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The exact 2-domination number of generalized Petersen graphs

XUE-GANG CHEN* and XUE-SONG ZHAO

Department of Mathematics, North China Electric Power University, Beijing 102206, China
*Corresponding author. E-mail: gxcxdm@163.com

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Abstract. Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is a 2-dominating set of G if each vertex in $V - S$ is adjacent to at least two vertices in S. The 2-domination number of G is the cardinality of the smallest 2-dominating set of G. In this paper, we shall prove that the 2-domination number of generalized Petersen graphs $P(5k+1, 3)$, $P(5k+2, 3)$ and $P(5k+3, 3)$ is $4k+2$, $4k+3$ and $4k+4$, respectively. This proves one conjecture due to Bakhshesh *et al.* (*Proc. Indian Acad. Sci. (Math. Sci.)* **128** (2018) 17).

Keywords. 2-Domination number; generalized Petersen graph.

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1. Introduction

Graph theory terminology not presented here can be found in [1]. Let $G = (V, E)$ be a graph with $|V| = n$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph G is clear from context, we simply write $d(v)$, $N(v)$ and $N[v]$, respectively. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph induced by $S \subseteq V$ is denoted by $G[S]$. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance among pairs of vertices in G. For any subset $S \subseteq V$, let $E(S, V - S) = \{e \in E(G) : e = uv, u \in S, v \in V - S\}$.

A subset $S \subseteq V$ is a *k-dominating set* of G if each vertex in $V - S$ is adjacent to at least k vertices in S. The minimum size of a k-dominating set in G is called *k-domination number* of G and is denoted by $\gamma_k(G)$. A k-dominating set of G with the minimum cardinality is called a γ_k-set of G. The k-domination number of a graph was introduced by Fink and Jacobson [3].

The *generalized Petersen graph* $P(n, k) = (V, E)$ is defined as follows: $V = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$, $E = \bigcup_{i=1}^{n} \{(v_i, u_i), (v_i, v_{i+1}), (u_i, u_{i+k})\}$, where the subscripts are taken modulo n.

The problem of finding the domination number of generalized Petersen graphs has been considered by some researchers in [4] and [5]. The 2-domination number of generalized Petersen graphs was considered by Cheng [2]. He determined the exact value on the 2-domination number of generalized Petersen graphs $P(5k, 2)$, $P(5k+3, 2)$ and $P(5k+4, 2)$. In particular, he gave the following.

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PROPOSITION 1

For $k > 0$ *and* $0 \leq t < k$, $\gamma_2(P(5k, 5t+3)) = 4k$.

Bakhshesh *et al.* [1] determined the exact value on the 2-domination number of generalized Petersen graphs $P(5k+1, 2)$ and $P(5k+2, 2)$. In particular, he gave the following.

PROPOSITION 2

For each $k > 0$, $\gamma_2(P(5k+4, 3)) = 4k+4$.

Furthermore, they gave a good lower and upper bounds on the 2-domination number of generalized Petersen graphs $P(5k+1, 3)$, $P(5k+2, 3)$ and $P(5k+3, 3)$, and they gave one conjecture on the 2-domination number of generalized Petersen graphs $P(5k+1, 3)$, $P(5k+2, 3)$ and $P(5k+3, 3)$.

PROPOSITION 3

For each $k > 0$,

$$4k+1 \leq \gamma_2(P(5k+1, 3)) \leq 4k+2,$$
$$4k+2 \leq \gamma_2(P(5k+2, 3)) \leq 4k+3,$$
$$4k+3 \leq \gamma_2(P(5k+3, 3)) \leq 4k+4.$$

Conjecture 1. For each $k > 0$,

$$\gamma_2(P(5k+1, 3)) = 4k+2,$$
$$\gamma_2(P(5k+2, 3)) = 4k+3,$$
$$\gamma_2(P(5k+3, 3)) = 4k+4.$$

In this paper, we shall prove that the 2-domination number of generalized Petersen graphs $P(5k+1, 3)$, $P(5k+2, 3)$ and $P(5k+3, 3)$ is $4k+2$, $4k+3$ and $4k+4$, respectively. This proves the above conjecture due to Bakhshesh *et al.*

2. Main results

Let k be a positive integer. In order to give the exact 2-domination number of generalized Petersen graphs $P(5k+1, 3)$, we first consider the special case for $k = 1$.

