Proc. Indian Acad. Sci. (Math. Sci.) (2020) 130:53
https://doi.org/10.1007/s12044-020-0568-7

Lifting to two-term relative maximal rigid subcategories in triangulated categories

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MS received 5 August 2019; revised 25 November 2019; accepted 24 February 2020

Abstract. Let $\mathscr{C}$ be a triangulated category with shift functor [1] and $\mathcal{R}$ a contravariantly rigid subcategory of $\mathscr{C}$. We show that a tilting subcategory of $\operatorname{mod}\mathcal{R}$ lifts to a two-term maximal $\mathcal{R}[1]$-rigid subcategory of $\mathscr{C}$. As an application, our result generalizes a result by Xie and Liu (*Proc. Amer. Math. Soc.* **141**(**10**) (2013) 3361–3367) for maximal rigid objects and a result by Fu and Liu (*Comm. Algebra* **37**(**7**) (2009) 2410–2418) for cluster tilting objects.

Keywords. Tilting subcategories; maximal rigid objects; cluster tilting objects.

Mathematics Subject Classification. 18E30, 16D90.

1. Introduction

Let k be an algebraically closed field and Λ be a finite-dimensional algebra. Let $\operatorname{mod}\Lambda$ be the category of finite-dimensional right Λ-modules. For an Λ-module T, let $\operatorname{add} T$ denote the full subcategory of $\operatorname{mod}\Lambda$ with objects all direct summands of direct sums of copies of T. Then T is called a tilting module in $\operatorname{mod}\Lambda$ if

- $\operatorname{pd}_\Lambda T \leq 1$;
- $\operatorname{Ext}^1_\Lambda(T, T) = 0$;
- there exists an exact sequence $0 \to \Lambda \to T^0 \to T^1 \to 0$, with T^0, T^1 in $\operatorname{add} T$.

Fu and Liu proved the following result.

Theorem 1.1 [**8**, **Theorem 3.3**]. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category with a cluster tilting object T and let Λ be the endomorphism algebra of T. If M is a tilting module over Λ, then M lifts to a cluster tilting object in $\mathscr{C}$.*

Note that each cluster tilting object is a maximal rigid in a 2-Calabi–Yau triangulated category. But the converse is not true, in general. The counter-examples can be found in tube categories [7] or in categories of Cohen–Macaulay modules over an isolated hypersurface singularity [4]. Xie and Liu [14] showed that tilting modules over such algebras lift to maximal rigid objects in the corresponding 2-Calabi–Yau triangulated category. Namely, they proved the following.

Published online: 08 September 2020

Theorem 1.2 **[14, Theorem 2.2]**. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category with a maximal rigid object T and let Λ be the endomorphism algebra of T. If M is a tilting module over Λ, then M lifts to a maximal rigid object in $\mathscr{C}$.*

The notion of tilting module was generalized to abelian categories by Beligiannis [3]. We recall this definition here: Let $\mathcal{A}$ be an abelian category with enough projective objects. A contravariantly finite subcategory $\mathcal{M}$ of $\mathcal{A}$ is called a tilting subcategory if

- $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) = 0$.
- $\mathsf{pd}_{\mathcal{A}} M \leq 1$, for any $M \in \mathcal{M}$.
- for any projective object P in $\mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow P \longrightarrow M_0 \longrightarrow M_1 \longrightarrow 0$$

where $M_0, M_1 \in \mathcal{M}$.

In this note, we give a similar result of Fu and Liu [8] and Xie and Liu [14]. Namely, we prove the following.

Theorem 1.3 **(see Theorem 3.3 for more details)**. *Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ be a contravariantly rigid subcategory of $\mathscr{C}$. If $\mathcal{M}$ is a tilting subcategory of $\mathsf{mod}\,\mathcal{R}$, then $\mathcal{M}$ lifts to a two-term maximal $\mathcal{R}[1]$-rigid subcategory of $\mathscr{C}$.*

As an application, we obtain the following important conclusion.

Theorem 1.4 **(see Theorem 3.4 for more details)**. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category and $\mathcal{R}$ be a contravariantly maximal rigid subcategory of $\mathscr{C}$. If $\mathcal{M}$ is a tilting subcategory of $\mathsf{mod}\,\mathcal{R}$, then $\mathcal{M}$ lifts to a maximal rigid subcategory of $\mathscr{C}$.*

This article is organized as follows. In Section 2, we recall some elementary definitions and facts on cluster tilting theory and τ-tilting theory that will be used. In Section 3, we prove the main results of this article, and give an application. We conclude this section with some conventions.

