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On Ricci solitons whose potential is convex

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Abstract. In this paper, we consider the Ricci curvature of a Ricci soliton. In particular, we have showed that a complete gradient Ricci soliton with non-negative Ricci curvature possessing a non-constant convex potential function having finite weighted Dirichlet integral satisfying an integral condition is Ricci flat and also it isometrically splits a line. We have also proved that a gradient Ricci soliton with non-constant concave potential function and bounded Ricci curvature is non-shrinking and hence the scalar curvature has at most one critical point.

Keywords. Ricci soliton; scalar curvature; Ricci flat; convex function; critical point; Riemannian manifold.

Mathematics Subject Classification. 53C20, 53C21, 53C44.

1. Introduction and preliminaries

In 1982, Hamilton [8] introduced the concept of Ricci flow. The Ricci flow is defined by an evolution equation for metrics on the Riemannian manifold (M, g_0):

$$\frac{\partial}{\partial_t} g(t) = -2\,\mathrm{Ric}, \quad g(0) = g_0.$$

A complete Riemannian manifold (M, g) of dimension $n \geq 2$ with Riemannian metric g is called a Ricci soliton if there exists a vector field X satisfying

$$\mathrm{Ric} + \frac{1}{2}£_X g = \lambda g, \tag{1}$$

where λ is a constant and £ denotes the Lie derivative. The vector field X is called potential vector field. The Ricci solitons are self-similar solutions to the Ricci flow. Ricci solitons are natural generalization of Einstein metrics, which have been significantly studied in differential geometry and geometric analysis. A Ricci soliton is an Einstein metric if the vector field X is zero or Killing. Throughout the paper, by M, we mean an n-dimensional, $n \geq 2$, complete Riemannian manifold endowed with Riemannian metric g. Let $C^\infty(M)$ be the ring of smooth functions on M. If X is the gradient of some function $u \in C^\infty(M)$, such a manifold is called a *gradient Ricci soliton*, and then (1) reduces to the form

$$\nabla^2 u + \mathrm{Ric} = \lambda g, \tag{2}$$

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where $\nabla^2 u$ is the Hessian of u and the function u is called potential function. The Ricci soliton (M, g, X, λ) is called shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Each type of Ricci solitons determines some unique topology of the manifold. For example, if the scalar curvature of a complete gradient shrinking Ricci soliton is bounded, then the manifold has finite topological type [5]. Munteanu and Wang [11] proved that an n-dimensional gradient shrinking Ricci soliton with non-negative sectional curvature and positive Ricci curvature must be compact (for more results, see [10,12]). Perelman [13] proved that a compact Ricci soliton is always gradient Ricci soliton. For the detailed treatment on Ricci solitons and their interaction to Ricci flow, we refer to [2,4]. A smooth function $\varphi : M \to \mathbb{R}$ is said to be convex [15,16] if for any $p \in M$ and for any vector $v \in T_p M$,

$$\langle \operatorname{grad} \varphi, v \rangle_p \leq \varphi(\exp_p v) - \varphi(p).$$

If φ is convex, then $-\varphi$ is called concave.

The paper is arranged as follows: in section 2, we have proved that a complete non-compact gradient Ricci soliton with non-negative Ricci curvature possessing a non-constant convex function with finite weighted Dirichlet integral satisfying an integral condition is Ricci flat and also it isometrically splits a line. We have also deduced a corollary relating to the Ricci soliton and harmonic function. In section 3, we have proved that if in a complete gradient Ricci soliton, the potential function is a non-constant concave function with bounded Ricci curvature, then the scalar curvature possesses at most one critical point (see Theorem 3.3).

2. Ricci soliton and Ricci flat manifold

Lemma 2.1. *Let (M, g) be a complete Riemannian manifold with non-negative Ricci curvature. If $u \in C^\infty(M)$ is a non-constant convex function with finite weighted Dirichlet integral, i.e.*,

$$\int_{M-B(p,r)} d(x, p)^{-2} |\nabla u|^2 < \infty, \tag{3}$$

and also satisfies the relation

$$\int_{M-B(p,r)} d(x, p)^{-2} u < \infty, \tag{4}$$

where $B(p, r)$ is an open ball with center p and radius r, then the Hessian of u vanishes in M.