Lemma 1. $\gamma_2(P(6, 3)) = 6$.

Proof. Let D be a γ_2-set of $P(6, 3)$. It follows that $|D \cap \{v_1, v_2, v_3\}| \geq 1$ and $|D \cap \{v_4, v_5, v_6\}| \geq 1$. If $|D \cap \{u_1, u_2, u_3, u_4, u_5, u_6\}| \geq 4$, then $\gamma_2(P(6, 3)) = |D| \geq 6$. Without loss of generality, we can assume that $|D \cap \{u_1, u_2, u_3, u_4, u_5, u_6\}| \leq 3$. So $|(V-D) \cap \{u_1, u_2, u_3, u_4, u_5, u_6\}| \geq 3$. Since $d(u_i) = 2$ for $i = 1, 2, \cdots, 6$, it follows that $|(V-D) \cap \{u_j, u_{j+3}\}| \leq 1$, where $j = 1, 2, 3$. So $|(V-D) \cap \{u_1, u_2, u_3, u_4, u_5, u_6\}| = 3$, i.e., $|D \cap \{u_1, u_2, u_3, u_4, u_5, u_6\}| = 3$ and $|D \cap \{u_j, u_{j+3}\}| = 1$, where $j = 1, 2, 3$. If

$|D \cap \{v_1, v_2, v_3\}| \geq 2$ or $|D \cap \{v_4, v_5, v_6\}| \geq 2$, then $\gamma_2(P(6,3)) = |D| \geq 6$. Hence, we can assume that $|D \cap \{v_1, v_2, v_3\}| = 1$ and $|D \cap \{v_4, v_5, v_6\}| = 1$, i.e., $|D| = 5$. Without loss of generality, we can assume that $u_1 \notin D$. Then $v_1, u_4 \in D$. Since $|D \cap \{v_1, v_2, v_3\}| = 1$, it follows that $v_2, v_3 \notin D$. In order to 2-dominate the vertex v_2, $u_2 \in D$. Since $|D \cap \{u_2, u_5\}| = 1$, $u_5 \notin D$. In order to 2-dominate the vertex u_5, $v_5 \in D$. Hence $v_4, v_6 \notin D$. So v_3 is not 2-dominated by D, which is a contradiction.

Hence, in all cases it follows that $\gamma_2(P(6,3)) \geq 6$. By Proposition 3, $\gamma_2(P(6,3)) = 6$. □

Lemma 2. *For each $k \geq 2$, $\gamma_2(P(5k+1,3)) = 4k+2$.*

Proof. Let D be a γ_2-set of $P(5k+1,3)$. Let $D' = \{v \in D : |N(v) \cap (V-D)| \leq 2\}$. We will discuss it from the following cases:

Case 1. $D' \neq \emptyset$. Let $v \in D'$. Then $|N(v) \cap (V-D)| \leq 2$ and since $d(v) = 3$, $N(v) \cap D \neq \emptyset$. Let $w \in N(v) \cap D$. Since $d(w) = 3$, it follows that $|N(w) \cap (V-D)| \leq 2$ which means $w \in D'$. Hence, if $D' \neq \emptyset$, then $|D'| \geq 2$. Since every vertex in $V-D$ is adjacent to at least two vertices in D, it follows that $2|V-D| \leq |E(D, V-D)| \leq 3(|D|-|D'|) + 2|D'|$. So, $2(10k+2-|D|) \leq 3|D| - 2$. So, $\gamma_2(P(5k+1,3)) = |D| \geq 4k + \frac{6}{5}$. That is $\gamma_2(P(5k+1,3)) \geq 4k+2$.