Throughout this article, k is an algebraically closed field. When we say that $\mathscr{C}$ is a category, we always assume that $\mathscr{C}$ is a Hom-finite Krull–Schmidt k-linear category. We write $\mathscr{C}(X, Y)$ for the set of morphisms of from X to Y. Let $\mathscr{C}$ be an additive category. When we say that $\mathcal{R}$ is a subcategory of $\mathscr{C}$, we always assume that $\mathcal{R}$ is a full subcategory which is closed under isomorphisms, direct sums and direct summands. We denote by $[\mathcal{R}]$ the ideal of $\mathscr{C}$ consisting of morphisms which factor through objects in $\mathcal{R}$. For two subcategories $\mathscr{X}$ and $\mathscr{Y}$ of a triangulated category $\mathscr{C}$, we denote by $\mathscr{X} * \mathscr{Y}$ the collection of objects in $\mathscr{C}$ consisting of all such $M \in \mathscr{C}$ with triangles $X \longrightarrow M \longrightarrow Y \longrightarrow X[1]$, where $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. We denote by $\mathscr{X} \vee \mathscr{Y}$ the smallest subcategory of $\mathscr{C}$ containing $\mathscr{X}$ and $\mathscr{Y}$.

Let $\mathscr{C}$ be a triangulated category with a shift functor [1]. Recall that $\mathscr{C}$ admits a Serre functor $\mathbb{S}$ if there exists a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(X,\ Y) \simeq D\,\mathrm{Hom}_{\mathscr{C}}(Y,\ \mathbb{S}X) \quad \text{for any } X, Y \in \mathscr{C},$$

where $D = \mathrm{Hom}_k(-, k)$ is the usual duality over k. If $\mathbb{S} \simeq [2]$, $\mathscr{C}$ is called 2-Calabi–Yau. In [13], Reiten and Van den Bergh proved that if $\mathscr{C}$ has a Serre functor $\mathbb{S}$, then $\mathscr{C}$ has Auslander–Reiten triangles. Moreover, if τ is the Auslander–Reiten translation, then $\mathbb{S} \simeq \tau[1]$. For $X, Y \in \mathscr{C}$, we put as usual $\mathrm{Ext}^1_{\mathscr{C}}(X, Y) = \mathrm{Hom}(X, Y[1])$.

2. Preliminaries

In this section, we review some useful notation and results. We first recall the definition of cluster tilting subcategories and related subcategories from [6,10–12].

DEFINITION 2.1

Let $\mathscr{C}$ be a triangulated category

(i) A subcategory $\mathcal{R}$ of $\mathscr{C}$ is called rigid if $\mathrm{Ext}^1_{\mathscr{C}}(\mathcal{R},\mathcal{R}) = 0$.
(ii) A subcategory $\mathcal{R}$ of $\mathscr{C}$ is called maximal rigid if it is rigid and maximal with respect to the property: If $M \in \mathscr{C}$ satisfying $\mathrm{Ext}^1_{\mathscr{C}}(\mathcal{R} \vee \mathsf{add}\, M, \mathcal{R} \vee \mathsf{add}\, M) = 0$, then $M \in \mathcal{R}$.
(iii) A contravariantly finite subcategory $\mathcal{R}$ of $\mathscr{C}$ is called cluster tilting if
$$\mathcal{R} = \{M \in \mathscr{C} \mid \mathrm{Ext}^1_{\mathscr{C}}(\mathcal{R}, M) = 0\}.$$
(iv) An object R in $\mathscr{C}$ is called rigid, maximal rigid, or cluster tilting if $\mathsf{add}\, R$ is rigid, maximal rigid, or cluster tilting respectively.

One can see that any cluster tilting subcategory is maximal rigid, but the converse is not true, in general [4,7]. But Zhou and Zhu [17, Theorem 2.6] proved that the converse is true if the 2-Calabi–Yau triangulated categories admit a cluster tilting subcategory. More precisely, they show the following.

Lemma 2.2 [17, *Theorem* 2.6]. *Let $\mathscr{C}$ be a* 2-*Calabi–Yau triangulated category with a cluster tilting subcategory. Then every contravariantly finite maximal rigid subcategory is cluster tilting.*

The following lemma can be found in [5, Proposition I.1.7] and [17, Corollary 2.5].

