Note that if $d_{G_1}(g(v, 0), g(w, 0)) > 2$, then these vertices have no common neighbor in the graph B. Also, if $d_{G_1}(g(v, 0), g(w, 0)) = 2$, then $d_G(v, w) = 2$, and hence v, w have $c(v, w) \neq a_0$ common neighbor(s) in the graph G. Hence $(v, 0)$ and $(w, 0)$ have $c(v, w)$ common neighbor(s) in the graph B, and therefore $g(v, 0), g(w, 0)$ have $c(v, w) \neq a_0$ common neighbor(s) in the graph B. We now deduce that $d_{G_1}(g(v, 0), g(w, 0)) = 1$. It follows that $h = g|_U$ is an automorphism of the graph G_1. Thus, the condition (ii) of Definition 3.4 holds for the graph G_1.

Now, suppose that ϕ is an automorphism of the graph G_1. Then there is an automorphism f of the graph G such that $\phi = d_f$. We now define the mapping e_ϕ on the set $V(B)$ by the following rule:

$$(*) \qquad e_\phi(v, i) = \begin{cases} (f(v), 0), & \text{if } i = 0 \\ (f(v), 1), & \text{if } i = 1. \end{cases}$$

It is easy to see that e_ϕ is an extension of the automorphism ϕ to an automorphism of the graph B. We now deduce that the graph G_1 is a faithful representation of the graph B.

On the other hand, it is easy to see that the mapping $t : V(B) \to V(B)$, which is defined by the rule,

$$(**) \qquad t(v, i) = \begin{cases} (v, 0), & \text{if } i = 1 \\ (v, 1), & \text{if } i = 0, \end{cases}$$

is an automorphism of the graph B of order 2. Hence $\langle t \rangle \cong \mathbb{Z}_2$. Also, it is easy to see that for every automorphism ϕ of the graph G_1 we have $te_\phi = e_\phi t$. We now conclude by

Theorem 3.9, that

$$\mathrm{Aut}(G \times K_2) = \mathrm{Aut}(B) \cong \mathrm{Aut}(G_1) \rtimes \langle t \rangle \cong \mathrm{Aut}(G) \times \langle t \rangle \cong \mathrm{Aut}(G) \times \mathbb{Z}_2.$$

□

As an application of Theorem 3.17, we show that the Johnson graph $J(n, k)$ is a stable graph. Since $J(n, k) \cong J(n, n-k)$ in the sequel, we assume that $k \leq \frac{n}{2}$.

Theorem 3.18. *Let n, k be positive integers with $k \leq \frac{n}{2}$. If $n \neq 6$, then the Johnson graph $J(n, k)$ is a stable graph.*

Proof. We know that the vertex set of the graph $J(n, k)$ is the set of k-subsets of $[n] = \{1, 2, 3, \ldots, n\}$ in which two vertices v and w are adjacent if and only if $|v \cap w| = k - 1$. If $k = 1$, then $J(n, k) \cong K_n$, the complete graph on n vertices. It is easy to see that if $X = K_n$, then the bipartite double cover of X is isomorphic with the bipartite Kneser graph $H(n, 1)$. From Theorem 3.9 (or [18]), we know that $\mathrm{Aut}(H(n, 1)) \cong \mathrm{Sym}([n]) \times \mathbb{Z}_2 \cong \mathrm{Aut}(K_n) \times \mathbb{Z}_2$. Hence the Johnson graph $J(n, k)$ is a stable graph when $k = 1$. We now assume that $k \geq 2$. We let $G = J(n, k)$. It is easy to see that G is an irreducible graph. It can be shown that if v, w are vertices in G, then $d(v, w) = k - |v \cap w|$ [2]. Hence, G is a connected graph. It is easy to see that the girth of the Johnson graph $J(n, k)$ is 3. Therefore, G is a non-bipartite graph. It is clear that when $d(v, w) \geq 3$, then v, w have no common neighbors. We now consider two other possible cases, that is, (i) $d(v, w) = 2$ or (ii) $d(v, w) = 1$. Let $c(v, w)$ denote the number of common neighbors of v, w in G. In the sequel, we show that if $d(v, w) = 2$, then $c(v, w) = 4$, and if $d(v, w) = 1$, then $c(v, w) = n - 2$.

