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# On ramification index of composition of complete discrete valuation fields

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**Abstract.** For an extension $L/K$ of discrete valuation fields, let $e_{L/K}, \mathfrak{O}_L$ denote the ramification index and valuation ring of $L/K$ respectively. Let $K$ be a complete discrete valuation field and $L_1/K$, $L_2/K$ be finite linearly disjoint extensions over $K$. We show that if $\mathfrak{O}_{L_1L_2} = \mathfrak{O}_{L_1}\mathfrak{O}_{L_2}$ or $\gcd(e_{L_1/K}, e_{L_2/K}) = 1$, and one of the residue fields $l_1/k$, $l_2/k$ is separable, then $e_{L_1L_2/L_1} = e_{L_2/K}$. The analogous results for the residue degrees are also true.

**Keywords.** Complete discrete valuation fields; ramification indices; linearly disjoint extensions.

**Mathematics Subject Classification.** Primary: 11S; Secondary: 11S15, 11S20.

## 1. Introduction

An extension $E \supset K$ is an extension of discrete valuation fields if the discrete valuation $\omega$ on $E$ is equivalent to an extension of the discrete valuation $\upsilon$ on $K$. The ramification index $e_{E/K}$ of $E/K$ is defined by $\pi_K = \pi_E^{e_{E/K}}$, where $\pi_K, \pi_E$ are uniformizers of $K$ and $E$ respectively. Let $F \subset K \subset L$ be extensions of complete discrete valuation fields. It is a well-known fact that $e_{L/F} = e_{L/K}e_{K/F}$. The residue degree $f_{E/K}$ of the extension $E/K$ is defined by $[\frac{O_E}{(\pi_E)} : \frac{O_K}{(\pi_K)}]$, where $O_E$, $O_K$ are valuation ring $E$ and $K$ respectively .

Let $L_1$, $L_2$ be finite extensions over a complete discrete valuation field $K$. Extend the discrete valuation $\upsilon$ on $K$ to the discrete valuations $\omega_1$ and $\omega_2$ over $L_1$ and $L_2$ respectively.

The $e_1, e_2, e_1^*, e_2^*, e$ in Figure 1 indicates ramification indices of the corresponding extensions. We know that $e = e_1e_1^* = e_2e_2^*$, hence $e = e_1e_2 \Leftrightarrow e_1 = e_2^* \Leftrightarrow e_2 = e_1^*$. In this article, we show that under certain conditions on $L_1/K$ and $L_2/K$, $e = e_1e_2$.

We know that $L_1$, $L_2$ are linearly disjoint and finite extensions over $K$ if and only if the dimension of the composite field $L_1L_2$ of $L_1$ and $L_2$ is the product of the dimensions of $L_1$ and $L_2$ over $K$. That is, $[L_1L_2 : K] = [L_1 : K][L_2 : K]$. We also have that if $L_1/K$, $L_2/K$ are complete discrete valuation fields, then $[L_1L_2 : K] = e_{L_1L_2/K}f_{L_1L_2/K} = e_{L_1/K}f_{L_1/K}e_{L_2/K}f_{L_2/K}$ ([3], Chapter 2, Section 4). The natural question is whether the ramification indices can be equal on both sides and also whether the residue degrees can be equal on both sides. Our main purpose in this paper is to show that under certain



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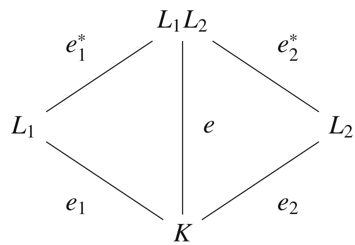


**Figure 1.** Ramification indices of $L_1, L_2, L_1L_2$ over $K$.

assumptions, $e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K}$ and $f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}$ when $L_1/K, L_2/K$ are linearly disjoint over $K$.

*Remark* 1. Linearly disjoint assumption is must in our theorems. Because, if $L_1/K$ and $L_2/K$ are not linearly disjoint, then $[L_1L_2 : K] \neq [L_1 : K][L_2 : K]$. On the other hand, $e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K}$ and $f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}$ will imply that $[L_1L_2 : K] = [L_1 : K][L_2 : K]$, which is a contradiction.