Case 2. $D' = \emptyset$. That is, $|N(v) \cap (V-D)| = 3$ for every vertex $v \in D$. Let $\bar{D} = \{w \in V-D : |N(w) \cap D| = 3\}$ and $|\bar{D}| = l$. So $2|(V-D)-\bar{D}| + 3|\bar{D}| = |E(D, V-D)| = 3|D|$. Hence, $2(10k+2-|D|-l)+3l = 3|D|$. So $\gamma_2(P(5k+1,3)) = |D| = 4k + \frac{4+l}{5}$. If $l \geq 2$, then $\gamma_2(P(5k+1,3)) = |D| \geq 4k+2$. If $l = 0$, then $\gamma_2(P(5k+1,3)) = |D| = 4k + \frac{4}{5}$, which is a contradiction. Hence, we can assume that $l = 1$. Without loss of generality, we can assume that $\bar{D} = \{v_4\}$ or $\bar{D} = \{u_4\}$.

Suppose that $\bar{D} = \{v_4\}$. Then $v_3, u_4, v_5 \in D$. Since $D' = \emptyset$, it follows that $u_3, v_6, u_7 \notin D$. Since $l = 1$, it follows that $|\{u_6, v_7\} \cap D| = 1$. If $u_6 \in D$ and $v_7 \notin D$, then v_7 is not 2-dominated by D. If $u_6 \notin D$ and $v_7 \in D$, then u_6 is not 2-dominated by D. In all cases, there is a contradiction.

Suppose that $\bar{D} = \{u_4\}$. Then $u_1, v_4, u_7 \in D$. Since $D' = \emptyset$, it follows that $v_1, v_3, v_5, v_7 \notin D$. Since $v_1, v_3 \notin D$, it follows that $v_2 \in D$ and $u_2 \notin D$. Since $u_2, v_5 \notin D$, it follows that $u_5 \in D$. Since $l = 1$, it follows that $v_6 \notin D$. Hence v_6 is not 2-dominated by D, which is a contradiction.

Hence, in all cases it follows that $\gamma_2(P(5k+1,3)) \geq 4k+2$. By Proposition 3, $\gamma_2(P(5k+1,3)) = 4k+2$. □

By Lemma 1 and Lemma 2, we have the following.

Theorem 1. *For each $k > 0$, $\gamma_2(P(5k+1,3)) = 4k+2$.*

Theorem 2. *For each $k > 0$, $\gamma_2(P(5k+2,3)) = 4k+3$.*

Proof. Let D be a γ_2-set of $P(5k+2,3)$. Let $D' = \{v \in D : |N(v) \cap (V-D)| \leq 2\}$. We will discuss it from the following cases:

Case 1. $|D'| \geq 3$. Since every vertex in $V-D$ is adjacent to at least two vertices in D, it follows that $2|V-D| \leq |E(D, V-D)| \leq 3(|D|-|D'|) + 2|D'|$. So, $2(10k+4-|D|) \leq$

$3|D| - |D'|$. So, $\gamma_2(P(5k+2,3)) = |D| \geq 4k + \frac{8+|D'|}{5}$. Since $|D'| \geq 3$, it follows that $\gamma_2(P(5k+2,3)) \geq 4k+3$.

Case 2. $|D'| = 2$. Let $\bar{D} = \{w \in V - D : |N(w) \cap D| = 3\}$ and $|\bar{D}| = l$. So $2|(V-D) - \bar{D}| + 3|\bar{D}| = |E(D, V-D)| = 3(|D| - |D'|) + 2|D'|$. Hence, $2(10k + 4 - |D| - l) + 3l = 3|D| - 2$. So $\gamma_2(P(5k+2,3)) = |D| = 4k + 2 + \frac{l}{5}$. If $l \geq 1$, then $\gamma_2(P(5k+2,3)) \geq 4k+3$. Hence, we can assume that $l = 0$. That is, $\bar{D} = \emptyset$. Since $|D' \cap \{v_1, v_2, \ldots, v_{5k+2}\}| = 0, 1, 2$, without loss of generality, we can assume that $D' = \{v_3, v_4\}$, $D' = \{v_5, u_5\}$ or $D' = \{u_4, u_7\}$. First, we assume that $k \geq 2$.

