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On Betti numbers in the linear strand and regularity of triangular graphs

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Abstract. To every finite simple graph G, we associate the so-called edge ideal $I(G)$, which is a square-free quadratic monomial ideal generated by the monomials corresponding to the edges of G. In this paper, we determine all the initial graded Betti numbers of edge ideals of triangular graphs and alternate triangular graphs in terms of the underlying graph. We also compute the regularity of edge ideals of triangular and alternate triangular graphs.

Keywords. Edge ideals; Betti numbers; regularity.

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1. Introduction

Throughout this paper, let $R = \Bbbk[x_1, x_2, \ldots, x_n]$ be a homogeneous polynomial ring in variables $x_1, x_2, \ldots, x_n$ with $\deg x_i = 1$ over a field $\Bbbk$. Let $G = (V, E)$ be a finite simple graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and edge set $E(G) \subseteq V(G) \times V(G)$. Then the quadratic square-free monomial ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)) \subseteq R$$

associated to G is known as the *edge ideal* of G. The assignment $G \mapsto I(G)$ is a natural one-to-one correspondence between the family of simple graphs and the family of square-free quadratic monomial ideals. The edge ideals were first studied by Villarreal [17]. The edge ideals were studied to find the relation between homological invariants of edge ideals and invariants associated to graphs; see [6,8,13,17–19] and references therein.

Like any homogeneous ideal in polynomial ring R, the edge ideal $I(G)$ in R corresponding to the graph G admits a minimal graded free resolution (see [3])

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$$0 \to \bigoplus_{j=1}^{s_p} R(-j)^{\beta_{p,j}} \to \cdots \to \bigoplus_{j=1}^{s_i} R(-j)^{\beta_{i,j}} \to \cdots \to$$
$$\bigoplus_{j=1}^{s_1} R(-j)^{\beta_{1,j}} \to R \to R/I(G) \to 0$$

of $R/I(G)$ over R, where $R(-j)$ is the graded free R-module of rank 1 whose d-th graded component is given by $R(-j)_d = R_{d-j}$, the $(d-j)$-th graded component of R. The number of generators of the i-th syzygy module in degree j is $\beta_{i,j}$ and is called i-th *graded Betti number* of $R/I(G)$ in degree j. For the sake of brevity, we write $\beta_{i,j}(G)$ for $\beta_{i,j}(R/I(G))$. The graded Betti numbers of the form $\beta_{i,i+1}$ appear in the first row of the Betti diagram of the minimal $\mathbb{N}$-graded free resolution of $I(G)$ and this row of Betti numbers is known as the *linear strand* (see [3,16] and references therein). The graded betti numbers $\beta_{i,i+1}$ which appear in the first non-zero row of the Betti diagram are also called as the *initial graded Betti numbers* (see [15]).

The *Castelnuovo–Mumford regularity* of $R/I(G)$, denoted by $\mathrm{reg}(R/I(G))$ (in short, we write $\mathrm{reg}(G)$ for $\mathrm{reg}(R/I(G))$, is defined by

$$\mathrm{reg}(G) = \max\{j - i \mid \beta_{i,j}(G) \neq 0\}.$$

One can see from the minimal graded free resolution that $\mathrm{reg}(I(G)) = \mathrm{reg}(R/I(G)) + 1$. The regularity and projective dimension of a forest (which is a graph with no cycles) was first characterized by Zheng [19]. Later Hà and Van Tuyl [5] extended this characterization of regularity to that for chordal graphs. Jacques [8] has computed these invariants for some well known graphs such as lines, cycles and bipartite graphs. Regularity is an important homological invariant in commutative algebra which, roughly speaking, measures the width of the minimal graded free resolution of modules. The combinatorial characterization of graphs with regularity 2 is often referred to as *Fröberg's characterization*. Fröberg [4, Theorem 1] proved that if G is a finite simple graph, then reg $(I(G)) = 2$ if and only if G is a co-chordal graph. However, graphs whose edge ideals have regularity 3 are yet to be characterized. It has been proved by Hà and Van Tuyl [5] that the regularity of edge ideals of chordal graphs is one more than the induced matching number of the underlying graph. Rather and Singh [15] computed the initial graded Betti numbers of edge ideals of crown graphs. Katzman [10] proved some results on non-vanishing graded Betti numbers. For more on regularity and its connections to other algebraic or geometric invariants, we refer to [3]. Betti numbers, regularity and other homological invariants of edge ideals have been widely studied (see [1,5,8–12,19] and references therein).

The triangular graphs constitute an interesting class of chordal graphs. Many combinatorial properties of triangular graphs are obtained in [14]. In this paper, we study the homological invariants of edge ideals of these graphs. We compute the initial graded Betti numbers of triangular and alternate triangular graphs using Hochster's formula and compute their regularity using the notion of induced matching number and Theorem 2.1. The computation of other graded Betti numbers of triangular graphs involves the computation of higher reduced homologies of the independent complex and that is not so easy to compute.

We will now describe the organization of the paper. In Section 2, we recall some definitions and introduce some notations that will be used in subsequent sections. We also recall a well-known result by Hochster [7] in this section which is a main tool for computing Betti numbers of Stanley–Reisner rings. In Section 3, we introduce triangular and alternate triangular graphs and compute the initial graded Betti numbers of their edge ideals (see Theorem 3.2 and Theorem 3.9). We also compute the regularity of edge ideals of the triangular and the alternate triangular graphs; see Theorem 3.6 and Theorem 3.15.

2. Preliminaries

We begin by collecting some definitions on graphs and some known results about the Betti numbers of edge ideals. We also fix some notations and terminology for their use in forthcoming sections.