Lemma 2.3. *Let $\mathscr{C}$ be a* 2-*Calabi–Yau triangulated category and $\mathcal{R}$ a contravariantly finite maximal rigid subcategory of $\mathscr{C}$. Then every rigid object belongs to $\mathcal{R} * \mathcal{R}[1]$.*

The notion of two-term relative maximal rigid subcategories and related subcategories are introduced by Zhou and Zhu in [16].

DEFINITION 2.4 [16, Definition 2.3]

Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ a rigid subcategory of $\mathscr{C}$.

(i) A subcategory $\mathscr{X}$ in $\mathscr{C}$ is called $\mathcal{R}[1]$-*rigid*if $[\mathcal{R}[1]](\mathscr{X}, \mathscr{X}[1]) = 0$.
Any $\mathcal{R}[1]$-rigid subcategory in $\mathcal{R} * \mathcal{R}[1]$ is called *two-term* $\mathcal{R}[1]$-rigid.
(ii) A subcategory $\mathscr{X} \subseteq \mathcal{R} * \mathcal{R}[1]$ is called *two-term* $\mathcal{R}[1]$-*maximal rigid* if $\mathscr{X}$ is $\mathcal{R}[1]$-rigid and for any $M \in \mathcal{R} * \mathcal{R}[1]$,
$$[\mathcal{R}[1]](\mathscr{X} \vee \mathsf{add}\, M, (\mathscr{X} \vee \mathsf{add}\, M)[1]) = 0 \text{ implies } M \in \mathscr{X}.$$
In this case, $\mathscr{X}$ is also called the two-term relative maximal rigid subcategories.
(iii) A subcategory $\mathscr{X} \subseteq \mathcal{R} * \mathcal{R}[1]$ is called two-term weak $\mathcal{R}[1]$-cluster tilting if $\mathcal{R} \subseteq \mathscr{X}[-1] * \mathscr{X}$ and
$$\mathscr{X} = \{M \in \mathcal{R} * \mathcal{R}[1] \mid [\mathcal{R}[1]](M, \mathscr{X}[1]) = 0 \quad \text{and} \quad [\mathcal{R}[1]](\mathscr{X}, M[1]) = 0\}.$$

(iv) An object X is called two-term $\mathcal{R}[1]$-rigid, two-term maximal $\mathcal{R}[1]$-rigid or two-term weak $\mathcal{R}[1]$-cluster tilting if add X is two-term $\mathcal{R}[1]$-rigid, two-term maximal $\mathcal{R}[1]$-rigid or two-term weak $\mathcal{R}[1]$-cluster tilting respectively.

The following result is essentially already proved in [15, Proposition 3.4]. We give the proof for the convenience of the reader.

Lemma 2.5. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category and $\mathcal{R}$ a rigid subcategory of $\mathscr{C}$. If $\mathscr{X}$ is a subcategory of $\mathscr{C}$ which is contained $\mathcal{R} * \mathcal{R}[1]$, then $\mathscr{X}$ is $\mathcal{R}[1]$-rigid if and only if $\mathscr{X}$ is rigid.*

Proof. It is obvious that any rigid subcategory is $\mathcal{R}[1]$-rigid.

Now we assume that $\mathscr{X}$ is $\mathcal{R}[1]$-rigid. For any two objects $X, X' \in \mathscr{X}$, since $X \in \mathscr{X} \subseteq \mathcal{R} * \mathcal{R}[1]$, there exists a triangle

$$R_1 \xrightarrow{f} R_0 \xrightarrow{g} X \xrightarrow{h} R_1[1],$$

where $R_0, R_1 \in \mathcal{R}$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(-, X'[1])$ to the above triangle, we have the following long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathscr{C}}(R_1[1], X'[1]) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(h,\, X'[1])} \mathrm{Hom}_{\mathscr{C}}(X, X'[1]) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(g,\, X'[1])} \mathrm{Hom}_{\mathscr{C}}(R_0, X'[1]) \to \cdots .$$