Suppose that $D' = \{v_3, v_4\}$. Then $v_2, u_3, u_4, v_5 \notin D$. since $\bar{D} = \emptyset$, it follows that $|\{u_5, v_6\} \cap D| = 1$. If $u_5 \in D$ and $v_6 \notin D$, then $u_2 \notin D$. In order to 2-dominate the vertex v_6, $\{u_6, v_7\} \subseteq D$ and $u_7 \notin D$. In order to 2-dominate the vertex u_4, $u_1 \in D$. So $v_1 \notin D$. Then v_2 is not 2-dominated by D, which is a contradiction. If $u_5 \notin D$ and $v_6 \in D$, then $u_6, v_7 \notin D$. Since $u_4, v_7 \notin D$, it follows that $u_7 \in D$. Since $\bar{D} = \emptyset$, it follows that $v_8 \notin D$. In order to 2-dominate the vertex v_8, $\{u_8, v_9\} \subseteq D$. So $u_9 \notin D$. Then u_6 is not 2-dominated by D, which is a contradiction.

Suppose that $D' = \{v_5, u_5\}$. Then $v_4, v_6, u_2, u_8 \notin D$. since $\bar{D} = \emptyset$, it follows that $|\{u_6, v_7\} \cap D| = 1$. If $u_6 \in D$ and $v_7 \notin D$, then $u_3 \notin D$. In order to 2-dominate the vertex v_7, $\{u_7, v_8\} \subseteq D$. So $u_4, u_8 \notin D$. In order to 2-dominate the vertex u_4, $u_1 \in D$. So $v_1 \notin D$. Since $v_1, u_2 \notin D$, $v_2 \in D$. So $v_3 \notin D$. Then v_4 is not 2-dominated by D, which is a contradiction. If $u_6 \notin D$ and $v_7 \in D$, then $u_7, v_8 \notin D$. Since $u_8, v_8 \notin D$, it follows that $u_9 \in D$ and $v_9 \notin D$. Then u_6 is not 2-dominated by D, which is a contradiction.

Suppose that $D' = \{u_4, u_7\}$. Then $v_4, v_7 \notin D$. since $\bar{D} = \emptyset$, it follows that $|\{v_3, v_5\} \cap D| = 1$. If $v_3 \in D$ and $v_5 \notin D$, then $u_5, v_6 \in D$ and $v_7, u_8 \notin D$. Since $\bar{D} = \emptyset$, $v_8 \notin D$. Then v_8 is not 2-dominated by D, which is a contradiction. If $v_3 \notin D$ and $v_5 \in D$, then $u_5, v_6 \notin D$. Since $v_3, v_4 \notin D$, $\{v_2, u_3\} \subseteq D$ and $u_6 \notin D$. Then v_6 is not 2-dominated by D, which is a contradiction.

By a similar way as above, if $k = 1$, there is a contradiction.

Case 3. $|D'| = 0$. That is, $|N(v) \cap (V-D)| = 3$ for every vertex $v \in D$. Let $\bar{D} = \{w \in V - D : |N(w) \cap D| = 3\}$ and $|\bar{D}| = l$. So $2|(V-D) - \bar{D}| + 3|\bar{D}| = |E(D, V-D)| = 3|D|$. Hence, $2(10k + 4 - |D| - l) + 3l = 3|D|$. So $\gamma_2(P(5k+2,3)) = |D| = 4k + \frac{8+l}{5}$. If $l \geq 3$, then $\gamma_2(P(5k+2,3)) \geq 4k+3$. If $l = 0$ or $l = 1$, then $\gamma_2(P(5k+2,3)) = 4k + \frac{8+l}{5}$, which is a contradiction. Hence, we can assume that $l = 2$. First, we assume that $k \geq 2$.