(i) If $d(v, w) = 2$, then $|v \cap w| = k - 2$. Let $v \cap w = u$. Then $v = u \cup \{i_1, i_2\}$, $w = u \cup \{j_1, j_2\}$, where $i_1, i_2, j_1, j_2 \in [n]$, $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$. Let $x \in V(G)$. It is easy to see that if $|x \cap u| < k - 2$, then x can not be a common neighbor of v, w. Hence, if x is a common neighbor of v and w, then x is of the form $x = u \cup \{r, s\}$, where $r \in \{i_1, i_2\}$ and $s \in \{j_1, j_2\}$. We now deduce that the number of common neighbors of v and w in the graph G is 4.

(ii) We now assume that $d(v, w) = 1$. Then $|v \cap w| = k - 1$. Let $v \cap w = u$. Then $v = u \cup \{r\}$, $w = u \cup \{s\}$, where $r, s \in [n]$, $r \neq s$. Let $x \in V(G)$. It is easy to see that if $|x \cap u| < k - 2$, then x can not be a common neighbor of v, w. Hence, if x is a common neighbor of v and w, then $|x \cap u| = k - 1$ or $|x \cap u| = k - 2$. In the first step, we assume that $|x \cap u| = k - 1$. Then x is of the form $x = u \cup \{y\}$, where $y \in [n] - (v \cup w)$. Since, $|v \cup w| = k + 1$, then the number of such x's is $n - k - 1$. We now assume that $|x \cap u| = k - 2$. Hence, x is of the form $x = t \cup \{r, s\}$, where t is a $(k-2)$-subset of the $(k-1)$-set u. Therefore the number of such x's is $\binom{k-1}{k-2} = k - 1$. Our argument follows that if v and w are adjacent, then we have $c(v, w) = n - k - 1 + k - 1 = n - 2$.

Noting that $n - 2 \neq 4$, we conclude from Theorem 3.18 that the Johnson graph $J(n, k)$ is a stable graph when $n \neq 6$. □

Although, Theorem 3.18, does not say anything about the stability of the Johnson graph $J(6, k)$, we show by the next result that this graph is a stable graph.

PROPOSITION 3.19

The Johnson graph $J(6, k)$ is a stable graph.

Proof. When $k = 1$ the assertion is true, and hence we assume that $k \in \{2, 3\}$. In the first step, we show that the Johnson graph $J(6, 2)$ is a stable graph. Let $B = J(6, 2) \times K_2$. We show that $\mathrm{Aut}(B) \cong \mathrm{Sym}([6]) \times \mathbb{Z}_2$, where $[6] = \{1, 2, \ldots, 6\}$. It is clear that B is a bipartite irreducible graph. Let $V = V(B)$ be the vertex-set of the graph B. Then $V = V_0 \cup V_1$, where $V_i = \{(v, i) | v \subset [6], |v| = 2\}, i \in \{0, 1\}$. Let $G_1 = (V_0, E_1)$ be the graph with the vertex-set V_0 in which two vertices $(v, 0), (w, 0)$ are adjacent whenever $|v \cap w| = 1$. It is clear that G_1 is isomorphic with the Johnson graph $J(6, 2)$. Hence, we have $\mathrm{Aut}(G_1) \cong \mathrm{Sym}([6])$. We show that G_1 is a faithful representation of the graph B. By what we saw in (*) of the proof of Theorem 3.17, it is clear that if h is an automorphism of the graph G_1, then h can be extended to an automorphism e_h of the graph B. Thus, the condition (i) of Definition 3.4, holds for the graph G_1.