2.1 *Graphs and independent complexes*

Consider a finite simple graph $G = (V, E)$. We say that a subgraph G' of G is an *induced subgraph* on a subset V' of $V(G)$ if $V(G') = V'$ and $E(G') = \{\{u, v\} \in E(G) \mid u, v \in V')\}$. We write $G' = G[V']$ when G' is an induced subgraph on V'. The *complement* of G, denoted by G^c, is the graph on the same vertex set as of G with edge set $E(G^c) = \{\{u, v\} \mid u, v \in V(G), \{u, v\} \notin E(G)\}$. The number of edges incident to a vertex $v \in V(G)$ in G is called the *degree* of v. We call a connected graph G a *t-cycle* ($|V(G)| = t \geq 3$), if all vertices of G are of degree 2. A t-cycle is denoted by C_t, where t is called the length of the cycle. A graph G on m vertices is called a *complete graph*, denoted by $\mathcal{K}_m$, if any two vertices in $V(G)$ are adjacent (i.e. connected by an edge). Given a graph G, a subset $W \subset V(G)$ with $|W| = m$ is called a *clique* of size m if $G[W]$ is the complete graph $\mathcal{K}_m$. A subset M of vertices of G is said to be *independent* if no two vertices of M are adjacent in G. For $n \geq 2$, we define a *path* P_n of length n as a graph on n-vertices $\{x_1, x_2, \ldots, x_n\}$ and edges $\{x_i, x_{i+1}\}$ for all $i = 1, 2, \ldots, n-1$. We call a vertex an *ear* if it is adjacent to precisely two adjacent vertices in a graph. Clearly, an ear has degree two.

A graph G is said to be *chordal* if G has no induced cycles of length greater than three. However, if every induced cycle in both G and G^c has length at most 4, then G is called *weakly chordal*. We call G a *co-chordal* graph if the complement of G is chordal.

Let G be a graph in which all vertices have degree one. Then G must consist of $2m$ vertices and m disjoint edges for some $m \geq 1$. We denote such a graph by $m\mathcal{K}_2$.

Two distinct edges $e = \{x_k, x_\ell\}$ and $e' = \{x_r, x_s\}$ in a graph G are said to form a *gap* if $\{x_k, x_\ell\} \cap \{x_r, x_s\} = \emptyset$ and the induced subgraph $G[\{x_k, x_\ell, x_r, x_s\}]$ on the vertex set $\{x_k, x_\ell, x_r, x_s\} \subset V(G)$ consists of only two edges e and e'. Sometimes we call two edges of a graph G *3-disjoint* in G if they form a gap. A subset $\{e_1, e_2, \ldots, e_m\} \subset E(G)$ of edges is said to be *pairwise 3-disjoint* subset, if e_i and e_j, $1 \leq i \neq j \leq m$ are 3-disjoint.

DEFINITION 1

The *induced matching number* of a graph G is the maximal m such that $m\mathcal{K}_2$ is an induced subgraph of G. We denote this number by $\mu(G)$.

The following result was first proved for forests in [19, Theorem .18] and later extended to chordal graphs in [5, Theorem 6.8].

Theorem 2.1. *Let G be a chordal graph and $\mu(G)$ be the maximum size of induced matching in G. Then*

$$\operatorname{reg}(G) = \mu(G).$$

DEFINITION 2

A *simplicial complex* Δ on vertex set $V = \{x_1, x_2, \ldots, x_n\}$ is a collection of subsets of V, called *faces*, such that $\{x_i\} \in \Delta$ for each $i = 1, 2, \ldots, n$ and if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

A face $F \in \Delta$ of cardinality $|F| = i + 1$ is called a *face of dimension i* or an *i-face* of Δ. If Δ is a simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_n\}$ and every subset of V is in Δ, then Δ is called a simplex. The *dimension* of Δ, denoted by $\dim(\Delta)$, is the maximum of dimension of all its faces. Let $W \subseteq V$. A subcomplex of Δ on W is said to be an *induced subcomplex*, denoted by $\Delta[W]$, if

$$\Delta[W] = \{F \in \Delta \mid F \subseteq W\}.$$

The *i-skeleton* of the simplicial complex Δ, denoted by $\Delta^{(i)}$, is the simplicial complex given by

$$\Delta^{(i)} = \{F \in \Delta \mid \dim(F) \leq i\}.$$

One can see that the 1-skeleton of a simplicial complex Δ is a simple graph provided $V(\Delta) \neq \emptyset$. The connectivity in Δ is defined as the connectivity in $\Delta^{(1)}$ as a graph: two vertices in Δ are connected if and only if they are connected in $\Delta^{(1)}$. We denote by $\operatorname{comp}(\Delta)$, the connected components of Δ.

DEFINITION 3

Let Δ be a simplicial complex on the vertex set $\{x_1, x_2, \ldots, x_n\}$. The *Stanley–Reisner ideal* of Δ is a squarefree monomial ideal I_Δ of $R = \Bbbk[x_1, x_2, \ldots, x_n]$ generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ such that $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ is a non-face of Δ. The quotient ring $\Bbbk[\Delta] = R/I_\Delta$ is called the *Stanley–Reisner* ring of the simplicial complex Δ.

To every finite simple graph G on the vertex set $\{x_1, x_2, \ldots, x_n\}$, one can associate two simplicial complexes given as follows:

(i) The *independent complex* $\Delta(G)$, whose faces are the independent subsets of the vertex set of G, that is,

$$\Delta(G) = \{\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \subset V(G) \mid \text{no } \{x_{i_j}, x_{i_k}\} \text{ is an edge of } G\}.$$

(ii) The *clique complex* Δ_G, whose faces are the cliques of G.

Remark 2.2. One can see that the edge ideal $I(G)$ of G is same as the Stanley–Reisner ideal $I_{\Delta(G)}$ of the independent complex $\Delta(G)$.

We shall make use of the following theorem, known as Hochster's formula, as a main tool for computing the graded Betti numbers of the Stanley–Reisner ring $\Bbbk[\Delta]$.

Theorem 2.3 [7, Hochster's formula]. *The i-th graded Betti number $\beta_{i,j}(\Bbbk[\Delta])$ of the Stanley–Reisner ring $\Bbbk[\Delta] = R/I_\Delta$ in degree j is given by*

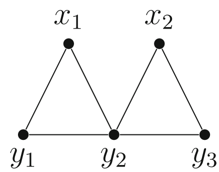

Figure 1. T_2.