Since $\mathscr{X}$ is $\mathcal{R}[1]$-rigid, we obtain that $\mathrm{Hom}_{\mathscr{C}}(g, X'[1])$ is an injective. It follows that the morphism

$$D\,\mathrm{Hom}_{\mathscr{C}}(g, X'[1])\colon D\,\mathrm{Hom}_{\mathscr{C}}(R_0, X'[1]) \to D\,\mathrm{Hom}_{\mathscr{C}}(X, X'[1])$$

is surjective. By the 2-Calabi–Yau property, we have the following commutative diagram:

$$\begin{array}{ccc} D\,\mathrm{Hom}_{\mathscr{C}}(R_0, X'[1]) & \xrightarrow{D\,\mathrm{Hom}_{\mathscr{C}}(g, X'[1])} & D\,\mathrm{Hom}_{\mathscr{C}}(X, X'[1]) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_{\mathscr{C}}(X', R_0[1]) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(X', g[1])} & \mathrm{Hom}_{\mathscr{C}}(X', X[1]). \end{array}$$

Thus we get that $\mathrm{Hom}_{\mathscr{C}}(X', g[1])\colon \mathrm{Hom}_{\mathscr{C}}(X', R_0[1]) \to \mathrm{Hom}_{\mathscr{C}}(X', X[1])$ is surjective. Since $\mathscr{X}$ is a $\mathcal{R}[1]$-rigid, we have that the morphism $\mathrm{Hom}_{\mathscr{C}}(X', g[1])$ is zero. Therefore, $\mathrm{Hom}_{\mathscr{C}}(X', X[1]) = 0$ and then $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, \mathscr{X}[1]) = 0$. This shows that $\mathscr{X}$ is rigid. □

Lemma 2.6. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category and $\mathcal{R}$ a contravariantly finite maximal rigid subcategory of $\mathscr{C}$. Then $\mathscr{X}$ is a two-term $\mathcal{R}[1]$-maximal rigid subcategory of $\mathscr{C}$ if and only if $\mathscr{X}$ is a maximal rigid subcategory of $\mathscr{C}$.*

Proof. We first show the 'only if' part. Let $\mathscr{X}$ be a two-term $\mathcal{R}[1]$-maximal rigid subcategory of $\mathscr{C}$.

By Lemma 2.5, we have that $\mathscr{X}$ is rigid. Assume that $M \in \mathscr{C}$ satisfies

$$\mathrm{Ext}^1_{\mathscr{C}}(\mathcal{R} \vee \mathsf{add}\, M, \mathcal{R} \vee \mathsf{add}\, M) = 0.$$

Then $[\mathcal{R}[1]](\mathscr{X} \vee \mathsf{add}\, M, (\mathscr{X} \vee \mathsf{add}\, M)[1]) = 0$ and $\mathrm{Ext}^1_{\mathscr{C}}(M, M)$=0, that is, M is a rigid object. By Lemma 2.3, we obtain $M \in \mathcal{R} * \mathcal{R}[1]$. Since $\mathscr{X}$ is a two-term $\mathcal{R}[1]$-maximal rigid, we have $M \in \mathscr{X}$. This shows that $\mathscr{X}$ is a maximal rigid subcategory of $\mathscr{C}$.

To prove the 'if' part. Suppose that $\mathscr{X}$ is a maximal rigid subcategory of $\mathscr{C}$. Then $\mathscr{X}$ is rigid and then $\mathscr{X}$ is $\mathcal{R}[1]$-rigid. By Lemma 2.3, we get $\mathscr{X} \subseteq \mathcal{R} * \mathcal{R}[1]$. Assume that $M \in \mathcal{R} * \mathcal{R}[1]$ satisfies $[\mathcal{R}[1]](\mathscr{X} \vee \mathsf{add}\, M, (\mathscr{X} \vee \mathsf{add}\, M)[1]) = 0$, that is, $\mathscr{X} \vee \mathsf{add}\, M$ is a $\mathcal{R}[1]$-rigid. By Lemma 2.5, we deduce that $\mathscr{X} \vee \mathsf{add}\, M$ is a rigid, namely,

$$\mathrm{Ext}^1_{\mathscr{C}}(\mathcal{R} \vee \mathsf{add}\, M, \mathcal{R} \vee \mathsf{add}\, M) = 0.$$

It follows that $M \in \mathscr{X}$ since $\mathscr{X}$ is a maximal rigid. This shows that $\mathscr{X}$ is a two-term $\mathcal{R}[1]$-maximal rigid subcategory of $\mathscr{C}$. □

Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ a contravariantly rigid subcategory of $\mathscr{C}$. We write $\mathsf{Mod}\, \mathcal{R}$ for the abelian group of contravariantly additive functor from $\mathcal{R}$ to the category of abelian group, and $\mathsf{mod}\, \mathcal{R}$ for the full subcategory of finitely presentation functor, see [1]. There exists a functor

$$\begin{aligned} \mathbb{F}\colon \mathscr{C} &\longrightarrow \mathsf{Mod}\, \mathcal{R} \\ M &\longmapsto \mathrm{Hom}_{\mathscr{C}}(-, M) \mid_{\mathcal{R}}, \end{aligned}$$

sometimes known as the restricted Yoneda functor.