Let $a = (v, 0)$ and $b = (w, 0)$ be two adjacent vertices in the graph G_1, that is, $|v \cap w| = 1$. Let $N(a, b)$ denote the set of common neighbors of a and b in the graph B. Let $X(a, b) = \{a, b\} \cup N(a, b) \cup t(N(a, b))$, where t is the automorphism of the graph B defined by the rule $t(v, i) = (v, i^c), i^c \in \{0, 1\}, i^c \neq i$. Let $\langle X(a, b)\rangle$ be the subgraph induced by the set $X(a, b)$ in the graph B. It can be shown that if a, b are adjacent vertices in G_1, that is, $|v \cap w| = 1$, then $\langle X(a, b)\rangle$ has a vertex of degree 0. On the other hand, when a, b are not adjacent vertices in G_1, that is, $|v \cap w| = 0$, then $\langle X(a, b)\rangle$ has no vertices of degree 0. In the rest of the proof, we let $\{x, y\} = xy$. For example, let $r = (12, 0)$ and $s = (13, 0)$ be two adjacent vertices of G_1. Then $X(r, s) = \{(12, 0), (13, 0), (14, 1), (15, 1), (16, 1), (23, 1), (14, 0), (15, 0), (16, 0), (23, 0)\}$. Now, in the graph $\langle X(r, s)\rangle$ the vertex $(23, 0)$ is a vertex of degree 0. Whereas, if we let $r = (12, 0), u = (34, 0)$, then r, u are not adjacent in the graph G_1. Then $X(r, u) = \{(12, 0), (34, 0), (13, 1), (14, 1), (23, 1), (24, 1), (13, 0), (14, 0), (23, 0), (24, 0)\}$. Now, it is clear that the graph $\langle X(r, u)\rangle$ has no vertices of degree 0.

Note that the graph G_1 is isomorphic with the Johnson graph $J(6, 2)$, and hence G_1 is a distance-transitive graph. Now if c, d are two adjacent vertices in the graph G_1, then there is an automorphism f in $\mathrm{Aut}(G_1)$ such that $f(r) = c$ and $f(s) = d$. Let e_f be the extension of f to an automorphism of the graph B. Therefore, $\langle X(c, d)\rangle = \langle X(e_f(r), e_f(s))\rangle = e_f(\langle X(r, s)\rangle)$ has a vertex of degree 0. This argument also shows that if p, q are non-adjacent vertices in the graph G_1, then $\langle X(p, q)\rangle$ has no vertices of degree 0.

Now, let g be an automorphism of the graph B such that $g(V_0) = V_0$. We show that $g|_{V_0}$ is an automorphism of the graph G_1. Let $a = (v, 0)$ and $b = (w, 0)$ be two adjacent vertices of the graph G_1, that is, $|v \cap w| = 1$. Then $\langle X(a, b)\rangle$ has a vertex of degree 0. Hence, $g(\langle X(a, b)\rangle) = \langle X(g(a), g(b))\rangle$ has a vertex of degree 0. Then $g(a)$ and $g(b)$ are adjacent in the graph G_1. We now deduce that if g is an automorphism of the graph B such that $g(V_0) = V_0$, then $g|_{V_0}$ is an automorphism of the graph G_1. Therefore, the condition (ii) of Definition 3.4, holds for the graph G_1. Therefore, G_1 is a faithful representation of the graph B. Note that t is an automorphism of the graph B of order 2 such that $t(V_0) = V_1$ and $t(V_1) = V_0$. Also, we have $tf = ft, f \in \mathrm{Aut}(G_1)$. We now conclude by Theorem 3.9 that

$$\mathrm{Aut}(B) \cong \mathrm{Aut}(G_1) \rtimes \langle t\rangle \cong \mathrm{Aut}(G_1) \times \langle t\rangle \cong \mathrm{Aut}(G) \times \mathbb{Z}_2 \cong \mathrm{Sym}([6]) \times \mathbb{Z}_2.$$

Therefore, the graph $G = J(6, 2)$ is a stable graph.

By a similar argument, we can show that the graph $J(6, 3)$ is a stable graph. □

Combining Theorem 3.18 and Proposition 3.19, we obtain the following result.

Theorem 3.20. *The Johnson graph $J(n, k)$ is a stable graph.*

4. Conclusion

In this paper, we gave a method for finding the automorphism groups of connected bipartite irreducible graphs (Theorems 3.8 and 3.9). Then by our method, we explicitly determined the automorphism groups of some classes of bipartite irreducible graphs, including the graph $S(q, n, k)$ which is a derived graph from the Grassman graph $G(q, n, k)$ (Theorem 3.15). Also, we provided a sufficient ascertainable condition such that when a connected non-bipartite irreducible graph G satisfies this condition, then G is a stable graph (Theorem 3.17). Finally, we showed that the Johnson graph $J(n, k)$ is a stable graph (Theorem 3.20).

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