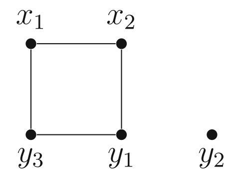

Figure 2. $\Delta(T_2)$.

$$\beta_{i,j}(\Bbbk[\Delta]) = \sum_{W \subseteq V, |W|=j} \dim_{\Bbbk} \tilde{H}_{j-i-1}(\Delta[W]; \Bbbk). \tag{2.1}$$

PROPOSITION 2.4 [8]

Given a finite simple graph G on vertex set $V(G)$, for any subset W of $V(G)$ such that $G[W]$ is a path of length 3, *we have*

$$\dim_{\Bbbk} \tilde{H}_i(\Delta[W]; \Bbbk) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

3. Triangular and alternate triangular graphs

3.1 *Betti numbers and regularity of triangular graphs*

DEFINITION 4

A *triangular graph* T_n (also called a *triangular snake graph*) on $2n + 1$ vertices, $n \geq 1$, is a simple connected graph with vertex set $V(T_n) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n+1}\}$ and edge set $E(T_n) = \{\{x_i, y_i\}, \{x_i, y_{i+1}\}, \{y_i, y_{i+1}\} \mid 1 \leq i \leq n\}$.

A triangular graph can also be viewed as a graph obtained from a path $P_{n+1} = y_1 y_2 \cdots y_{n+1}$ of length $n + 1$ by adding an ear x_i to each edge $\{y_i, y_{i+1}\}$, where $1 \leq i \leq n$. A triangular graph has $2n + 1$ vertices and $3n$ edges. Figure 1 shows the triangular graph T_2. The corresponding simplicial complex $\Delta(T_2)$ of the triangular graph T_2 is pictorially given in Figure 2.

Lemma 3.1. *Let T_n be a triangular graph and $W \subseteq V(T_n)$ with $|W| \geq 6$. Then $\Delta[W]$ is connected, where $\Delta = \Delta(T_n)$.*

Proof. We shall prove this, by contradiction. Suppose that there exists a subset $W \subset V(T_n)$ with $|W| \geq 6$ such that $\Delta[W]$ has more than one connected components. One can see that $\Delta[W]$ can have at most three connected components, otherwise T_n will contain a clique of size more than three, which is not the case. Also, $\Delta[W]$ cannot have two or three connected components, otherwise there is either a vertex in $V(T_n)$ having degree more than four or a cycle of length greater than three in T_n, which is not possible in either case. Thus $\Delta[W]$ is connected, for any subset W of $V(T_n)$ with $|W| \geq 6$.

Theorem 3.2. *Let T_n be a triangular graph. Then the initial graded Betti numbers $\beta_{i,i+1}(T_n)$ are given by*

$$\beta_{i,i+1}(T_n) = \begin{cases} 3n, & \text{if } i = 1, \\ 6n-4, & \text{if } i = 2, \\ 4n-4, & \text{if } i = 3, \\ n-1, & \text{if } i = 4, \\ 0, & \text{if } i \geq 5, \end{cases}$$

Proof. By Hochster's formula, we have

$$\beta_{i,i+1}(T_n) = \sum_{W \subseteq V,\ |W|=i+1} \dim_{\mathbb{k}} \tilde{H}_0(\Delta[W]; \mathbb{k}),$$

where $\Delta = \Delta(T_n)$. We know that $\dim_{\mathbb{k}} \tilde{H}_0(\Delta; \mathbb{k})$ is one less than the number of connected components of Δ. Therefore, $\beta_{i,i+1}(T_n)$ is non-zero if the number of connected components of $\Delta[W]$ is greater than one. Also, we have seen that $\Delta[W]$ has at most three connected components for any subset W of $V(T_n)$. We have the following cases:

Case 1. *For* $i = 1$, since $\beta_{1,2}(T_n)$ gives the number of edges in T_n, we see that $\beta_{1,2}(T_n) = 3n$.

Case 2. For $i = 2$, we count all the subsets $W \subseteq V(T_n)$ $(|W| = 3)$, such that $\Delta[W]$ consists of more than one connected component. One can see that $\Delta[W]$, where $W \subseteq V(T_n)$ is of the form $\{x_k, y_k, y_{k+1}\}$ $(1 \leq k \leq n)$, consists of exactly three connected components and hence contribute 2 to $\beta_{2,3}(T_n)$. There are exactly n such subsets W of $V(T_n)$, hence making a total contribution of $2n$ to $\beta_{2,3}(T_n)$. Also, the subsets $W \subseteq V(T_n)$ $(|W| = 3)$ such that $\Delta[W]$ consists of exactly two connected components are those subsets of $V(T_n)$ corresponding to which $T_n[W]$ is a path of length 3. The subsets of the form $\{y_k, y_{k+1}, y_{k+2}\}$, $\{x_{k+1}, y_k, y_{k+1}\}$, $\{x_k, y_{k+1}, y_{k+2}\}$ and $\{x_k, y_{k+1}, x_{k+1}\}$ $(1 \leq k \leq n-1)$ are the only subsets of $V(T_n)$ on which the induced subgraph is a path of length 3. Each of these subsets are $n-1$ in number. Using Proposition 2.4, the total contribution to $\beta_{2,3}(T_n)$ by these subsets is $4(n-1)$. Therefore, $\beta_{2,3}(T_n) = 2n + 4(n-1) = 6n - 4$.

Case 3. For $i = 3$, we are interested in all those subsets $W \subset V(T_n)$ $(|W| = 4)$ such that $\Delta[W]$ consists of more than one connected components. One can easily verify that $\Delta[W]$ has non-zero 0-th reduced homology (i.e. $\Delta[W]$ has more than one connected components) if and only if W is one of the following four forms: $\{x_k, y_k, y_{k+1}, y_{k+2}\}$, $\{x_{k+1}, y_{k+2}, y_{k+1}, y_k\}$, $\{x_k, y_k, y_{k+1}, x_{k+1}\}$ and $\{x_{k+1}, y_{k+2}, y_{k+1}, x_k\}$, where $1 \leq k \leq n-1$. There are exactly $n-1$ subsets of $V(T_n)$ of each of the above four forms of subsets.

If $W \subset V(T_n)$ is one of the above forms of subsets, then $G[W]$ is isomorphic to the graph G given in Figure 3. In that case, $\Delta[W]$ will consist two connected components (a path of length 3 and a vertex) and hence contribute 1 to $\beta_{3,4}(T_n)$.

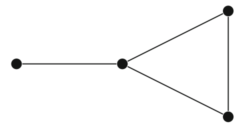

Figure 3. G.

