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A note on bounds of the *p*-length of a *p*-soluble group

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Abstract. Suppose that the finite group $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B. By using the Wielandt lenghts of Sylow p-subgroups of A and B, we present a new bound of the p-length of G. Some known results are improved.

Keywords. p-Soluble; p-length; mutually permutable product; Wielandt length..

Mathematics Subject Classification. 20D15, 20D20.

1. Introduction

All groups considered here are finite. Let G be a group. By $\pi(G)$, we denote the set of all prime divisors of $|G|$. For a prime $p \in \pi(G)$, by G_p, we mean a Sylow p-subgroup of G. The other notations and terminologies used are standard (see [5,7]).

The p-length of a p-soluble group is an important invariant parameter. As is well-known, the celebrated Hall–Higman theorem has established basic theorem on the p-length of a p-soluble group G, showing that the p-length of G is bounded above by $c(G_p)$, the nilpotent class of G_p [6]. Many scholars have also investigated on this invariant parameter. For instance, the readers can refer to [3–6].

The product of two p-soluble subgroups need not be p-soluble, in general. However, Ballester-Bolinches *et al*. in [1] showed that if G is a mutually permutable product of two p-soluble subgroups, then G is still a p-soluble group. The so-called group $G = AB$ is a mutually permutable product of A and B if $AC = CA$ for any subgroup C of B and $BD = DB$ for any subgroup D of A [1].

Recall that the Wielandt subgroup $\omega(G)$ of a group G is the intersection of the normalisers of all subnormal subgroups of G. The Wielandt series of G,

$$\omega_0(G), \omega_1(G), \ldots, \omega_n(G) = G$$

are defined inductively by setting $\omega_0(G) = 1$ and $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$. The smallest positive integer n such that $\omega_n(G) = G$ is called the Wielandt length of G and is denoted by $wl(G)$ [9].

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We continue to study the p-length of a p-soluble group. We focus our attention on the product of two mutually permutable p-soluble subgroups. By using the Wielandt lengths of Sylow p-subgroups of the factors, we obtain a new bound of the p-length of the given group.

Theorem 1.1. *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. Then $l_p(G) \leq \max\{wl(A_p), wl(B_p)\}$.*

A group G is called a Hamiltonian group if every subgroup of G is normal in G.

COROLLARY 1.2

Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. If both A_p and B_p are Hamiltonian groups, then $l_p(G) \leq 1$.

2. Preliminaries

Let G be a p-soluble group. Define the upper p-series of G as follows:

$$1 = P_0(G) \trianglelefteq M_0(G) \trianglelefteq P_1(G) \trianglelefteq M_1(G) \trianglelefteq \cdots \trianglelefteq P_n(G) \trianglelefteq M_n(G) = G,$$

where $M_i(G)/P_i(G) = O_{p'}(G/P_i(G))$, $P_i(G)/M_{i-1}(G) = O_p(G/M_{i-1}(G))$. The integer n is called the p-length of G and is denoted by $l_p(G)$ (refer [7, XI, 6.1]).

Lemma 2.1 ([1, *Theorem* 4.1.15]). *Let the group G be the product of mutually permutable subgroups A and B. If A and B are p-soluble, then G is p-soluble.*

Lemma 2.2 ([1, *Lemma* 4.1.10]). *Let the group G be the product of mutually permutable subgroups A and B. If N is a normal subgroup of G, then G/N is a mutually permutable product of AN/N and BN/N.*

Lemma 2.3 ([7, *VI*, 6.4]). *Let G be a p-soluble group.*

(1) *If $N \trianglelefteq G$, then $l_p(G/N) \leq l_p(G)$.*
(2) *If $U \leq G$, then $l_p(U) \leq l_p(G)$.*
(3) *Let N_1 and N_2 be two normal subgroups of G. Then*

$$l_p(G/(N_1 \cap N_2)) \leq \max\{l_p(G/N_1), l_p(G/N_2)\}.$$

(4) $l_p(G/\Phi(G)) = l_p(G)$.

Lemma 2.4 ([1, *Theorem* 4.3.11]). *Let the non-trivial group G be the product of mutually permutable subgroups A and B. Then $A_G B_G$ is not trivial.*

Lemma 2.5 ([1, *Lemma* 4.3.3]). *Let the group G be the product of mutually permutable subgroups A and B. Then*

(1) *If N is a minimal normal subgroup of G, then $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$.*

(2) *If N is a minimal normal subgroup of G contained in A and $B \cap N = 1$, then $N \leq C_G(A)$ or $N \leq C_G(B)$. If, furthermore, N is not cyclic, then $N \leq C_G(B)$.*

Lemma 2.6 ([1, *Corollary* 4.1.25]). *Let the group G be the product of mutually permutable subgroups A and B. Then A' and B' are subnormal in G.*

Lemma 2.7. *Let P be a p-group, $Q \leq P$ and $N \trianglelefteq P$, where p is a prime. Then*

(1) $wl(P) \leq c(P)$.
(2) $wl(Q) \leq wl(P)$.
(3) $wl(P/N) \leq wl(P)$.