Theorem 2.7 [10, Proposition 6.2]. *The functor* $\mathbb{F} = \mathrm{Hom}_{\mathscr{C}}(\mathcal{R}, -)$ *induces an equivalence*

$$(\mathcal{R} * \mathcal{R}[1])/[\mathcal{R}[1]] \xrightarrow{\sim} \mathsf{mod}\, \mathcal{R}.$$

Note that if $\mathcal{R}$ *is a cluster tilting subcategory of* $\mathscr{C}$, *then* $\mathcal{R} * \mathcal{R}[1] = \mathscr{C}$.

The notion of support τ-tilting pairs was introduced by Iyama *et al.* [9].

DEFINITION 2.8 [9, Definition 1.3]

Let $\mathcal{R}$ be an additive category.

(i) Let $\mathcal{M}$ be a subcategory of $\mathsf{mod}\, \mathcal{R}$. A class $\{ P_1 \stackrel{\pi^M}{\to} P_0 \to M \to 0 \mid M \in \mathcal{M}\}$ of projective presentations in $\mathsf{mod}\, \mathcal{R}$ is said to have *Property* (*S*) if

$$\mathrm{Hom}_{\mathsf{mod}\, \mathcal{R}}(\pi^M, M') : \mathrm{Hom}_{\mathsf{mod}\, \mathcal{R}}(P_0, M') \to \mathrm{Hom}_{\mathsf{mod}\, \mathcal{R}}(P_1, M')$$

is surjective for any $M, M' \in \mathcal{M}$.

(ii) A subcategory $\mathcal{M}$ of $\mathsf{mod}\, \mathcal{R}$ is said to be τ-*rigid* if there is a class of projective presentations $\{P_1 \to P_0 \to M \to 0 \mid M \in \mathcal{M}\}$ which has Property (S).

(iii) A τ-*rigid pair* of $\mathsf{mod}\, \mathcal{R}$ is a pair $(\mathcal{M}, \mathcal{E})$, where $\mathcal{M}$ is a τ-rigid subcategory of $\mathsf{mod}\, \mathcal{R}$ and $\mathcal{E} \subseteq \mathcal{R}$ is a subcategory with $\mathcal{M}(\mathcal{E}) = 0$, that is, $M(E) = 0$ for each $M \in \mathcal{M}$ and $E \in \mathcal{E}$.

(iv) A τ-rigid pair $(\mathcal{M}, \mathcal{E})$ is *support τ-tilting* if $\mathcal{E} = \mathrm{Ker}\,(\mathcal{M})$ and for each $R \in \mathcal{R}$, there exists an exact sequence $\mathcal{R}(-, R) \xrightarrow{f} M^0 \to M^1 \to 0$ with $M^0, M^1 \in \mathcal{M}$ such that f is a left $\mathcal{M}$-approximation. In this case, $\mathcal{M}$ is called a *support τ-tilting subcategory* of $\mathsf{mod}\,\mathcal{R}$.

Zhou and Zhu [17] showed that there exists a bijection between the set of two-term weak $\mathcal{R}[1]$-cluster tilting subcategories of $\mathscr{C}$ and the set of support τ-tilting pairs of $\mathsf{mod}\,\mathcal{R}$.

Theorem 2.9 [16, Theorem 4.5]. *Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ a contravariantly rigid category of $\mathscr{C}$. The functor*

$$\mathbb{F}\colon \mathscr{C} \to \mathsf{mod}\,\mathcal{R}$$

induces a bijection

$$\Phi\colon \mathscr{X} \longmapsto \big(\mathbb{F}(\mathscr{X}), \mathcal{R} \cap \mathscr{X}[-1]\big)$$

from the first of the following sets to the second:

(*i*) *Two-term weak $\mathcal{R}[1]$-cluster tilting subcategories of $\mathscr{C}$.*
(*ii*) *Support τ-tilting pairs of $\mathsf{mod}\,\mathcal{R}$.*

3. Main results

In order to prove our main results, we need the following lemmas.