In the context of complete discrete valuation fields, we have the following result:
If $L_1/K$ and $L_2/K$ are finite extensions over a complete discrete valuation field $K$ and they are separable extensions with at least one of the extensions, $L_1/K$ or $L_2/K$ is tamely ramified, then $e_{L_1L_2/K} = \operatorname{lcm}(e_{L_1/K}, e_{L_2/K})$ (Theorem 2.1, [1]). In the following Theorems 2 and 3, we do not assume separability condition on $L_1/K$ and $L_2/K$ as well as $p \nmid [L_1 : K]$ or $[L_2 : K]$. We also prove the analogous results on residue degrees.

**Theorem 2.** *Let $K$ be a complete discrete valuation field. Let $L_1/K$ and $L_2/K$ be finite extensions of $K$ and also linearly disjoint over $K$. Let $k, l_1, l_2$ denote the residue fields of $K$, $L_1$ and $L_2$ respectively. Assume one of $l_1/k$, $l_2/k$ is a separable extension and $e_{L_1/K}$, $e_{L_2/K}$ are coprime. Then*

$$e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K}, \quad f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}.$$

In the above theorem, we assume $e_{L_1/K}$, $e_{L_2/K}$ are coprime. We replace it by $\mathfrak{D}_{L_1L_2} = \mathfrak{D}_{L_1}\mathfrak{D}_{L_2}$ in the following theorem.

**Theorem 3.** *Let $K$ be a complete discrete valuation field. Let $L_1/K$ and $L_2/K$ be finite extensions of $K$ and also linearly disjoint over $K$. Assume one of $l_1/k, l_2/k$ is a separable extension and $\mathfrak{D}_{L_1L_2} = \mathfrak{D}_{L_1}\mathfrak{D}_{L_2}$. Then*

$$e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K}, \quad f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}.$$

We prove our Theorems 2 and 3 in Section 3

## 2. Preliminaries

If we consider an extension $L$ of a complete discrete valuation field $K$, then we consider only the valuation on $L$ which is equivalent to the valuation on $L$ extended from $K$. Throughout the paper, this notion is followed.

*Lemma* 4. *Let $K$ be a complete discrete valuation field and $L_1/K$, $L_2/K$ be finite extensions of $K$ which are linearly disjoint over $K$. Then $e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K} \Leftrightarrow f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}$.*

*Proof.* We have

$$[L_1L_2 : K] = [L_1 : K][L_2 : K] \\ \Rightarrow e_{L_1L_2/K}f_{L_1L_2/K} = e_{L_1/K}f_{L_1/K}e_{L_2/K}f_{L_2/K}.$$

Hence $e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K} \Leftrightarrow f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}$. □

*Lemma* 5. *Let $L_1/K$, $L_2/K$ be extensions of $K$ which are linearly disjoint over $K$. If $e_{L_1L_2/K} \geq e_{L_1/K}e_{L_2/K}$ and $f_{L_1L_2/K} \geq f_{L_1/K}f_{L_2/K}$, then $e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K}$ and $f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}$.*

*Proof.* Since $L_1/K, L_2/K$ are extensions of $K$ which are linearly disjoint over $K$. we have $[L_1L_2; K] = [L_1 : K][L_2 : K]$. Hence $e_{L_1L_2/K}f_{L_1L_2/K} = e_{L_1/K}f_{L_1/K}e_{L_2/K}f_{L_2/K}$. We also have $e_{L_1L_2/K} \geq e_{L_1/K}e_{L_2/K}$ and $f_{L_1L_2/K} \geq f_{L_1/K}f_{L_2/K}$. This implies that $e_{L_1L_2/K} = e_{L_1/K}e_{L_2/K}$ and $f_{L_1L_2/K} = f_{L_1/K}f_{L_2/K}$. □

## 3. One of the residue field of $L_1/K$, $L_2/K$ is separable

We setup some background for proving Theorems 2 and 3.