Suppose that $\bar{D} \cap \{v_1, v_2, \ldots, v_{5k+2}\} \neq \emptyset$. Without loss of generality, we can assume that $v_4 \in \bar{D}$. Then $v_3, u_4, v_5 \in D$. Since $D' = \emptyset$, it follows that $u_3, u_5, v_6, u_7 \notin D$. Since $u_3, v_6 \notin D$, it follows that $u_6 \in D$ and $u_9 \notin D$. Since $v_6, u_7 \notin D$, it follows that $v_7 \in D$ and $v_8 \notin D$. Since $u_5, v_8 \notin D$, it follows that $u_8 \in D$. Since $l = 2$ and $v_4, v_6 \in \bar{D}$, $v_9 \notin D$. Then v_9 is not 2-dominated by D, which is a contradiction.

Suppose that $\bar{D} \cap \{v_1, v_2, \ldots, v_{5k+2}\} = \emptyset$. Without loss of generality, we can assume that $u_4 \in \bar{D}$. Then $u_1, v_4, u_7 \in D$. Since $D' = \emptyset$, it follows that $v_1, v_3, v_5, v_7 \notin D$. Since $v_1, v_3 \notin D$, it follows that $v_2 \in D$ and $u_2 \notin D$. Since $v_5, v_7 \notin D$, it follows that $v_6 \in D$ and $u_6 \notin D$. Since $v_3, u_6 \notin D$, it follows that $u_3 \in D$. Then $v_3 \in \bar{D}$, which is a contradiction.

By a similar way as above, if $k = 1$, there is a contradiction.

Hence, in all cases it follows that $\gamma_2(P(5k+2,3)) \geq 4k+3$. By Proposition 3, $\gamma_2(P(5k+2,3)) = 4k+3$. □

Theorem 3. *For each $k > 0$, $\gamma_2(P(5k+3,3)) = 4k+4$.*

Proof. Let D be a γ_2-set of $P(5k+3, 3)$. Let $D' = \{v \in D : |N(v) \cap (V - D)| \leq 2\}$. We will discuss it from the following cases:

Case 1. $|D'| \geq 4$. Since every vertex in $V - D$ is adjacent to at least two vertices in D, it follows that $2|V - D| \leq |E(D, V - D)| \leq 3(|D| - |D'|) + 2|D'|$. So, $2(10k + 6 - |D|) \leq 3|D| - |D'|$. So, $\gamma_2(P(5k+3, 3)) = |D| \geq 4k + \frac{12+|D'|}{5}$. Since $|D'| \geq 4$, it follows that $\gamma_2(P(5k+3, 3)) \geq 4k+4$.

Case 2. $|D'| = 3$. Then there exists a vertex $w \in D'$ such that $|N(w) \cap (V - D)| \leq 1$. So, $2|V - D| \leq |E(D, V - D)| \leq 3(|D| - |D'|) + 5$. So, $2(10k + 6 - |D|) \leq 3|D| - 4$. So, $\gamma_2(P(5k+3, 3)) = |D| \geq 4k + \frac{16}{5}$. That is $\gamma_2(P(5k+3, 3)) \geq 4k + 4$.

Case 3. $|D'| = 2$. Let $\bar{D} = \{w \in V - D : |N(w) \cap D| = 3\}$ and $|\bar{D}| = l$. So $2|(V - D) - \bar{D}| + 3|\bar{D}| = |E(D, V - D)| = 3(|D| - |D'|) + 2|D'|$. Hence, $2(10k + 6 - |D| - l) + 3l = 3|D| - 2$. So $\gamma_2(P(5k+3, 3)) = |D| = 4k + \frac{14+l}{5}$. If $l \geq 2$, then $\gamma_2(P(5k+3, 3)) \geq 4k+4$. If $l = 0$, then $\gamma_2(P(5k+3, 3)) = 4k + \frac{14}{5}$, which is a contradiction. Hence, we can assume that $l = 1$. Since $|D' \cap \{v_1, \ldots, v_{5k+3}\}| = 0, 1, 2$, without loss of generality, we can assume that $D' = \{v_3, v_4\}$, $D' = \{v_5, u_5\}$ or $D' = \{u_4, u_7\}$. In the following, we only give the proof for the case for $k \geq 2$. By a similar way, it is easy to prove the case for $k = 1$.