Therefore, $\beta_{3,4}(T_n)$ is the total number of such subsets. That is, $\beta_{3,4}(T_n) = 4(n-1)$.

Case 4. For $i = 4$, $\Delta[W]$ (where $W \subseteq V(T_n)$ and $|W| = 5$) consists of two connected components (i.e. has non-zero 0-th reduced homology) if and only if W is of the form $\{x_k, x_{k+1}, y_k, y_{k+1}, y_{k+2}\}$ $(1 \leq k \leq n-1)$. In this case, $G[W]$ will be isomorphic to the graph given in Figure 1. Because there are $n-1$ such subsets, $\beta_{4,5}(T_n) = n-1$.

Case 5. For $i \geq 5$, $\Delta[W]$ (where $W \subseteq V(T_n)$ and $|W| \geq 6$) is connected by Lemma 3.1. Thus $\dim_{\Bbbk} \tilde{H}_i(\Delta[W]; \Bbbk) = 0$ for all $i \geq 5$, so $\beta_{i,i+1}(T_n) = 0$ for $i \geq 5$. □

Theorem 3.3 **[10]**. *Let G be a finite simple graph on $V(G)$. Then the graded Betti number $\beta_{i,2i}(G)$ is same as the number of subsets W of $V(G)$ for which $G[W]$ consists of i pairwise 3-disjoint edges.*

Theorem 3.4. *Let T_n be a triangular graph. Then $\beta_{2,4}(T_n) = \frac{9n^2-33n+32}{2}$.*

Proof. By Theorem 3.3, we see that $\beta_{2,4}(T_n)$ is same as the number of distinct pairwise 3-disjoint subsets Y of $E(T_n)$, where Y consists of exactly two edges. Thus we shall count all subsets W of $V(T_n)$ such that the induced subgraph $T_n[W]$ consists of two disjoint edges which do form a gap, i.e., which are 3-disjoint. One can see that if $W \subseteq V(T_n)$ is of the form $\{x_r, y_r, x_s, y_{s+1}\}$, where $1 \leq r \leq n-1$ and $r+1 \leq s \leq n$, then $T_n[W]$ corresponds to a pairwise 3-disjoint subset of $E(T_n)$ that consists of exactly two edges. The number of such subsets of $V(T_n)$ are equal to $\sum_{i=1}^{n-1} i$. Also, the induced subgraph $T_n[W]$, where W is of the form $\{x_r, y_r, x_s, y_s\}$, $\{x_r, y_r, y_s, y_{s+1}\}$, $\{x_r, y_{r+1}, x_s, y_{s+1}\}$ or $\{y_r, y_{r+1}, x_s, y_{s+1}\}$ $(1 \leq r \leq n-2$ and $r+2 \leq s \leq n)$ corresponds to a pairwise 3-disjoint subset of $E(T_n)$ that consists of exactly two edges. The total number of subsets of $V(T_n)$ of each such form is $\frac{n^2-3n+2}{2}$ and hence making a total contribution of $2(n^2-3n+2)$ to $\beta_{2,4}(T_n)$. Furthermore, if $W \subset V(T_n)$ is of the form $\{x_r, y_{r+1}, x_s, y_s\}$, $\{x_r, y_{r+1}, y_s, y_{s+1}\}$, $\{y_r, y_{r+1}, x_s, y_s\}$ or $\{y_r, y_{r+1}, y_s, y_{s+1}\}$, where $1 \leq r \leq n-3$ and $r+3 \leq s \leq n$, then again $T_n[W]$ corresponds to a pairwise 3-disjoint subset of $E(T_n)$ that consists of exactly two edges. The total number of subsets of $V(T_n)$ of each such form is $\frac{n^2-5n+6}{2}$ and hence making a total contribution of $2(n^2-5n+6)$ to $\beta_{2,4}(T_n)$. Therefore,

$$\beta_{2,4}(T_n) = \sum_{i=1}^{n-1} i + 2(n^2-3n+2) + 2(n^2-5n+6) = \frac{9n^2-33n+32}{2}.$$

□

Example 3.5. Consider the triangular graph T_3 on vertex set $V(T_3) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$. Then for the edge ideal $I(T_3) = \{x_1y_1, x_1y_2, x_2y_2, x_2y_3, x_3y_3, x_3y_4, y_1y_2, y_2y_3, y_3y_4\}$, the minimal graded free resolution of $R/I(T_3)$ is given by

$$0 \to R[-7]^2 \to \begin{array}{c} R[-6]^9 \\ \oplus \\ R[-5]^2 \end{array} \longrightarrow \begin{array}{c} R[-5]^{14} \\ \oplus \\ R[-4]^8 \end{array} \longrightarrow \begin{array}{c} R[-4]^7 \\ \oplus \\ R[-3]^{14} \end{array} \to R[-2]^9 \to R \to 0.$$

Using Theorem 3.2, the Betti numbers $\beta_{i,i+1}(T_3),\ i = 1, 2, 3, 4$ are given by

$$\begin{aligned} &\beta_{1,2}(T_3) = 3.3 = 9, \\ &\beta_{2,3}(T_3) = 6.3 - 4 = 14, \\ &\beta_{3,4}(T_3) = 4.3 - 4 = 8, \\ &\beta_{4,5}(T_3) = 3 - 1 = 2. \end{aligned}$$

One can verify that the Betti numbers $\beta_{i,i+1}(T_3),\ i = 1, 2, 3, 4$ are same as computed below in the Betti table of the minimal graded free resolution of $R/I(T_3)$ with the help of Singular 2.0 [2].

```
0   1   2   3   4   5
0: 1   -   -   -   -   -
1: -   9  14   8   2   -
2: -   -   7  14   9   2
total: 1   9  21  22  11   2
```

which is read as in Figure 4.

Theorem 3.6. *Let T_n be a triangular graph. Then the regularity of T_n is given by*

$$\operatorname{reg}(T_n) = \begin{cases} \frac{n+1}{2} & \textit{if } \ n \textit{ is odd}, \\ \frac{n+2}{2} & \textit{if } \ n \textit{ is even}. \end{cases}$$

```
          0  1  ···  i  ···
total:--------------
     0:
     ⋮           ⋮
     j:  ···  β_{i,i+j}(T_3)  ···
     ⋮           ⋮
```

Figure 4. The Betti diagram of T_3.