Proof. Since $u \in C^\infty(M)$ is a non-constant convex function on M, it follows that M is non-compact [16]. Now, we consider the cut-off function, introduced in [3], $\varphi_r \in C_0^2(B(p, 2r))$ for $r > 0$ such that

$$\begin{cases} 0 \leq \varphi_r \leq 1 & \text{in } B(p, 2r), \\ \varphi_r = 1 & \text{in } B(p, r), \\ |\nabla \varphi_r|^2 \leq \frac{C}{r^2} & \text{in } B(p, 2r), \\ \Delta \varphi_r \leq \frac{C}{r^2} & \text{in } B(p, 2r). \end{cases}$$

Then for $r \to \infty$, we have $\Delta\varphi_r^2 \to 0$ as $\Delta\varphi_r^2 \leq \frac{C}{r^2}$. Since u is a smooth convex function, u is also subharmonic [6], i.e., $\Delta u \geq 0$. Now using integration by parts, we have

$$\int_M u\Delta\varphi_r^2 = \int_M \Delta u\varphi_r^2. \tag{5}$$

Since $\varphi_r \equiv 1$ in $B(p, r)$, using (5), we get

$$\int_{B(p,r)} \Delta u = 0.$$

Again, using the integration by parts and also by our assumption, we obtain

$$0 \leq \int_{B(p,2r)} \varphi_r^2 \Delta u = \int_{B(p,2r)-B(p,r)} u\Delta\varphi_r^2 \leq \int_{B(p,2r)-B(p,r)} u\frac{C}{r^2} \to 0,$$

as $r \to \infty$. Hence we have

$$\int_M \Delta u = 0.$$

But $\Delta u \geq 0$. Therefore, $\Delta u = 0$ in M, i.e., u is a harmonic function. The Bochner formula [1] for the Riemannian manifold is written as

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + g(\nabla u, \nabla\Delta u) + \mathrm{Ric}(\nabla u, \nabla u).$$

Since u is harmonic, so $\Delta u = 0$. Therefore, the above equation reduces to

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u). \tag{6}$$

Combining φ_r^2 with (6) and then integrating, we obtain

$$\int_M \{|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\}\varphi_r^2 = \int_M \frac{1}{2}\Delta|\nabla u|^2\varphi_r^2.$$

Using integration by parts, we get

$$\int_M \frac{1}{2}\Delta|\nabla u|^2\varphi_r^2 = \int_M \frac{1}{2}|\nabla u|^2\Delta\varphi_r^2.$$

Then the above equation and the property of φ_r together imply

$$\int_{B(p,2r)-B(p,r)} \frac{1}{2}\Delta|\nabla u|^2\varphi_r^2 \leq \int_{B(p,2r)-B(p,r)} \frac{C}{2r^2}|\nabla u|^2 \to 0$$

as $r \to \infty$. And also in $B(p, r)$, we have

$$\int_{B(p,r)} \{|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\} = \int_{B(p,r)} \frac{1}{2}|\nabla u|^2\Delta\varphi_r^2 = 0,$$

since $\varphi_r^2 \equiv 1$ in $B(p, r)$. Therefore,

$$\begin{aligned}&\int_M \{|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\}\\ &\quad = \lim_{r\to\infty}\left(\int_{B(p,2r)-B(p,r)} + \int_{B(p,r)}\right)\{|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\}\\ &\quad = 0, \end{aligned} \tag{7}$$

which implies that $\nabla^2 u = 0$. □

Lemma 2.2 [14, *Lemma* 2.3]. *Let u be a smooth function in a complete Riemannian manifold (M, g). Then the following conditions are equivalent*:

(*i*) *u is an affine function,*
(*ii*) *Hessian of u vanishes everywhere in M,*
(*iii*) *∇u is a Killing vector field with $|\nabla u|$ is constant.*

Theorem 2.3 [9, Theorem 1]. *If a complete Riemannian manifold (M, g) admits a non-constant smooth affine function, then M is isometric to $N \times \mathbb{R}$ for a totally geodesic submanifold N of M.*

Theorem 2.4. *Let M be a complete Riemannian manifold with non-negative Ricci curvature. If M admits a non-constant convex function satisfying* (3) *and* (4), *then M is isometric to the Riemannian product $N \times \mathbb{R}$, where N is a totally geodesic submanifold of M.*

Proof. In view of Lemma 2.1, it follows that Hessian of u vanishes. Again, Lemma 2.2 implies that u is an affine function. Therefore, using Theorem 2.3, we conclude that M is isometric to the Riemannian product $N \times \mathbb{R}$, where N is a totally geodesic submanifold of M. □

Theorem 2.5. *Let (M, g, u) be a complete gradient Ricci soliton with non-negative Ricci curvature. If u is a non-constant convex function on M satisfying* (3) *and* (4), *then M is Ricci flat. Moreover, ∇u is a Killing vector field with $|\nabla u|$ is constant.*

Proof. From Lemma 2.1, we get $\nabla^2 u = 0$ and $\mathrm{Ric}(\nabla u, \nabla u) = 0$, since Ricci curvature is non-negative. Now from (2), we get

$$\mathrm{Ric}(\omega, \omega) = \lambda g(\omega, \omega), \quad \text{for } \omega \in TM,$$

which implies that

$$\mathrm{Ric}(\nabla u, \nabla u) = \lambda g(\nabla u, \nabla u) = 0.$$

Therefore, we conclude that $\lambda = 0$. And hence, (2) implies that Ricci curvature of M vanishes in M, i.e., M is a Ricci flat manifold. Also, Lemma 2.1 and Lemma 2.2 together imply that ∇u is Killing vector field with $|\nabla u|$ is constant. □

COROLLARY 2.6

Let (M, g, u) be a complete non-compact gradient Ricci soliton satisfying (2) with non-negative Ricci curvature. If $u \in C^\infty(M)$ is a harmonic function with finite weighted Dirichlet integral, i.e.,

$$\int_{M-B(p,r)} d(x,p)^{-2}|\nabla u|^2 < \infty,$$

then M is a Ricci flat manifold.