Proof.

(1) It is easy to show that $Z_i(P) \leq \omega_i(P)$ for any integer $i \geq 0$, so $wl(P) \leq c(P)$.
(2) We only need to prove that $\omega_i(P) \cap Q \leq \omega_i(Q)$ holds for any integer $i \geq 0$. Obviously the inequality holds when $i = 0$. Suppose $\omega_{i-1}(P) \cap Q \leq \omega_{i-1}(Q)$ and let $x \in \omega_i(P) \cap Q$. Then, for any $\omega_{i-1}(Q) \leq U \leq Q$, we have $U^x \omega_{i-1}(P) = U \omega_{i-1}(P)$. It follows that

$$U^x \omega_{i-1}(P) \cap Q = U \omega_{i-1}(P) \cap Q,$$

namely,

$$U^x (\omega_{i-1}(P) \cap Q) = U(\omega_{i-1}(P) \cap Q).$$

Note that $\omega_{i-1}(P) \cap Q \leq \omega_{i-1}(Q)$. We have $U^x = U$. So $x \in \omega_i(Q)$ and $\omega_i(P) \cap Q \leq \omega_i(Q)$.
(3) We will show that $\omega_i(P)N/N \leq \omega_i(P/N)$ holds for any integer $i \geq 0$. Clearly, the inequality holds when $i = 0$. Suppose $\omega_{i-1}(P)N/N \leq \omega_{i-1}(P/N)$ and $xN \in \omega_i(P)N/N$, where $x \in \omega_i(P)$. Then $U^x \omega_{i-1}(P) = U \omega_{i-1}(P)$ for any $N \leq U \leq P$. Consequently,

$$(U^x/N)(\omega_{i-1}(P)N/N) = (U/N)(\omega_{i-1}(P)N/N).$$

Since $\omega_{i-1}(P)N/N \leq \omega_{i-1}(P/N) \trianglelefteq P/N$, we have

$$(U^x/N)(\omega_{i-1}(P/N)) = (U/N)(\omega_{i-1}(P/N)).$$

Hence $xN \in \omega_i(P/N)$ and $\omega_i(P)N/N \leq \omega_i(P/N)$. Thus (3) follows. □

3. Proof of Theorem 1.1

Proof. Suppose that Theorem 1.1 is false and let G be a minimal counter-example. Then

(1) *G is p-soluble*. This follows from Lemma 2.1.

(2) *G has a unique minimal normal subgroup N and G/N satisfies the hypotheses of Theorem* 1.1. In particular, $O_{p'}(G) = \Phi(G) = 1$. Suppose that N is a minimal normal subgroup of G. Consider $\bar{G} = G/N$ together with $\bar{A} = AN/N$ and $\bar{B} = BN/N$. Clearly,

$\bar{A}_p = A_pN/N \in \mathrm{Syl}_p(\bar{A})$, $\bar{B}_p = B_pN/N \in \mathrm{Syl}_p(\bar{B})$. By Lemma 2.2, $\bar{G}$ is mutually the product of two p-soluble subgroups $\bar{A}$ and $\bar{B}$, hence $\bar{G}$ satisfies the hypotheses of Theorem 1.1. The choice of G and Lemma 2.7 imply that

$$l_p(\bar{G}) \leq \max\{wl(\bar{A}_p), wl(\bar{B}_p)\} \leq \max\{wl(A_p), wl(B_p)\}.$$

If L is minimal normal in G with $L \neq N$, then we also have

$$l_p(G/L) \leq \max\{wl(A_p), wl(B_p)\}.$$

By Lemma 2.3,

$$l_p(G) \leq \max\{l_p(G/N), l_p(G/L)\} \leq \max\{wl(A_p), wl(B_p)\},$$

a contradiction. Thereby N is the unique minimal normal subgroup of G. Moreover, if $N \leq O_{p'}(G)$ or $N \leq \Phi(G)$, then

$$l_p(G) = l_p(\bar{G}) \leq \max\{wl(A_p), wl(B_p)\},$$

also a contradiction. Thus $O_{p'}(G) = \Phi(G) = 1$.

(3) *$N = O_p(G) = C_G(N)$ and G has a complement M to N.* By (1) and (2), we have $O_p(G) \neq 1$, hence $N \leq O_p(G)$. Now that $\Phi(G) = 1$, G has a maximal subgroup M such that $G = NM$ and $N \cap M = 1$. It follows that $N = O_p(G)$ and $N = C_G(N)$, by [5].