Proof. Let $V(T_n) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n+1}\}$ be the vertex set of the triangular graph T_n. Since T_n is chordal, then by Theorem 2.1, the regularity of T_n is same as the induced matching number of T_n. We have the following two cases:

Case 1. If n is odd, then one can see that the subset $E_1 = \{\{x_{2j-1}, y_{2j-1}\} \mid 1 \leq j \leq \frac{n+1}{2}\}$ of the edge set $E(T_n)$ is a pairwise 3-disjoint subset. We claim that E_1 is the maximal pairwise 3-disjoint subset of $E(T_n)$. Suppose not, then there exists an edge $e = \{x_\ell, y_m\}$, where ℓ, m are not both odd, (or $e_j = \{y_j, y_{j+1}\},\ 1 \leq j \leq n$) in $E(T_n)$ such that $E_1 \cup e$ (or $E_1 \cup e_j$) is a maximal pairwise 3-disjoint subset of $E(T_n)$. Suppose ℓ is odd (or j is odd), then the edges e and $\{x_\ell, y_\ell\} \in E_1$ (or the edges e_j and $\{x_j, y_j\} \in E_1$) are adjacent and hence not maximal 3-disjoint subset. On the other hand, if ℓ, m are both even, then the edges e and $\{x_{\ell+1}, y_{m+1}\} \in E_1$ are again adjacent and hence not pairwise 3-disjoint. Therefore, we conclude that E_1 is a maximal pairwise 3-disjoint subset of $E(T_n)$. The result is analogous when we assume m (or $j+1$) is odd. Since any triangle in T_n can contribute at most one edge to any 3-disjoint subset of $E(T_n)$ and any three consecutive triangles of T_n can contribute at most two edges to any 3-disjoint subset of $E(T_n)$, one can determine that E_1 has size atleast that of all other pairwise 3-disjoint subsets of $E(T_n)$. This implies that $\text{reg}(T_n) = \frac{n+1}{2}$, since $|E_1| = \frac{n+1}{2}$.

Case 2. If n is even, then the subset $E_2 = \{\{x_{2j-1}, y_{2j-1}\} \mid 1 \leq j \leq \frac{n}{2}\} \cup \{x_n, y_{n+1}\}$ of the edge set $E(T_n)$ is a pairwise 3-disjoint subset. We shall show that E_2 is the maximal pairwise 3-disjoint subset of $E(T_n)$. Suppose not, then there exists an edge $e = \{x_\ell, y_m\}$ (or $e_j = \{y_j, y_{j+1}\},\ 1 \leq j \leq n$) that do not belong to E_2 such that $E_2 \cup e$ (or $E_2 \cup e_j$) is a 3-disjoint subset of $E(T_n)$. Suppose ℓ is odd (or j is odd), one can conclude that E_2 is a maximal pairwise 3-disjoint subset of $E(T_n)$, since the edges e and $\{x_\ell, y_\ell\} \in E_2$ (or the edges e_j and $\{x_j, y_j\} \in E_2$) are adjacent and hence not pairwise 3-disjoint subset. Further, if ℓ, m are both even, then the edges e and $\{x_{\ell+1}, y_{m+1}\} \in E_2$ are again adjacent. Same result can be obtained when we assume m (or $j+1$) is odd. By the same argument as in Case 1, one can see that E_2 has size atleast that of all other pairwise 3-disjoint subsets of $E(T_n)$. Thus, $\text{reg}(T_n) = \frac{n+2}{2}$, since $|E_2| = \frac{n+2}{2}$. □

3.2 *Betti numbers of alternate triangular graphs*

DEFINITION 5

An *alternate triangular graph* (also called an *alternate triangular snake graph*) A_n is a simple connected graph which is obtained from a path $P_n = y_1 y_2 \cdots y_n$ by alternatively adding an ear x_i to the edge $\{y_i, y_{i+1}\}$.

An alternate triangular graph is one of following three types depending on the length of the path P_n and the choice of adding an ear to the alternate edges of the path P_n.

Case 1. If the length n of the path P_n is even, then we have the following two types of alternate triangular graphs:

Type 1. If we add an ear alternatively to the path P_n from the first edge of P_n, then the alternate triangular graph so obtained is given in Figure 5.

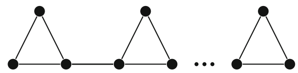

Figure 5. $A_n^{'}$.

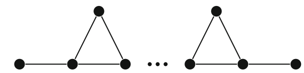

Figure 6. $A_n^{''}$.

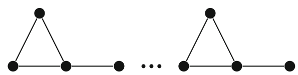

Figure 7. $A_n^{'''}$.

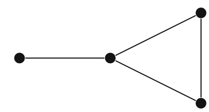

Figure 8. G.

We denote an alternate triangular graph of Type 1 by A'_n. The number of vertices and edges in the alternate triangular graph A'_n are $\frac{3n}{2}$ and $2n - 1$ respectively.

Type 2. If we add an ear alternatively to the path P_n from the second edge, the alternate triangular graph obtained in this way is given in Figure 6.

We shall denote an alternate triangular graph of Type 2 by A''_n. The number of vertices and edges in the alternate triangular graph A''_n are $\frac{3n-2}{2}$ and $2n - 3$ respectively.

Case 2. If the length n of the path P_n is odd, then only one type of alternate triangular graph is obtained as described below:

Type 3. If we add an ear alternatively to the path P_n from the first or second edge, then the alternate triangular graph so obtained is isomorphic to the graph given in Figure 7.

We shall denote an alternate triangular graph of Type 3 by A'''_n. The number of vertices and edges in the alternate triangular graph A'''_n are $\frac{3n-1}{2}$ and $2n - 2$ respectively.

Remark 3.7. Observe that the alternate triangular graph A''_n can be obtained from the alternate triangular graph A'_{n-2} on the vertex set $\{x_1, \dots, x_{\frac{n}{2}-1}, y_2, \dots, y_{n-1}\}$ by adding two edges $\{y_1, y_2\}$ and $\{y_{n-1}, y_n\}$ to A'_{n-2}. Similarly, A'''_n can be obtained from the alternate triangular graph A'_{n-1} on the vertex set $\{x_1, \dots, x_{\frac{n-1}{2}}, y_1, \dots, y_{n-1}\}$ by adding the edge $\{y_{n-1}, y_n\}$ to A'_{n-1}.