*Lemma* 6. *Let $L/K$ be an extension of complete discrete valuation fields such that $l/k$ is a separable extension, where $l, k$ are residue fields of $L$, $K$ respectively. Then there exists an extension $M$ of $K$ such that $M/K$ is unramified and $L/M$ is a totally ramified extension such that $[M : K] = [l : k]$.*

*Proof.* Let $\mathfrak{P}$ and $\mathfrak{p}$ be the prime ideals of $\mathfrak{O}_L$ and $\mathfrak{O}_K$ respectively. Since $l/k$ is a finite separable extension by the primitive element theorem ([2], Theorem 14.4.25), we have $l = k(\bar{\alpha})$. Let $\bar{g} \in k[x]$ be the minimal polynomial of $\bar{\alpha}$. Then $[l : k] = \deg(\bar{g})$. Let $g(x) \in \mathfrak{O}_K[x]$ be the lift of $\overline{g(x)}$ to $\mathfrak{O}_K[x]$ such that $\deg(g) = \deg(\bar{g})$. Since $\bar{g}$ is irreducible, $g(x) \in \mathfrak{O}_K[x]$ is irreducible. By Hensel's lemma, choose a root $\beta$ of $g(x)$ in $\mathfrak{O}_L$ such that $\beta + \mathfrak{p} = \bar{\alpha}$. Set $M = K(\beta)$ is the desired unramified extension. □

*Lemma* 7. *Let $K$ be a complete discrete valuation field. Let $L_1/K$ and $L_2/K$ be finite extensions of $K$ and also linearly disjoint over $K$. Assume one of $l_1/k$, $l_2/k$ is a separable extension. Then $f_{L_1L_2/K} \geq f_{L_1/K}f_{L_2/K}$.*

*Proof.* Let $L_1/K$ be the extension of discrete valuation such that $l_1/k$ is separable. Then by Lemma 6, there exists $M_1 \subset L_1$ such that $M_1/K$ is unramified and $L_1/M_1$ is totally ramified. Hence $e_{M_1/K} = 1$, $f_{L_1/K} = f_{M_1/K}$. Therefore, $M_1L_2/L_2$ is unramified over $L_2$ (Proposition II.7.2, [4]), hence we have $e_{M_1L_2/L_2} = 1$. Consider

$$e_{M_1L_2/K} = e_{M_1L_2/L_2}e_{L_2/K} = e_{L_2/K} = e_{M_1/K}e_{L_2/K}.$$

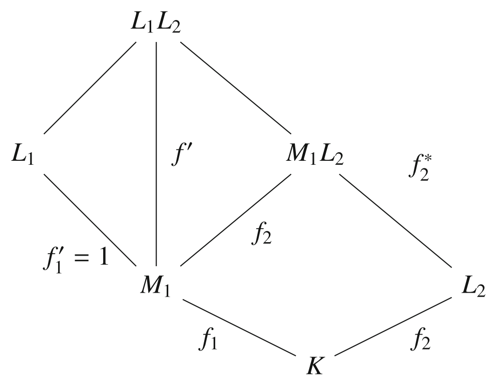


**Figure 2.** Residual degrees of sub extensions of $L_1L_2$.

Applying Lemma 4 for $M_1/K$ and $L_2/K$, we have $f_{M_1L_2/K} = f_{M_1/K} f_{L_2/K}$ which implies that $f_{L_2/K} = f_{M_1L_2/M_1}$ (Figure. 2).

Now look at $f_{L_1L_2/K} = f_{L_1L_2/M_1L_2} f_{M_1L_2/M_1} f_{M_1/K}$. This implies $f_{L_1L_2/K} = f_{L_1L_2/M_1L_2} f_{L_2/K} f_{L_1/K}$. Hence $f_{L_1L_2/K} \geq f_{L_1/K} f_{L_2/K}$. □

*Proof of Theorem* 2. Observe that $e_{L_1/K} | e_{L_1L_2/K}$ and $e_{L_2/K} | e_{L_1L_2/K}$ so $\text{lcm}(e_{L_1/K}, e_{L_2/K}) = e_{L_1/K} e_{L_2/K}$ divides $e_{L_1L_2/K}$. Hence $e_{L_1L_2/K} \geq e_{L_1/K} e_{L_2/K}$. Now the theorem is evident from Lemmas 7 and 5. □