Case 3.1. $D' = \{v_3, v_4\}$. Then $v_2, u_3, u_4, v_5 \notin D$. It follows that $\{u_5, v_6\} \cap D \neq \emptyset$.

If $u_5 \in D$ and $v_6 \notin D$, then $u_2 \notin D$. In order to 2-dominate the vertex v_6, $\{u_6, v_7\} \subseteq D$ and $u_7 \notin D$. In order to 2-dominate the vertex u_4, $u_1 \in D$ and $v_1 \notin D$. Then v_2 is not 2-dominated by D, which is a contradiction.

If $u_5 \notin D$ and $v_6 \in D$, then $u_6, v_7 \notin D$. Since $u_4, v_7 \notin D$, it follows that $u_7 \in D$. If $v_8 \notin D$, then $u_8, v_9 \in D$ and $u_9 \notin D$. Then u_6 is not 2-dominated by D, which is a contradiction. Hence, we can assume that $v_8 \in D$. Then $u_8 \notin D$. Then u_5 is not 2-dominated by D, which is a contradiction.

If $u_5 \in D$ and $v_6 \in D$, then $u_6, v_7, u_8 \notin D$. Since $u_4, v_7 \notin D$, it follows that $u_7 \in D$. Since $v_5 \in \bar{D}$ and $l = 1$, it follows that $v_8 \notin D$. Then v_8 is not 2-dominated by D, which is a contradiction.

Case 3.2. $D' = \{v_5, u_5\}$. Then $v_4, v_6, u_2, u_8 \notin D$. It follows that $\{u_6, v_7\} \cap D \neq \emptyset$.

If $v_7 \in D$ and $u_6 \notin D$, then $u_7, v_8 \notin D$. In order to 2-dominate the vertex v_8, $v_9 \in D$. So $u_9 \notin D$. Then u_6 is not 2-dominated by D, which is a contradiction.

If $v_7 \notin D$ and $u_6 \in D$, then $u_3 \notin D$. In order to 2-dominate the vertex v_7, $v_7 \in D$. So $u_4 \notin D$. In order to 2-dominate the vertex u_4, $u_1 \in D$. So $v_1 \notin D$. Hence, $v_2 \in D$ and $v_3 \notin D$. Then v_3 is not 2-dominated by D, which is a contradiction.

If $v_7 \in D$ and $u_6 \in D$, then $u_3, u_7 \notin D$. Since $v_4, u_7 \notin D$, $u_4 \in D$. Since $v_6 \in \bar{D}$ and $l = 1$, $v_3 \notin D$. Then v_3 is not 2-dominated by D, which is a contradiction.

Case 3.3. $D' = \{u_4, u_7\}$. Then $u_1, v_4, v_7, u_{10} \notin D$. It follows that $\{v_3, v_5\} \cap D \neq \emptyset$.

If $v_3 \in D$ and $v_5 \notin D$, then $u_3 \notin D$. Since $v_5, v_7 \notin D$, $v_6 \in D$ and $u_6 \notin D$. Since $u_3, u_6 \notin D$, $u_9 \in D$ and $v_9 \notin D$. Since $v_7, v_9 \notin D$, $v_8 \in D$ and $u_8 \notin D$. Since $v_7 \in \overline{D}$ and $l = 1$, $v_{10} \notin D$. Then v_{10} is not 2-dominated by D, which is a contradiction.