Lemma 3.1. *Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ be a contravariantly finite rigid subcategory of $\mathscr{C}$. Then $\mathsf{mod}\,\mathcal{R}$ is an abelian category with enough projective objects.*

Proof. We first show that $\mathcal{R}$ has a weak kernel. For any morphism $a\colon R_1 \to R_2$ in $\mathcal{R}$, there exists a triangle

$$M \xrightarrow{b} R_1 \xrightarrow{a} R_2 \xrightarrow{c} M[1].$$

Since $\mathcal{R}$ is contravariantly finite, there exists a right $\mathcal{R}$-approximation $d\colon R_0 \to M$ of M. We claim that $bd\colon R_0 \to R_1$ is a weak kernel of a. In fact, we have $a(bd) = (ab)d = 0$. Let $u\colon R \to R_2$ be in $\mathcal{R}$ such that $au = 0$. Then there exists a morphism $e\colon R \to M_1$ such that $u = be$. Since d is a right $\mathcal{R}$-approximation of M, there exists a morphism $f\colon R \to R_0$ such that $e = df$. It follows that $u = be = (bd)f$. This shows that $\mathcal{R}$ has a weak kernel. By [2, Proposition 2.1], we obtain that $\mathsf{mod}\,\mathcal{R}$ is an abelian category.

For any object $M \in \mathsf{mod}\,\mathcal{R}$, by Theorem 2.7, there exists an object $X \in \mathcal{R} * \mathcal{R}[1]$ such that $\mathbb{F}(X) = M$. Since $X \in \mathcal{R} * \mathcal{R}[1]$, there exists a triangle

$$R_3 \xrightarrow{x} R_4 \xrightarrow{y} X \xrightarrow{z} R_3[1],$$

with $R_3, R_4 \in \mathcal{R}$. Applying the functor $\mathbb{F} = \mathrm{Hom}_{\mathscr{C}}(\mathcal{R}, -)$ to the above triangle, we have the following exact sequence:

$$\mathbb{F}(R_3) \to \mathbb{F}(R_4) \to \mathbb{F}(X) = M \to \mathbb{F}(R_3[1]) = 0.$$

This shows that $\mathsf{mod}\,\mathscr{X}$ has enough projective objects. □

Lemma 3.2. *Let* $\mathscr{C}$ *be a triangulated category and* $\mathcal{R}$ *be a contravariantly rigid subcategory of* $\mathscr{C}$*. If* $\mathcal{M}$ *is a tilting subcategory of* $\mathsf{mod}\,\mathcal{R}$*, then* $(\mathcal{M}, 0)$ *is a support* τ*-tilting pair of* $\mathsf{mod}\,\mathcal{R}$*.*

Proof. We first show that $(\mathcal{M}, 0)$ is a τ-rigid pair of $\mathsf{mod}\,\mathcal{R}$.

For any object $M \in \mathcal{M}$, since $\mathsf{pd}\,M \leq 1$, we have a short exact sequence

$$0 \longrightarrow P_1 \xrightarrow{\pi^M} P_0 \longrightarrow M \longrightarrow 0,$$

where P_0, P_1 are two projective object in $\mathsf{mod}\,\mathcal{R}$. Applying the functor $\mathrm{Hom}_{\mathsf{mod}\,\mathcal{R}}(-, \mathcal{M})$ to the above short exact sequence, we get the following exact sequence

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{mod}\,\mathcal{R}}(P_0, \mathcal{M}) \xrightarrow{\circ\pi^M} \mathrm{Hom}_{\mathsf{mod}\,\mathcal{R}}(P_1, \mathcal{M})\\ &\quad\to \mathrm{Ext}^1_{\mathsf{mod}\,\mathcal{R}}(M, \mathcal{M}) = 0.\end{aligned}$$

This means that there exists a class of projective presentations $\{P_1 \xrightarrow{\pi^M} P_0 \to M \to 0 \mid M \in \mathcal{M}\}$ which has Property (S). Therefore $(\mathcal{M}, 0)$ is a τ-rigid pair of $\mathsf{mod}\,\mathcal{R}$ since $\mathcal{M}(0) = 0$.

Now we show that $(\mathcal{M}, 0)$ is a support τ-tilting pair of $\mathsf{mod}\,\mathcal{R}$.