*Proof of Theorem* 3. Let $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}, \mathfrak{p}$ be the unique prime ideals of $\mathfrak{O}_{L_1}, \mathfrak{O}_{L_2}, \mathfrak{O}_{L_1L_2}$ and $\mathfrak{O}_K$ respectively. We show there exists a surjective homomorphism between $\frac{\mathfrak{O}_K}{\mathfrak{p}}$ vector spaces $\frac{\mathfrak{O}_{L_1}}{\mathfrak{P}_1} \otimes \frac{\mathfrak{O}_{L_2}}{\mathfrak{P}_2}$ and $\frac{\mathfrak{O}_{L_1L_2}}{\mathfrak{P}}$.

Define the map $\phi : \frac{\mathfrak{O}_{L_1}}{\mathfrak{P}_1} \times \frac{\mathfrak{O}_{L_2}}{\mathfrak{P}_2} \to \frac{\mathfrak{O}_{L_1L_2}}{\mathfrak{P}}$ by $\phi(x + \mathfrak{P}_1, y + \mathfrak{P}_2) \mapsto xy + \mathfrak{P}$. Observe that $\phi$ is a bi-linear map. Hence there exists a homomorphism $\Psi : \frac{\mathfrak{O}_{L_1}}{\mathfrak{P}_1} \otimes \frac{\mathfrak{O}_{L_2}}{\mathfrak{P}_2} \to \frac{\mathfrak{O}_{L_1L_2}}{\mathfrak{P}}$. Observe that this map is surjective, since $\mathfrak{O}_{L_1L_2} = \mathfrak{O}_{L_1}\mathfrak{O}_{L_2}$. This implies that $f_{L_1L_2/K} \leq f_{L_1/K} f_{L_2/K}$. By using Lemma 7, we conclude that $f_{L_1L_2/K} = f_{L_1/K} f_{L_2/K}$. By using Lemma 4, we conclude that $e_{L_1L_2/K} = e_{L_1/K} e_{L_2/K}$. □

*Example.* We present an example of linearly disjoint extensions without $\mathfrak{O}_{L_1L_2} = \mathfrak{O}_{L_1}\mathfrak{O}_{L_2}$.

Let $K = \mathbb{Q}_3$ be the field of 3-adic numbers. Then $L_1 = K(\sqrt{3})$ and $L_2 = K(\sqrt{-3})$.

The equality $e_{L_1L_2} = e_{L_1/K} e_{L_2/K}$ does not hold in this example. Hence the assumption $\mathfrak{O}_{L_1L_2} = \mathfrak{O}_{L_1}\mathfrak{O}_{L_2}$ is a must in our Theorem 3.

The numbers in Figure 3 indicate the ramification indices of extensions with respect to the prime 3. Observe that $e \neq e_1e_2$.

We plan to remove the hypotheses of our theorems in our future work. In particular, we hope to generalize our theorems for linearly disjoint fields without the assumption $\mathfrak{O}_{L_1L_2} = \mathfrak{O}_{L_1}\mathfrak{O}_{L_2}$.

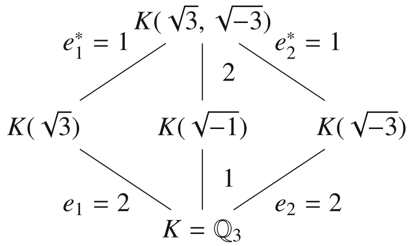


**Figure 3.** Ramification indices of extensions with respect to the prime 3.

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## References

[1] Chabert J–L and Halberstadt E, On Abhyankar's lemma about ramification indices, arXiv: 1805.08869
[2] Dummit D S and Foote R M, Abstract algebra (2003) (John Wiley & Sons)
[3] Lang S, Algebraic Number Theory (1994) (New York: Springer)
[4] Neukirch J, Algebraic Number Theory (1999) (Berlin, Heidelberg: Springer-Verlag)

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