If $v_3 \notin D$ and $v_5 \in D$, then $u_6 \notin D$. In order to 2-dominate the vertex v_3, $u_3 \in D$ and $u_6 \notin D$. Then v_6 is not 2-dominated by D, which is a contradiction.

If $v_3 \in D$ and $v_5 \in D$, then $v_2, u_3, u_5 \notin D$. Since $v_2, u_5 \notin D$, $u_2 \in D$. Since $v_4 \in \bar{D}$ and $l = 1$, $v_1 \notin D$. Then v_1 is not 2-dominated by D, which is a contradiction.

Case 4. $|D'| = 0$. That is ,$|N(v) \cap (V - D)| = 3$ for every vertex $v \in D$.

Let $\bar{D} = \{w \in V - D : |N(w) \cap D| = 3\}$ and $|\bar{D}| = l$. So $2|(V - D) - \bar{D}| + 3|\bar{D}| = |E(D, V - D)| = 3|D|$. Hence, $2(10k + 6 - |D| - l) + 3l = 3|D|$. So $\gamma_2(P(5k+3, 3)) = |D| = 4k + \frac{12+l}{5}$. If $l \geq 4$, then $\gamma_2(P(5k + 3, 3)) \geq 4k + 4$. If $l = 0, 1, 2$, then $\gamma_2(P(5k+3, 3)) = 4k + \frac{12+l}{5}$, which is a contradiction. Hence, we can assume that $l = 3$.

Suppose that $\bar{D} \cap \{v_1, v_2, \ldots, v_{5k+3}\} \neq \emptyset$. Without loss of generality, we can assume that $v_2 \in \bar{D}$. Then $v_1, u_2, v_3 \in D$. Since $D' = \emptyset$, it follows that $u_1, u_3, u_5, v_4 \notin D$. Since $u_1, v_4 \notin D$, it follows that $u_4 \in D$ and $u_7 \notin D$. Since $v_4, u_5 \notin D$, it follows that $v_5 \in D$ and $v_6 \notin D$. Since $v_6, u_7 \notin D$, it follows that $v_7 \in D$ and $v_8 \notin D$. Since $u_3, v_6 \notin D$, it follows that $u_6 \in D$. Since $u_5, v_8 \notin D$, it follows that $u_8 \in D$. If $k = 1$, then $\{v_2, v_4, v_6, v_8\} \subseteq \bar{D}$, which is a contradiction. Suppose that $k \geq 2$. Since $u_6 \in D$, $u_9 \notin D$. Since $l = 3$ and $v_2, v_4, v_6 \in \bar{D}$, $v_9 \notin D$. Then v_9 is not 2-dominated by D, which is a contradiction.

Suppose that $\bar{D} \cap \{v_1, v_2, \cdots, v_{5k+3}\} = \emptyset$. Without loss of generality, we can assume that $u_5 \in \bar{D}$. Then $u_2, v_5, u_8 \in D$. Since $D' = \emptyset$, it follows that $v_2, v_4, v_6, v_8 \notin D$. Since $v_2, v_4 \notin D$, it follows that $v_3 \in D$ and $u_3 \notin D$. Since $v_6, v_8 \notin D$, it follows that $v_7 \in D$ and $u_7 \notin D$. Since $u_3, v_6 \notin D$, it follows that $u_6 \in D$. Then $v_6 \in \bar{D}$, which is a contradiction.

Hence, in all cases it follows that $\gamma_2(P(5k + 3, 3)) \geq 4k + 4$. By Proposition 3, $\gamma_2(P(5k + 3, 3)) = 4k + 4$. □

By Proposition 1, Proposition 2 and Theorems 1, 2 and 3, we have the following.

Theorem 4. *For* $k > 0$,

$$\gamma_2(P(5k+i, 3)) = \begin{cases} 4k+i, & \text{if } i = 0, 4, \\ 4k+i+1, & \text{if } i = 1, 2, 3. \end{cases}$$

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