For any object $R \in \mathcal{R}$, $\mathcal{R}(-, R)$ is a projective object in $\mathsf{mod}\,\mathcal{R}$. Since $\mathcal{M}$ is a tilting subcategory of $\mathsf{mod}\,\mathcal{R}$, there exists a short exact sequence

$$0 \longrightarrow \mathcal{R}(-, R) \xrightarrow{f} M_0 \longrightarrow M_1 \longrightarrow 0,$$

where $M_0, M_1 \in \mathcal{M}$. Applying the functor $\mathrm{Hom}_{\mathsf{mod}\,\mathcal{R}}(-, \mathcal{M})$ to the above exact sequence, we have the following exact sequence:

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{mod}\,\mathcal{R}}(M_0, \mathcal{M}) \xrightarrow{\circ f} \mathrm{Hom}_{\mathsf{mod}\,\mathcal{R}}(\mathcal{R}(-, R), \mathcal{M})\\ &\quad\to \mathrm{Ext}^1_{\mathsf{mod}\,\mathcal{R}}(M_1, \mathcal{M}) = 0.\end{aligned}$$

This shows that f is a left $\mathcal{M}$-approximation of $\mathcal{R}(-, R)$.

If $\mathcal{M}(E) = 0$, where $E \in \mathcal{R}$, by the above discussion, there exists an exact sequence

$$0 \longrightarrow \mathcal{R}(-, E) \longrightarrow M_0 \longrightarrow M_1 \longrightarrow 0,$$

where $M^0, M^1 \in \mathcal{M}$. It follows that there exists an exact sequence

$$0 \longrightarrow \mathcal{R}(E, E) \longrightarrow M_0(E) \longrightarrow M_1(E) \longrightarrow 0.$$

Since $M_0(E) = 0$, we have $\mathcal{R}(E, E) = 0$ and then $E = 0$. Therefore, $\mathrm{Ker}\,(\mathcal{M}) = 0$.

This shows that $(\mathcal{M}, 0)$ is a support τ-tilting pair of $\mathsf{mod}\,\mathcal{R}$. □

We now prove our main result.

Theorem 3.3. *Let $\mathscr{C}$ be a triangulated category and $\mathcal{R}$ be a contravariantly rigid subcategory of $\mathscr{C}$. If $\mathcal{M}$ is a tilting subcategory of* $\operatorname{mod}\mathcal{R}$, *then $\mathcal{M}$ lifts to two-term maximal $\mathcal{R}[1]$-rigid subcategory of $\mathscr{C}$.*

Proof. Since $\mathcal{M}$ is a subcategory of $\operatorname{mod}\mathcal{R}$, by Theorem 2.7, there is a subcategory $\mathscr{X}$ of $\mathcal{R} * \mathcal{R}[1] \subseteq \mathscr{C}$ such that $\mathbb{F}\mathscr{X} = \mathcal{M}$. Because $\mathcal{M}$ is a tilting subcategory of $\operatorname{mod}\mathcal{R}$, by Lemma 3.2, we have that $(\mathcal{M}, 0)$ is a support τ-tilting pair of $\operatorname{mod}\mathcal{R}$. By Theorem 2.9, we know that $\mathscr{X}$ is a two-term weak $\mathcal{R}[1]$-cluster tilting subcategory of $\mathscr{C}$ and then $\mathscr{X}$ is two-term $\mathcal{R}[1]$-rigid.

Now we show that $\mathscr{X}$ is a two-term maximal $\mathcal{R}[1]$-rigid subcategory of $\mathscr{C}$.

For any $M \in \mathcal{R} * \mathcal{R}[1]$, it satisfies

$$[\mathcal{R}[1]](\mathscr{X} \vee \operatorname{add} M, (\mathscr{X} \vee \operatorname{add} M)[1]) = 0.$$

Then we have

$$[\mathcal{R}[1]](\mathscr{X}, M[1]) = 0 = [\mathcal{R}[1]](M, \mathscr{X}[1])$$

which implies that $M \in \mathscr{X}$ since $\mathscr{X}$ is a two-term weak $\mathcal{R}[1]$-cluster tilting subcategory.

This shows that $\mathscr{X}$ is a two-term maximal $\mathcal{R}[1]$-rigid subcategory of $\mathscr{C}$. □

This theorem immediately yields the following important conclusion.