Lemma 3.8. *Let A_n be an alternate triangular graph and $W \subseteq V(A_n)$ with $|W| \geq 5$. Then $\Delta[W]$ is connected*, *where $\Delta = \Delta(A_n)$.*

Proof. Proof by contradiction. Suppose that there exists a subset W of the vertex set of any one of the three types of alternate triangular graphs with $|W| \geq 5$ such that $\Delta[W]$ has more than one connected component. Since an alternate triangular graph does not contain a clique of size more than three as an induced subgraph, it follows that $\Delta[W]$ has at most three connected components. Also, $\Delta[W]$ can not have two or three connected components, otherwise there is either a vertex having degree more than three or a cycle of length greater than three in any of the alternate triangular graphs, which is not possible in either case. Thus, $\Delta[W]$ is connected, for any subset W of vertex set with $|W| \geq 5$. □

Theorem 3.9. *The initial graded Betti numbers $\beta_{i,i+1}$ of the alternate triangular graph A'_n are given by*

$$\beta_{i,i+1}(A'_n) = \begin{cases} 2n-1, & \text{if } i=1, \\ 3n-4, & \text{if } i=2, \\ n-2, & \text{if } i=3, \\ 0, & \text{if } i \geq 4. \end{cases}$$

Proof. Let A'_n be an alternate triangular graph on $\frac{3n}{2}$ vertices, $n \geq 1$, with vertex set $V(A'_n) = \{x_1, \dots, x_{\frac{n}{2}}, y_1, \dots, y_n\}$. Then by Hochster's formula, we have

$$\beta_{i,i+1}(A'_n) = \sum_{W \subseteq V, |W|=i+1} \dim_{\Bbbk} \tilde{H}_0(\Delta[W]; \Bbbk),$$

where $\Delta = \Delta(A'_n)$. We know that $\dim_{\Bbbk} \tilde{H}_0(\Delta; \Bbbk)$ is one less than the number of connected components of Δ. Therefore, $\beta_{i,i+1}(A'_n)$ is non-zero if the number of connected components of $\Delta[W]$ is greater than one. Also, we see that $\Delta[W]$ has at most three connected components for any subset W of $V(A'_n)$. We consider the following cases:

Case 1. For $i = 1$, we have $\beta_{1,2}(A'_n) = 2n - 1$, since $\beta_{1,2}(A'_n)$ is the number of edges in A'_n.

Case 2. For $i = 2$, we count all the subsets $W \subset V(A'_n)$ ($|W| = 3$), such that $\Delta[W]$ consists of more than one connected component. The subsets $W \subseteq V(A'_n)$ of the form $\{x_k, y_{2k-1}, y_{2k}\}$, $1 \leq k \leq \frac{n}{2}$ are the only subsets corresponding to which $\Delta[W]$ consists of exactly three connected components. These subsets contribute 2 to $\beta_{2,3}(A'_n)$. Since the induced subgraph of A'_n on the vertex set $\{x_k, y_{2k-1}, y_{2k}\}$, $1 \leq k \leq \frac{n}{2}$ corresponds to a 3 cycle in A'_n, we conclude that there are exactly $\frac{n}{2}$ such subsets of $V(A'_n)$. Hence, the total contribution of such subsets to $\beta_{2,3}(A'_n)$ is n. Next, we shall count all subsets W of $V(A'_n)$ such that $\Delta[W]$ consists of exactly two connected components. There are $\frac{n}{2} - 1$ subsets of $V(A'_n)$ of the form $\{x_k, y_{2k}, y_{2k+1}\}$ and $\{x_{k+1}, y_{2k+1}, y_{2k}\}$, where $1 \leq k \leq \frac{n}{2} - 1$ and $n - 2$ subsets of the form $\{y_k, y_{k+1}, y_{k+2}\}$, where $1 \leq k \leq n - 2$ corresponding to which the induced subgraph is a path of length 3. Using Proposition 2.4, the total contribution to $\beta_{2,3}(A'_n)$ by these subsets is $2(\frac{n}{2} - 1) + (n - 2) = 2n - 4$. Therefore, $\beta_{2,3}(A'_n) = n + (2n - 4) = 3n - 4$.

Case 3. For $i = 3$, we see that $\Delta[W]$, where $W \subseteq V(A'_n)$ and $|W| = 4$ has non-zero 0-th reduced homology if and only if W is either of the form $\{x_k, y_{2k-1}, y_{2k}, y_{2k+1}\}$ or $\{x_{k+1}, y_{2k+2}, y_{2k+1}, y_{2k}\}$, where $1 \leq k \leq \frac{n}{2} - 1$. In that case, the induced subgraph $A'_n[W]$ on such a subset is isomorphic to the graph G given in Figure 8.
The number of subsets W of $V(A'_n)$ of each of the above mentioned forms of subsets of A'_n is $\frac{n}{2} - 1$. In that case, $\Delta[W]$ will consist of two connected components and hence contribute 1 to $\beta_{3,4}(A'_n)$.

Therefore, $\beta_{3,4}(A'_n)$ is the total number of such subsets. That is, $\beta_{3,4}(A_{n,2n}) = n - 2$.