Proof. The proof is same as that of Theorem 2.5 except the part where we have proved the harmonicity of the function u and hence we omit. □

3. Ricci soliton and critical points

Theorem 3.1. *Let (M, g) be a complete gradient Ricci soliton satisfying (2). If $u \in C^\infty(M)$ is a non-constant concave function and (M, g) has bounded Ricci curvature, i.e., $|\mathrm{Ric}| \le K$ for some constant $K > 0$, then the Ricci soliton is non-shrinking.*

Proof. Since u is a non-trivial concave function in M, the function $-u$ is non-constant convex and it implies that the manifold M is non-compact. Let us consider a length minimizing normal geodesic $\gamma : [0, t_0] \to M$ for some arbitrarily large $t_0 > 0$. Take $p = \gamma(0)$ and $X(t) = \gamma'(t)$ for $t > 0$. Then X is the unit tangent vector along γ. Now integrating (2) along γ, we get

$$\begin{aligned}\int_0^{t_0} \mathrm{Ric}(X, X) &= \int_0^{t_0} \lambda g(X, X) - \int_0^{t_0} \nabla^2 u(X, X)\\ &= \lambda t_0 - \int_0^{t_0} \nabla^2 u(X, X). \qquad (8)\end{aligned}$$

Again, by the second variation of arc length, we have

$$\int_0^{t_0} \varphi^2 \mathrm{Ric}(X, X) \le (n-1) \int_0^{t_0} |\varphi'(t)|^2 dt, \qquad (9)$$

for every non-negative function φ defined on $[0, t_0]$ with $\varphi(0) = \varphi(t_0) = 0$. We now choose the function φ as follows:

$$\varphi(t) = \begin{cases} t & t \in [0, 1],\\ 1 & t \in [1, t_0 - 1],\\ t_0 - t & t \in [t_0 - 1, t_0]. \end{cases}$$

Then

$$\int_0^{t_0} \text{Ric}(X, X)dt = \int_0^{t_0} \varphi^2\text{Ric}(X, X)dt + \int_0^{t_0} (1-\varphi^2)\text{Ric}(X, X)dt$$
$$\leq (n-1)\int_0^{t_0} |\varphi'(t)|^2 dt + \int_0^{t_0} (1-\varphi^2)\text{Ric}(X, X)dt$$
$$\leq 2(n-1) + \sup_{B(p,1)} |\text{Ric}| + \sup_{B(\gamma(t_0),1)} |\text{Ric}|. \tag{10}$$

Combining equations (2) and (10), we get

$$\lambda t_0 - \int_0^{t_0} \nabla^2 u(X, X) \leq 2(n-1) + \sup_{B(p,1)} |\text{Ric}| + \sup_{B(\gamma(t_0),1)} |\text{Ric}|$$
$$= 2(n-1) + 2K. \tag{11}$$

Therefore, taking limit $t_0 \to \infty$ on both sides of (11), we can write

$$\lim_{t_0\to\infty} \lambda t_0 - \lim_{t_0\to\infty} \int_0^{t_0} \nabla^2 u(X, X) \leq 2(n-1) + 2K. \tag{12}$$

Now $\lim_{t_0\to\infty} \int_0^{t_0} \nabla^2 u(X, X) \leq 0$, since u is a concave function. If $\lambda > 0$, then $\lim_{t_0\to\infty} \lambda t_0 = +\infty$, which contradicts inequality (12). Thus $\lambda \leq 0$, i.e., the Ricci soliton is non-shrinking. □

Lemma 3.2 [7]. *Let (M, g) be a steady gradient Ricci soliton with positive Ricci curvature. Then there is at most one critical point of R.*

Theorem 3.3. *Let (M, g) be a complete non-compact gradient Ricci soliton satisfying*

$$\nabla^2 u + \text{Ric} = \lambda g,$$

with $\lambda \geq 0$. If $u \in C^\infty(M)$ is a non-constant concave function and Ricci curvature of M satisfies $0 < \text{Ric} \leq K$ for some constant $K > 0$, then there is at most one critical point of the scalar curvature R.

Proof. Since $\lambda \geq 0$, by using Theorem 3.1, we can prove that $\lambda = 0$. Therefore, M is a steady Ricci soliton. Now using Lemma 3.2, the result easily follows. □

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