Theorem 3.4. *Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category and $\mathcal{R}$ be a contravariantly maximal rigid subcategory of $\mathscr{C}$. If $\mathcal{M}$ is a tilting subcategory of* $\operatorname{mod}\mathcal{R}$, *then $\mathcal{M}$ lifts to a maximal rigid subcategory of $\mathscr{C}$.*

Proof. This follows from Lemma 2.6 and Theorem 3.3. □

As a special case of Thoerem 3.4, we have the following.

COROLLARY 3.5 [14, Theorem 2.2]

Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category with a maximal rigid object T and let Λ be the endomorphism algebra of T. If M is a tilting module over Λ, then M lifts to a maximal rigid object in $\mathscr{C}$.

As a special case of Corollary 3.5, we have the following.

COROLLARY 3.6 [8, Theorem 3.3]

Let $\mathscr{C}$ be a 2-Calabi–Yau triangulated category with a cluster tilting object T and let Λ be the endomorphism algebra of T. If M is a tilting module over Λ, then M lifts to a cluster tilting object in $\mathscr{C}$.

Proof. This follows from Lemma 2.2 and Corollary 3.5. □

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant Nos 11901190 and 11671221), the Hunan Provincial Natural Science Foundation of China (Grant No. 2018JJ3205) and the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 19B239). The author would like to thank the referee for reading the paper carefully and for many suggestions.

References

[1] Auslander M, Representation theory of Artin algebras I, *Comm. Algebra* **1** (1974) 177–268
[2] Auslander M, Coherent functors, Proc. Conf. Categorical Algebra (1965) (California: La Jolla) pp. 189–231, Springer, New York, 1966
[3] Beligiannis A, Rigid objects, triangulated subfactors and abelian localizations, *Math. Z.* **274(3-4)** (2013) 841–883
[4] Burban I, Iyama O, Keller B and Reiten I, Cluster tilting for one-dimensional hypersurface singularities, *Adv. Math.* **217(6)** (2008) 2443–2484
[5] Buan A, Iyama O, Reiten I and Scott J. Cluster structure for 2-Calabi–Yau categories and unipotent groups, *Compos. Math.* **145(4)** (2009) 1035–1079
[6] Buan A, Marsh R, Reineke M, Reiten I and Todorov G, Tilting theory and cluster combinatorics, *Adv. Math.* **204(2)** (2006) 572–618
[7] Buan A, Marsh R and Vatne D, Cluster structure from 2-Calabi–Yau categories with loops, *Math. Z.* **265(4)** (2010) 951–970
[8] Fu C and Liu P, Lifting to cluster-tilting objects in 2-Calabi-Yau triangulated categories, *Comm. Algebra* **37(7)** (2009) 2410–2418
[9] Iyama O, Jørgensen P and Yang D, Intermediate co-t-structures, two-term silting objects, τ-tilting modules, and torsion classes, *Algebra Number Theory*, **8(10)** (2014) 2413–2431
[10] Iyama O and Yoshino Y, Mutation in triangulated categories and rigid Cohen–Macaulay modules, *Invent. Math.* **172(1)** (2008) 117–168
[11] Keller B and Reiten I, Cluster-tilted algebras are Gorenstein and stably Calabi–Yau, *Adv. Math.* **211** (2007) 123–151
[12] Koenig S and Zhu B, From triangulated categories to abelian categories: Cluster tilting in a general framework, *Math. Z.* **258** (2008) 143–160
[13] Reiten I and Van den Bergh M, Noetherian hereditary abelian categories satisfying Serre duality, *J. Amer. Math. Soc.* **15(2)** (2002) 295–366
[14] Xie Y and Liu P, Lifting to maximal rigid objects in 2-Calabi–Yau triangulated categories, *Proc. Amer. Math. Soc.* **141(10)** (2013) 3361–3367
[15] Yang W and Zhu B, Relative cluster tilting objects in triangulated categories, *Trans. Amer. Math. Soc.* **371(1)** (2019) 387–412
[16] Zhou P and Zhu B, Two-term relative cluster tilting subcategories, τ-tilting modules and silting subcategories *J. Pure Appl. Algebra* **224(9)** (2020) 106365, 22 pp.
[17] Zhou Y and Zhu B, Maximal rigid subcategories in 2-Calabi–Yau triangulated categories, *J. Algebra* **348** (2011) 49–60

COMMUNICATING EDITOR: Amalendu Krishna