(4) *Neither A_p nor B_p is a Sylow p-subgroup of G.* If not, we may assume $A_p \in \mathrm{Syl}_p(G)$. Since $G = NM$, $A_p = N(A_p \cap M)$. Let $1 \neq a \in N$. Then a is of order p. By the definition of Wielandt subgroup, $\omega_1(A_p) \cap M$ acts on $\langle a \rangle$, by conjugation. Since $C_{\langle a \rangle}(\omega_1(A_p) \cap M) \neq 1$, we have $C_{\langle a \rangle}(\omega_1(A_p) \cap M) = \langle a \rangle$. Consequently, $\omega_1(A_p) \cap M \leq C_G(a)$. Since a is arbitrary in N, we have $\omega_1(A_p) \cap M \leq C_G(N) = N$ and so $\omega_1(A_p) \cap M = 1$. Furthermore,

$$A_p/N \cong A_p \cap M \cong (A_p \cap M)\omega_1(A_p)/\omega_1(A_p)$$

and

$$wl(A_p/N) = wl((A_p \cap M)\omega_1(A_p)/\omega_1(A_p)).$$

Now that G/N satisfies the hypotheses of Theorem 1.1, we obtain

$$l_p(G/N) \leq \max\{wl(A_p/N), wl(B_pN/N)\}.$$

Without loss of generality, we may assume $B_p \leq A_p$. By Lemma 2.7, we have

$$\max\{wl(A_p/N), wl(B_pN/N)\} = wl(A_p/N),$$

hence

$$l_p(G/N) \leq wl((A_p \cap M)\omega_1(A_p)/\omega_1(A_p)) \leq wl(A_p/\omega_1(A_p)) \leq wl(A_p) - 1.$$

Thus $l_p(G) \leq wl(A_p) = \max\{wl(A_p), wl(B_p)\}$, contradicting to the choice of G.

(5) $N \leq A \cap B$. By Lemma 2.4, $A_G B_G \neq 1$, so we may assume $N \leq A$, by (1). If $N \not\leq B$, then $N \cap B = 1$, by Lemma 2.5(1). If N is cyclic, then $N = C_G(N) \in \mathrm{Syl}_p(G)$, which is contrary to (4). Hence N is not cyclic and $N \leq C_G(B)$ by Lemma 2.5(2). Furthermore, $B \leq C_G(N) = N \leq A$ and so $G = AB = A$. Of course, $A_p \in \mathrm{Syl}_p(G)$, also a contradiction. Thereby $N \leq A \cap B$.

(6) *The final contradiction*. Denote $\bar{G} = G/N$, $\bar{A} = A/N$ and $\bar{B} = B/N$. Now that $\bar{G}$ satisfies the hypotheses of Theorem 1.1, we have $l_p(\bar{G}) \leq \max\{wl(\bar{A}_p), wl(\bar{B}_p)\}$ by the minimality of G. Clearly, $A_p = N(A_p \cap M)$. Choose $1 \neq a \in N$. Then a is of order p. By the definition of Wielandt subgroup, $\omega_1(A_p) \cap M$ acts on $\langle a \rangle$ by conjugation. Since $C_{\langle a \rangle}(\omega_1(A_p) \cap M) \neq 1$, $C_{\langle a \rangle}(\omega_1(A_p) \cap M) = \langle a \rangle$. Furthermore, $\omega_1(A_p) \cap M \leq C_G(a)$. Since a is arbitrary in N, $\omega_1(A_p) \cap M \leq C_G(N) = N$ and hence $\omega_1(A_p) \cap M = 1$. It follows that

$$\bar{A}_p \cong A_p \cap M \cong (A_p \cap M)\omega_1(A_p)/\omega_1(A_p)$$

and

$$wl(\bar{A}_p) = wl((A_p \cap M)\omega_1(A_p)/\omega_1(A_p)) \leq wl(A_p/\omega_1(A_p)) \leq wl(A_p) - 1.$$

Similarly, $wl(\bar{B}_p) \leq wl(B_p) - 1$. Thus

$$l_p(\bar{G}) \leq \max\{wl(\bar{A}_p), wl(\bar{B}_p)\} \leq \max\{wl(A_p), wl(B_p)\} - 1$$

and $l_p(G) \leq \max\{wl(A_p), wl(B_p)\}$. This is the final contradiction and the proof is complete. □

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References

[1] Ballester-Bolinches A, Esteban-Romero R and Asaad M, Products of Finite Groups (2010) (Berlin-New York: Walter de Gruyter)
[2] Cossey J and Li Y, On the structure of a mutually permutable product of finite groups, *Acta Math. Hungar.* **154(2)** (2018) 525–529
[3] Cossey J and Li Y, On the p-length of the mutually permutable product of two p-soluble groups, *Arch. Math.* **110** (2018) 533–537
[4] Felipe M J, Martínez-Paster A and Ortiz-Sotomayor V M, Prime power indices in factorised groups, *Mediterr. J. Math.* **14(6)** (2017) article 225
[5] Gorenstein D, Finite Groups (1968) (New York: Chelsea Pub. Co.)
[6] Hall P and Higman G, On the p-length of p-soluble groups and reduction theorems for Burnside's problem, *Proc. London Math. Soc.* **3(1)** (1956) 1–42
[7] Huppert B, Endiche Gruppen I (1967) (New York: Springer-Verlag)
[8] Qian G and Wang Y, On the conjugate class sizes and character degrees of finite groups, *J. Algebra Appl.* **13** (2014) 1–8
[9] Wielandt H, Über den normalisator der subnormalen untergruppen, *Math. Z.* **69(1)** (1958) 463–465

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