Case 4. For $i \geq 4$, we see that $\beta_{i,i+1}(A'_n) = 0$, since $\Delta[W]$ is connected for any subset $W \subseteq V(A'_n)$ with $|W| \geq i + 1$, by Lemma 3.8. □

Example 3.10. Consider the alternate triangular graph A'_6. Then for the edge ideal $I(A'_6) = \{x_1y_1, x_1y_2, x_2y_3, x_2y_4, x_3y_5, x_3y_6, y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_6\}$, the minimal graded free resolution of $R/I(A'_6)$ is given by

$$0 \to \begin{matrix} R[-9]^4 \\ \oplus \\ R[-8] \end{matrix} \to \begin{matrix} R[-8]^{15} \\ \oplus \\ R[-7]^{12} \end{matrix} \to \begin{matrix} R[-7]^{18} \\ \oplus \\ R[-6]^{42} \end{matrix} \to \begin{matrix} R[-6]^7 \\ \oplus \\ R[-5]^{56} \\ \oplus \\ R[-4]^4 \end{matrix} \to \begin{matrix} R[-4]^{25} \\ \oplus \\ R[-3]^{14} \end{matrix} \to R[-2]^{11} \to R \to 0.$$

Using Theorem 3.9, we have computed below the Betti numbers $\beta_{i,i+1}(A'_6)$, where $i = 1, 2, 3$ which are same as those computed below in the Betti table of the minimal free resolution of $R/I(A'_6)$ with the help of Singular 2.0 [2].

$$\beta_{1,2}(A'_6) = 2.6 - 11,$$
$$\beta_{2,3}(A'_6) = 3.6 - 4 = 14,$$
$$\beta_{3,4}(A'_6) = 6 - 2 = 4.$$

	0	1	2	3	4	5	6
0:	1	-	-	-	-	-	-
1:	-	11	14	4	-	-	-
2:	-	-	25	56	42	12	1
3:	-	-	-	7	18	15	4
total:	1	11	39	67	60	27	5

COROLLARY 3.11

The initial graded Betti numbers $\beta_{i,i+1}$ of the alternate triangular graph A''_n are given by

$$\beta_{i,i+1}(A''_n) = \begin{cases} 2n-3 & \text{if } i = 1 \\ 3n-6 & \text{if } i = 2 \\ n-2 & \text{if } i = 3 \\ 0 & \text{if } i \geq 4 \end{cases}.$$

Proof. By Hochster's formula, we have the following four cases to consider:

Case 1. For $i = 1$, we know that $\beta_{1,2}(A''_n)$ is the number of edges in A''. From Remark 3.7, we see that the number of edges in A''_n is equal to $2(n-2) - 1 + 2 = 2n - 3$. This implies that $\beta_{1,2}(A''_n) = \beta_{1,2}(A'_{n-2} + 2) = 2n - 3$.

Case 2. When $i = 2$, we see from Case 2 of Theorem 3.9 and Remark 3.7 that A''_n can be obtained from the alternate triangular graph A'_{n-2} by adding the two edges to A'_{n-2}. After doing so, the number of runs of length 3 in A'_{n-2} increases by 4. Therefore, $\beta_{2,3}(A''_n) = \beta_{2,3}(A'_{n-2}) + 4 = 3(n-2) - 4 + 4 = 3n - 6$.

Case 3. For $i = 3$, we shall make use of Case 3 of Theorem 3.9 and Remark 3.7. We see that, after adding the two edges to A'_{n-2}, the number of subsets $W \subseteq V(A'_{n-2})$ corresponding to which the induced subgraph $A'_{n-2}[W]$ is isomorphic to the graph G given in Figure 8, remains unaltered. Therefore, $\beta_{3,4}(A''_n) = \beta_{3,4}(A'_{n-2}) = n - 2$.

Case 4. For $i \geq 4$, we see that $\beta_{i,i+1}(A''_n) = 0$ for all $i \geq 4$, as in Case 4 of Theorem 3.9. □

Example 3.12. Consider the alternate triangular graph A''_6. Then for the edge ideal $I(A''_6) = \{x_1y_2, x_1y_3, x_2y_4, x_2y_5, y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_6\}$, the minimal graded free resolution of $R/I(A''_6)$ is given by

$$0 \to R[-8] \to R[-7]^8 \to R[-6]^{23} \longrightarrow \begin{matrix} R[-5]^{28} \\ \oplus \\ R[-4]^4 \end{matrix} \longrightarrow \begin{matrix} R[-4]^{12} \\ \oplus \\ R[-3]^{12} \end{matrix} \longrightarrow R[-2]^9 \to R \to 0.$$

Using Corollary 3.11, we have computed below the Betti numbers $\beta_{i,i+1}(A''_6)$, where $i = 1, 2, 3$ which are same as those computed below in the Betti table of the minimal free resolution of $R/I((A''_6))$ with the help of Singular 2.0 [2].

$$\begin{aligned} \beta_{1,2}(A''_6) &= 2.6 - 3 = 9, \\ \beta_{2,3}(A''_6) &= 3.6 - 6 = 12, \\ \beta_{3,4}(A''_6) &= 6 - 2 = 4. \end{aligned}$$

	0	1	2	3	4	5	6
0:	1	-	-	-	-	-	-
1:	-	9	12	4	-	-	-
2:	-	-	12	28	23	8	1
total:	1	9	24	32	23	8	1

COROLLARY 3.13

The initial graded Betti numbers $\beta_{i,i+1}$ of the alternate triangular graph A'''_n are given by

$$\beta_{i,i+1}(A'''_n) = \begin{cases} 2n - 2, & \text{if } i = 1, \\ 3n - 5, & \text{if } i = 2, \\ n - 2, & \text{if } i = 3, \\ 0, & \text{if } i \geq 4. \end{cases}$$

Proof. Again by Hochster's formula, we have the following four cases to consider:

Case 1. When $i = 1$, we know that $\beta_{1,2}(A'''_n)$ is same as the number of edges in A'''_n. One can note that by using Remark 3.7, the number of edges in A'''_n is equal to $2(n-1)-1+1 = 2n-2$. This implies that $\beta_{1,2}(A'''_n) = \beta_{1,2}(A'_{n-1} + 1) = 2n - 2$.

Case 2. When $i = 2$, using Case 2 of Theorem 3.9 and Remark 3.7, we see that A'''_n can be obtained from the alternate triangular graph A'_{n-1} by adding the edge to A'_{n-1}. After doing so, the number of runs of length 3 in A'_{n-1} increases by 2. Therefore, $\beta_{2,3}(A'''_n) = \beta_{2,3}(A'_{n-1}) + 2 = 3(n-1) - 4 + 2 = 3n - 5$.

Case 3. When $i = 3$, we shall make use of Case 3 of Theorem 3.9 and Remark 3.7. One can observe that after adding the edge to A'_{n-1}, the number of subsets $W \subset V(A'_{n-1})$ corresponding to which the induced subgraph $A'_{n-1}[W]$ is isomorphic to the graph given in Figure 8 remains unaltered. Therefore, $\beta_{3,4}(A'''_n) = \beta_{3,4}(A'_{n-1}) = n - 2$.

Case 4. For $i \geq 4$, we have $\beta_{i,i+1}(A'''_n) = 0$ for all $i \geq 4$ as in Case 4 of Theorem 3.9.

□

Example 3.14. Consider the alternate triangular graph A'''_5. Then for the edge ideal $I(A'''_5) = \{x_1y_1, x_1y_2, x_2y_3, x_2y_4, y_1y_2, y_2y_3, y_3y_4, y_4, y_5\}$, the minimal graded free resolution of $R/I(A'''_5)$ is given by

$$0 \to R[-7]^2 \to R[-6]^{10} \longrightarrow \begin{matrix} R[-5]^{16} \\ \oplus \\ R[-4]^3 \end{matrix} \longrightarrow \begin{matrix} R[-4]^{8} \\ \oplus \\ R[-3]^{10} \end{matrix} \longrightarrow R[-2]^8 \to R \to 0.$$

Using Corollary 3.13, we have computed below the Betti numbers $\beta_{i,i+1}(A'''_5)$, where $i = 1, 2, 3$ which are same as those computed below in the Betti table of the minimal free resolution of $R/I((A'''_5))$, with the help of Singular 2.0 [2].

$$\beta_{1,2}(A'''_5) = 2.5 - 2 = 8,$$
$$\beta_{2,3}(A'''_5) = 3.5 - 5 = 10,$$
$$\beta_{3,4}(A'''_5) = 5 - 2 = 3.$$

	0	1	2	3	4	5
0:	1	-	-	-	-	-
1:	-	8	10	3	-	-
2:	-	-	8	16	10	2
total:	1	8	18	19	10	2

Theorem 3.15. *For an alternate triangular graph A_n, the regularity is given by*

$$\operatorname{reg}(A'_n) = \frac{n}{2}, \quad \operatorname{reg}(A''_n) = \frac{n-2}{2} \quad \textit{and} \quad \operatorname{reg}(A'''_n) = \frac{n-1}{2}.$$

Proof. A_n being a chordal graph, by Theorem 2.1, the regularity of A_n is same as the induced matching number of A_n. To determine the induced matching number of A_n, we have the following cases to consider:

Case 1. If A_n is of Type 1, that is, $A_n = A'_n$, then clearly n is even. Let $V(A'_n) = \{x_1, x_2, \ldots, x_{\frac{n}{2}}, y_1, y_2, \ldots, y_n\}$. We claim the subset $\mathcal{E} = \left\{\{x_j, y_{2j-1}\} \mid 1 \le j \le \frac{n}{2}\right\}$ of the edge set $E(A'_n)$ is a maximal pairwise 3-disjoint subset. Indeed the edges in $\mathcal{E}$ are 3-disjoint. Suppose our claim is false, then there exists an edge $\{x_i, y_{2i}\}$, where $1 \le i \le \frac{n}{2}$ (or $e_j = \{y_j, y_{j+1}\}$, $1 \le j \le n$) that do not belong to $\mathcal{E}$ such that $\mathcal{E} \cup e$ (or $\mathcal{E} \cup e_j$) is a 3-disjoint subset of $E(A'_n)$. Suppose i is odd (or j is odd), then the edges e and $\{x_i, y_{2i-1}\} \in \mathcal{E}$ (or the edges e_j and $\{x_j, y_{2j-1}\} \in \mathcal{E}$) are adjacent. Therefore, we conclude that $\mathcal{E}$ is a maximal pairwise 3-disjoint subset of $E(T_n)$. The result is analogous when we assume i (or j) is even. Observe that the set $\mathcal{E}$ contains exactly one edge from each triangle of A'_n. Knowing the fact that any triangle in A'_n can contribute at most one edge to any 3-disjoint subset of $E(A'_n)$, we conclude that $\operatorname{reg}(A'_n) =$ number of triangles in $A'_n = \frac{n}{2}$.

Case 2. If A_n is of Type 2, that is, $A_n = A''_n$, then again n is an even integer. Since A''_n can be obtained from the alternate triangular graph A'_{n-2} on the vertex set

$$\{x_1, \ldots, x_{\frac{n-2}{2}}, y_2, \ldots, y_{n-1}\}$$

by adding two edges $\{y_1, y_2\}$ and $\{y_{n-1}, y_n\}$ to A'_{n-2}, one can see from Case 1 that the subset $\mathcal{E}' = \left\{\{x_j, y_{2j}\} \mid 1 \le j \le \frac{n-2}{2}\right\}$ of the edge set $E(A'_{n-2})$ is the largest pairwise 3-disjoint subset of $E(A'_{n-2})$. Therefore, $\mu(A'_{n-2}) = \frac{n-2}{2}$. Since the set $\mathcal{E}'$ contains exactly one edge from each triangle of A'_{n-2}, the induced matching number of A'_{n-2} do not alter by adding the edges $\{y_1, y_2\}$ and $\{y_{n-1}, y_n\}$ to A'_{n-2}. Therefore, we conclude that $\text{reg}(A''_n) = \frac{n-2}{2}$.

Case 3. If A_n is of Type 3, that is, $A_n = A'''_n$, then n is odd. Since A'''_n can be obtained from the alternate triangular graph A'_{n-1} on the vertex set $\{x_1, \ldots, x_{\frac{n-1}{2}}, y_1, \ldots, y_{n-1}\}$ by adding the edge $\{y_{n-1}, y_n\}$ to A'_{n-1}, one can notice from Case 1 that the subset $\mathcal{E}'' = \left\{\{x_j, y_{2j-1}\} \mid 1 \leq j \leq \frac{n-1}{2}\right\}$ of the edge set $E(A'_{n-1})$ is the largest pairwise 3-disjoint subset of $E(A'_{n-1})$. Therefore, $\mu(A'_{n-1}) = \frac{n-1}{2}$. Since the set $\mathcal{E}''$ contains exactly one edge from each triangle of A'_{n-1}, the induced matching number of A'_{n-1} do not alter by adding the edge $\{y_{n-1}, y_n\}$ to A'_{n-1}. Therefore, we have $\text{reg}(A'''_n) = \frac{n-1}{2}$. □

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